THE BEST CONSTANT OF SOBOLEV INEQUALITY WHICH CORRESPONDS TO SCHRÖDINGER OPERATOR WITH DIRAC DELTA POTENTIAL

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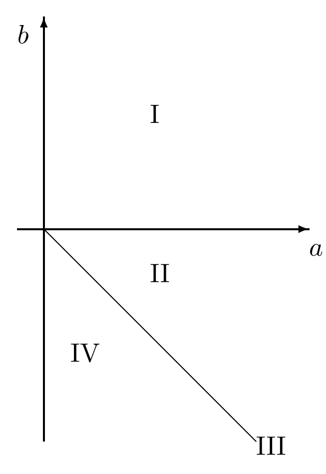
Abstract. We consider boundary value problems for the one-dimensional Schrödinger operator with Dirac delta potential. Green functions $G(x, y)$ are constructed by using the symmetric orthogonalization method, and their aspects as reproducing kernel are also investigated. As an application, the best constants of the corresponding Sobolev inequalities is expressed as the maximum of the diagonal value $G(y, y)$.

1 Conclusion

The present problem has two real parameters, a and b. We consider the following four cases.

I	$0 \leq b < \infty, \quad 0 < a < \infty$
II	$-\infty < b < 0, \quad \lvert b \rvert < a < \infty$
III	$-\infty < b < 0, \quad a = \lvert b \rvert$
IV	$-\infty < b < 0, \quad 0 < a < \lvert b \rvert$



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We introduce Sobolev space

$$H \;=\; H(a,b) \;=\; \left\{\, u(x) \;\middle|\; u(x),\;\; u'(x) \;\in\; L^2(-\infty,\infty),\right.$$
$$\left.\text{in case of III and IV, we require } \int_{-\infty}^{\infty} u(x)\, \exp(\, -|b||x|\,)\, dx \;=\; 0 \,\right\} \tag{1.1}$$

Sobolev inner product

$$(u,v)_H \;=\; \int_{-\infty}^{\infty} \Big[u'(x)\, \overline{v}'(x) \;+\; a^2\, u(x)\, \overline{v}(x) \Big] dx \;+\; 2b\, u(0)\, \overline{v}(0) \tag{1.2}$$

Sobolev energy

$$\|\, u \,\|_H^2 \;=\; (u,u)_H \;=\; \int_{-\infty}^{\infty} \Big[\, |\, u'(x)\,|^2 \;+\; a^2\, |\, u(x)\,|^2 \,\Big] dx \;+\; 2b\; |\, u(0)\,|^2 \tag{1.3}$$

and Sobolev functional

$$S(u) \;=\; \left(\sup_{-\infty<y<\infty} |\, u(y)\,| \right)^2 \Big/ \;\|\, u \,\|_H^2. \tag{1.4}$$

$(\cdot,\cdot)_H$ is proved to be an inner product of H afterwards. H is Hilbert space with an inner product $(\cdot,\cdot)_H$.

The purpose of the present paper is to find the supremum of Sobolev functional $S(u)$. The following two equivalent theorems were derived.

Theorem 1.1

$$\sup_{u\in H,\;\, u\not\equiv 0} S(u) \;=\; C(a,b) \;=\; \begin{cases} \dfrac{1}{2a} & (\text{I},\;\; \text{III},\;\; \text{IV}) \\[2ex] \dfrac{1}{2(a-|b|)} & (\text{II}) \end{cases} \tag{1.5}$$

Theorem 1.2 *There exists a positive constant C such that for any $u(x) \in H$ Sobolev inequality*

$$\left(\sup_{-\infty<y<\infty} |\, u(y)\,| \right)^2 \;\le\; C \left[\int_{-\infty}^{\infty} \Big[\, |\, u'(x)\,|^2 \;+\; a^2\, |\, u(x)\,|^2 \,\Big] dx \;+\; 2b\; |\, u(0)\,|^2 \right] \tag{1.6}$$

holds. The best constant $C(a,b)$ among such C is the same as that in (1.5).

The present paper is composed of five sections. In Section 2, we consider boundary value problems for one-dimensional Schrödinger operator with Dirac delta potential. In Section 3, Green functions are constructed. In particular, in cases III and IV, we use the symmetric orthogonalization method [2, 3, 4]. In Section 4, we show that Green functions are reproducing kernels for H and $(\cdot,\cdot)_H$ [1, 5]. Finally, Section 5 presents the proof of the main theorem, Theorem 1.2.

2 Boundary value problems

In this section, we explain the boundary value problem for second-order one-dimensional Schrödinger operator with Dirac delta potential.

We first survey the well-known case. For any $f(x) \in L^2(-\infty, \infty)$, the boundary value problem

$$\begin{cases} -u'' + a^2 u = f(x) \qquad (-\infty < x < \infty) & (2.1) \\ u(x),\ u'(x) \in L^2(-\infty, \infty) & (2.2) \end{cases}$$

has a unique solution expressed as

$$u(x) = \int_{-\infty}^{\infty} H(a\,;\,|x-y|)\,f(y)\,dy \qquad (-\infty < x < \infty) \tag{2.3}$$

$$H(a;x) = \frac{1}{2a}\exp(-ax) \qquad (0 \le x < \infty). \tag{2.4}$$

Note that

$$H'(a;x) = -a\,H(a;x), \qquad H''(a;x) = a^2\,H(a;x). \tag{2.5}$$

Before discussing the boundary value problem treated herein, we consider the following eigenvalue problem:

$$\begin{cases} -u'' + (a^2 + 2b\,\delta(x))\,u = \lambda\,u \qquad (-\infty < x < \infty) & (2.6) \\ u(x),\ u'(x) \in L^2(-\infty, \infty) & (2.7) \end{cases}$$

where $\delta(x)$ is Dirac delta function. If $b \ge 0$ then $\lambda = 0$ is not an eigenvalue. However, if $b < 0$ and $a = |b|$, $\lambda = 0$ is an eigenvalue and the corresponding eigenspace is one-dimensional. The normalized eigenfunction is given by

$$\varphi(x) = 2|b|^{3/2}\,H(\,|b|\,;\,|x|\,) \qquad (-\infty < x < \infty). \tag{2.8}$$

For $f(x) \in L^2(-\infty, \infty)$ satisfying the solvability condition

$$\text{S (Solvability condition)}: \begin{cases} \text{none} & (\text{I, II}) \\ \displaystyle\int_{-\infty}^{\infty} f(y)\,\varphi(y)\,dy = 0 & (\text{III, IV}) \end{cases} \tag{2.9}$$

we consider the following boundary value problem:

BVP

$$\begin{cases} -u'' + (a^2 + 2b\,\delta(x))\,u = f(x) \qquad (-\infty < x < \infty) & (2.10) \\ u(x),\ u'(x) \in L^2(-\infty, \infty) & (2.11) \\ \text{O (Orthogonality condition)}: \begin{cases} \text{none} & (\text{I, II}) \\ \displaystyle\int_{-\infty}^{\infty} u(x)\varphi(x)dx = 0 & (\text{III, IV}). \end{cases} & (2.12) \end{cases}$$

Theorem 2.1 *For any $f(x) \in L^2(-\infty, \infty)$ which satisfies the condition* S, BVP *possesses a unique solution $u(x)$ expressed as*

$$u(x) = \int_{-\infty}^{\infty} G(x,y)\,f(y)\,dy \qquad (-\infty < x < \infty) \tag{2.13}$$

where Green function $G(x,y)$ is given by Theorem 2.2.

Theorem 2.2 *Green function $G(x,y)$ is given by* (1) *and satisfies properties* (2)∼(8).

(1) $G(x,y) =$

$$\begin{cases} H(\,a\,;\,|x-y|\,) \,-\, \dfrac{2ab}{a+b}\, H(\,a\,;\,|x|\,)\, H(\,a\,;\,|y|\,) & \text{(I, II)} \\ H(\,a\,;\,|x-y|\,) \,-\, 2a\left(a|x| \,+\, a|y| \,+\, \dfrac{1}{2}\right) H(\,a\,;\,|x|\,)\, H(\,a\,;\,|y|\,) & \text{(III)} \\ H(\,a\,;\,|x-y|\,) \,-\, \dfrac{2a|b|}{|b|-a}\, H(\,a\,;\,|x|\,)\, H(\,a\,;\,|y|\,) + \\ \dfrac{4|b|^3}{|b|^2-a^2}\, H(\,|b|\,;\,|x|\,)\, H(\,|b|\,;\,|y|\,) & \text{(IV)} \end{cases}$$

$$(-\infty < x, y < \infty) \tag{2.14}$$

(2) $\partial_x G(x,y) =$

$$\begin{cases} -\,a\,\mathrm{sgn}(x-y)\, H(\,a\,;\,|x-y|\,) \,+\, \dfrac{2a^2 b}{a+b}\, \mathrm{sgn}(x)\, H(\,a\,;\,|x|\,)\, H(\,a\,;\,|y|\,) & \text{(I, II)} \\ -\,a\,\mathrm{sgn}(x-y)\, H(\,a\,;\,|x-y|\,) + \\ 2a^2\,\mathrm{sgn}(x)\left(a|x| \,+\, a|y| \,-\, \dfrac{1}{2}\right) H(\,a\,;\,|x|\,)\, H(\,a\,;\,|y|\,) & \text{(III)} \\ -\,a\,\mathrm{sgn}(x-y)\, H(\,a\,;\,|x-y|\,) \,+\, \dfrac{2a^2|b|}{|b|-a}\, \mathrm{sgn}(x)\, H(\,a\,;\,|x|\,)\, H(\,a\,;\,|y|\,) - \\ \dfrac{4|b|^4}{|b|^2-a^2}\, \mathrm{sgn}(x)\, H(\,|b|\,;\,|x|\,)\, H(\,|b|\,;\,|y|\,) & \text{(IV)} \end{cases}$$

$$(-\infty < x, y < \infty, \quad x \neq 0, y) \tag{2.15}$$

(3) $\partial_x^2 G(x,y) =$

$$\begin{cases} a^2\, H(\,a\,;\,|x-y|\,) \,-\, \dfrac{2a^3 b}{a+b}\, H(\,a\,;\,|x|\,)\, H(\,a\,;\,|y|\,) \,=\, a^2\, G(x,y) & \text{(I, II)} \\ a^2\, H(\,a\,;\,|x-y|\,) \,-\, 2a^3\left(a|x| \,+\, a|y| \,-\, \dfrac{3}{2}\right) H(\,a\,;\,|x|\,)\, H(\,a\,;\,|y|\,) \,= \\ a^2\, G(x,y) \,+\, \varphi(x)\,\varphi(y) & \text{(III)} \\ a^2\, H(\,a\,;\,|x-y|\,) \,-\, \dfrac{2a^3|b|}{|b|-a}\, H(\,a\,;\,|x|\,)\, H(\,a\,;\,|y|\,) + \\ \dfrac{4|b|^5}{|b|^2-a^2}\, H(\,|b|\,;\,|x|\,)\, H(\,|b|\,;\,|y|\,) \,=\, a^2\, G(x,y) \,+\, \varphi(x)\,\varphi(y) & \text{(IV)} \end{cases}$$

$$(-\infty < x, y < \infty, \quad x \neq 0, y) \tag{2.16}$$

(4) $G(0,y) =$

$$\begin{cases} \dfrac{a}{a+b}\, H(\,a\,;\,|y|\,) & \text{(I, II)} \\ -\left(a|y| - \dfrac{1}{2}\right) H(\,a\,;\,|y|\,) & \text{(III)} \\ -\dfrac{a}{|b|-a}\, H(\,a\,;\,|y|\,) + \dfrac{2|b|^2}{|b|^2-a^2}\, H(\,|b|\,;\,|y|\,) & \text{(IV)} \end{cases}$$

$$(-\infty < y < \infty) \tag{2.17}$$

(5) $\partial_x G(x,y)\Big|_{x=-0} - \partial_x G(x,y)\Big|_{x=+0} =$

$$\begin{cases} -\dfrac{2ab}{a+b}\, H(\,a\,;\,|y|\,) = -2b\, G(0,y) & \text{(I, II)} \\ -2a\left(a|y| - \dfrac{1}{2}\right) H(\,a\,;\,|y|\,) = 2a\, G(0,y) & \text{(III)} \\ -\dfrac{2a|b|}{|b|-a}\, H(\,a\,;\,|y|\,) + \dfrac{4|b|^3}{|b|^2-a^2}\, H(\,|b|\,;\,|y|\,) = 2|b|\, G(0,y) & \text{(IV)} \end{cases}$$

$$(-\infty < y < \infty) \tag{2.18}$$

(6)
$$\begin{cases} G(x,y)\Big|_{y=x-0} - G(x,y)\Big|_{y=x+0} = 0 \\ \partial_x G(x,y)\Big|_{y=x-0} - \partial_x G(x,y)\Big|_{y=x+0} = -1 \qquad \text{(I, II, III, IV)} \end{cases}$$

$$(-\infty < x < \infty) \tag{2.19}$$

(7)
$$\begin{cases} G(x,y)\Big|_{x=y+0} - G(x,y)\Big|_{x=y-0} = 0 \\ \partial_x G(x,y)\Big|_{x=y+0} - \partial_x G(x,y)\Big|_{x=y-0} = -1 \qquad \text{(I, II, III, IV)} \end{cases}$$

$$(-\infty < y < \infty) \tag{2.20}$$

(8)
$$\int_{-\infty}^{\infty} \varphi(x)\, G(x,y)\, dx = 0 \qquad \text{(III, IV)} \qquad (-\infty < y < \infty) \tag{2.21}$$

Expression (1) of Green function is derived in Section 3. Properties (2)∼(8) are shown through simple calculations.

3 Symmetric orthogonalization

In this section, we derive expression (1) (Theorem 2.2) of Green function by means of the symmetric orthogonalization method [2, 3, 4].

We first treat the simple cases I and II. Since the solution $u(x)$ of BVP satisfies

$$-u'' + a^2 u = f(x) - 2b\, u(0)\, \delta(x) \qquad (-\infty < x < \infty)$$

we have

$$u(x) \;=\; \int_{-\infty}^{\infty} H(\,a\,;\,|x-y|\,)\,\Big[f(y) \;-\; 2b\,u(0)\,\delta(y)\Big]\,dy \;=$$

$$\int_{-\infty}^{\infty} H(\,a\,;\,|x-y|\,)\,f(y)\,dy \;-\; 2b\,u(0)\,H(\,a\,;\,|x|\,). \tag{3.1}$$

Setting $x=0$, we have

$$u(0) \;=\; \frac{a}{a+b}\int_{-\infty}^{\infty} H(\,a\,;\,|y|\,)\,f(y)\,dy.$$

Thus, we obtain

$$u(x) \;=\; \int_{-\infty}^{\infty}\Big[H(\,a\,;\,|x-y|\,) \;-\; \frac{2ab}{a+b}\,H(\,a\,;\,|x|\,)\,H(\,a\,;\,|y|\,)\Big]\,f(y)\,dy.$$

Next, we consider case III. We assume that the function $u(x)$ satisfies the conditions (2.10) and (2.11). For (3.1) putting $b=-a$ and $x=0$, we have

$$\int_{-\infty}^{\infty} f(y)\,H(\,a\,;\,|y|\,)\,dy \;=\; 0 \tag{3.2}$$

and $u(0)$ is not determined. (3.2) is the necessary condition of the existence of classical solution to (2.10) and (2.11). The solution $u(x)$ is expressed as

$$u(x) \;=\; \int_{-\infty}^{\infty} H(\,a\,;\,|x-y|\,)\,f(y)\,dy \;+\; \alpha\,\varphi(x) \qquad (-\infty<x<\infty)$$

where α is a suitable constant.

Green function of BVP is constructed by the symmetric orthogonalizaion method starting from the above proto Green function $H(\,a\,;\,|x-y|\,)$. Namely, Green function $G(x,y)$ is constructed from $H(\,a\,;\,|x-y|\,)$ as follows:

$$\begin{aligned}
&G(x,y) \;=\; H(\,a\,;\,|x-y|\,) \;-\\
&\varphi(x)\int_{-\infty}^{\infty}\varphi(x')\,H(\,a\,;\,|x'-y|\,)\,dx' \;-\; \int_{-\infty}^{\infty} H(\,a\,;\,|x-y'|\,)\,\varphi(y')\,dy'\,\varphi(y) \;+\\
&\varphi(x)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi(x')\,H(\,a\,;\,|x'-y'|\,)\,\varphi(y')\,dx'\,dy'\,\varphi(y) \qquad (-\infty<x,y<\infty).
\end{aligned} \tag{3.3}$$

We next prove the following lemma.

Lemma 3.1 *If we introduce a function $\psi(x)$ by*

$$\psi(x) \;=\; \int_{-\infty}^{\infty} H(\,a\,;\,|x-y|\,)\,\varphi(y)\,dy \qquad (-\infty<x<\infty) \tag{3.4}$$

then we have

(1) $\psi(x) \;=\; a^{-1/2}\,(\,a|x| \;+\; 1\,)\,H(\,a\,;\,|x|\,) \qquad (-\infty<x<\infty)$ (3.5)

(2) $\displaystyle\int_{-\infty}^{\infty}\psi(x)\,\varphi(x)\,dx \;=\; \frac{3}{4a^2}.$ (3.6)

Proof of Lemma 3.1 We first introduce a fundamental solution of the heat equation given by

$$h(x,t) \;=\; (4\pi t)^{-1/2} \exp(\,-x^2/(4t)\,) \qquad (-\infty < x < \infty, \quad 0 < t < \infty). \tag{3.7}$$

The proto Green function is expressed as

$$H(\,a\,;\,|x|\,) \;=\; \int_0^\infty \exp(\,-a^2 t\,)\, h(x,t)\, dt \qquad (-\infty < x < \infty). \tag{3.8}$$

The function $\psi(x)$ is calculated as

$$\frac{1}{2a^{3/2}}\,\psi(x) \;=\; \int_{-\infty}^\infty H(\,a\,;\,|x-y|\,)\, H(\,a\,;\,|y|\,)\, dy \;=$$
$$\int_{-\infty}^\infty \int_0^\infty \exp(\,-a^2 t\,)\, h(x-y,t)\, dt \int_0^\infty \exp(\,-a^2 s\,)\, h(y,s)\, ds\, dy \;=$$
$$\int_0^\infty \int_0^\infty \exp(\,-a^2(t+s)\,) \int_{-\infty}^\infty h(x-y,t)\, h(y,s)\, dy\, dt\, ds \;=$$
$$\int_0^\infty \int_0^\infty \exp(\,-a^2(t+s)\,)\, h(x,t+s)\, dt\, ds \;=$$
$$(\tau = t+s,\ \sigma = t-s)$$
$$\frac{1}{2}\int_0^\infty \int_{-\tau}^\tau \exp(\,-a^2 \tau\,)\, h(x,\tau)\, d\sigma\, d\tau \;=\; \int_0^\infty \tau\, \exp(\,-a^2 \tau\,)\, h(x,\tau)\, d\tau \;=$$
$$-\,\frac{1}{2a}\,\partial_a H(\,a\,;\,|x|\,) \;=\; \frac{1}{2a^2}\,(\,a|x|+1\,)\, H(\,a\,;\,|x|\,).$$

Hence, we have

$$\psi(x) \;=\; a^{-1/2}(\,a|x|+1\,)\, H(\,a\,;\,|x|\,) \qquad (-\infty < x < \infty)$$

which proves (1). Then, (2) is shown from (1) through simple calculations. ■

From Lemma 3.1, Green function $G(x,y)$ is given by

$$G(x,y) \;=\; H(\,a\,;\,|x-y|\,) \,-\, \varphi(x)\,\psi(y) \,-\, \psi(x)\,\varphi(y) \,+\, \frac{3}{4a^2}\,\varphi(x)\,\varphi(y) \;=$$
$$H(\,a\,;\,|x-y|\,) \,-\, 2a\left(a|x| \,+\, a|y| \,+\, \frac{1}{2}\right) H(\,a\,;\,|x|\,)\, H(\,a\,;\,|y|\,)$$
$$(-\infty < x, y < \infty). \tag{3.9}$$

This shows (1) of Theorem 2.2 for case III.

Finally, we treat case IV. For any $f(x) \in L^2(-\infty,\infty)$ satisfying condition S, the boundary value problem

$$\begin{cases} -\,u'' \,+\, (\,a^2 \,-\, 2|b|\,\delta(x)\,)\,u \;=\; f(x) \qquad (-\infty < x < \infty) & (3.10) \\ u(x), u'(x) \;\in\; L^2(-\infty,\infty) & (3.11) \end{cases}$$

has a unique classical solution $u(x)$, which is expressed as

$$u(x) \;=\; \int_{-\infty}^\infty G_0(x,y)\, f(y)\, dy \qquad (-\infty < x < \infty) \tag{3.12}$$

where $G_0(x,y)$ is the proto Green function given by

$$G_0(x,y) \;=\; H(\,a\,;\,|x-y|\,) \;-\; \frac{2a|b|}{|b|-a}\,H(\,a\,;\,|x|\,)\,H(\,a\,;\,|y|\,) \qquad (-\infty < x, y < \infty). \tag{3.13}$$

Green function $G(x,y)$ is constructed by means of $G_0(x,y)$ as

$$G(x,y) \;=\; G_0(x,y) \;-\; \varphi(x)\int_{-\infty}^{\infty}\varphi(x')\,G_0(x',y)\,dx' \;-\; \int_{-\infty}^{\infty}G_0(x,y')\,\varphi(y')\,dy'\,\varphi(y) \;+$$
$$\varphi(x)\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi(x')\,G_0(x',y')\,\varphi(y')\,dx'\,dy'\,\varphi(y) \qquad (-\infty < x, y < \infty). \tag{3.14}$$

Before calculating the above integral, we prepare the following lemma.

Lemma 3.2

(1) $$\int_{-\infty}^{\infty}H(\,a\,;\,|x-y|\,)\,H(\,|b|\,;\,|y|\,)\,dy \;=\; \frac{1}{|b|^2-a^2}\Big(H(\,a\,;\,|x|\,) \;-\; H(\,|b|\,;\,|x|\,)\Big)$$
$$(-\infty < x < \infty) \tag{3.15}$$

(2) $$\int_{-\infty}^{\infty}H(\,a\,;\,|y|\,)\,H(\,|b|\,;\,|y|\,)\,dy \;=\; \frac{1}{2a|b|\,(a+|b|)} \tag{3.16}$$

Proof of Lemma 3.2 Since the proof of (2) is easy, we treat only (1).

$$\int_{-\infty}^{\infty}H(\,a\,;\,|x-y|\,)\,H(\,|b|\,;\,|y|\,)\,dy \;=$$
$$\int_{-\infty}^{\infty}\int_{0}^{\infty}\exp(\,-a^2t\,)\,h(x-y,t)\,dt\int_{0}^{\infty}\exp(\,-|b|^2s\,)\,h(y,s)\,ds\,dy \;=$$
$$\int_{0}^{\infty}\int_{0}^{\infty}\exp(\,-a^2t-|b|^2s\,)\int_{-\infty}^{\infty}h(x-y,t)\,h(y,s)\,dy\,dt\,ds \;=$$
$$\int_{0}^{\infty}\int_{0}^{\infty}\exp(\,-a^2t-|b|^2s\,)h(x,t+s)\,dt\,ds \;=$$
$$\frac{1}{2}\int_{0}^{\infty}\left(\int_{-\tau}^{\tau}\exp(\,-a^2(\tau+\sigma)/2-|b|^2(\tau-\sigma)/2\,)h(x,\tau)\,d\sigma\right)\,d\tau \;=$$
$$(\tau=t+s,\;\;\sigma=t-s)$$
$$\frac{1}{2}\int_{0}^{\infty}\int_{-\tau}^{\tau}\exp(\,(|b|^2-a^2)\sigma/2\,)\,d\sigma\,\exp(\,-(a^2+|b|^2)\tau/2\,)\,h(x,\tau)\,d\tau \;=$$
$$\frac{1}{|b|^2-a^2}\int_{0}^{\infty}\Big[\exp(\,-a^2\tau\,)-\exp(\,-|b|^2\tau\,)\Big]h(x,\tau)\,d\tau \;=$$
$$\frac{1}{|b|^2-a^2}\Big[H(\,a\,;\,|x|\,)-H(\,|b|\,;\,|x|\,)\Big]$$

which completes the proof. ■

The following lemma is a direct consequence of Lemma 3.2.

Lemma 3.3 *If we introduce a function $\psi(x)$ by*

$$\psi(x) \;=\; \int_{-\infty}^{\infty} G_0(x,y)\,\varphi(y)\,dy \qquad (-\infty < x < \infty) \tag{3.17}$$

then we have

(1) $$\psi(x) \;=\; -\,\frac{2|b|^{3/2}}{|b|^2-a^2}\,H(\,|b|\,;\,|x|\,) \;=\; -\,\frac{1}{|b|^2-a^2}\,\varphi(x) \qquad (-\infty < x < \infty) \tag{3.18}$$

(2) $$g_0 \;=\; \int_{-\infty}^{\infty} \psi(x)\,\varphi(x)\,dx \;=\; -\,\frac{1}{|b|^2-a^2}. \tag{3.19}$$

From Lemma 3.3, we have

$$\begin{aligned} &G(x,y) \;=\; G_0(x,y) \;-\; \varphi(x)\,\psi(y) \;-\; \psi(x)\,\varphi(y) \;+\; g_0\,\varphi(x)\,\varphi(y) \;=\\ &H(\,a\,;\,|x-y|\,) \;-\; \frac{2a|b|}{|b|-a}\,H(\,a\,;\,|x|\,)\,H(\,a\,;\,|y|\,) \;+\; \frac{4|b|^3}{|b|^2-a^2}\,H(\,|b|\,;\,|x|\,)\,H(\,|b|\,;\,|y|\,)\\ &(-\infty < x, y < \infty). \end{aligned} \tag{3.20}$$

This shows (1) of Theorem 2.2 for case IV. ■

4 Reproducing kernel

In this section, we show that Green functions $G(x,y)$ are also reproducing kernels for Hilbert space H equipped with its inner product $(\cdot,\cdot)_H$. First, we show the following lemma.

Lemma 4.1 $(\cdot,\cdot)_H$ *defined by* (1.2) *is an inner product of* H.

Proof of Lemma 4.1 It is sufficient to prove that $(u,u)_H = 0$ implies $u(x) = 0$ $(-\infty < x < \infty)$ in cases I $\sim$ IV.

Since this is obvious for case I, we consider case II. We first rewrite $(u,u)_H$ as

$$(u,u)_H \;=\; \int_{-\infty}^{\infty} \Big[\,|\,u'(x)\,|^2 \;+\; a^2\,|\,u(x)\,|^2\,\Big]\,dx \;-\; 2|b|\,|\,u(0)\,|^2 \;=\; \mathrm{I}_1 \;+\; \mathrm{I}_2 \;+\; \mathrm{I}_3$$

where

$$\mathrm{I}_1 \;=\; \int_{-\infty}^{\infty} \Big[\,|\,u'(x)\,|^2 \;+\; a^2\,|\,u(x)\,|^2\,\Big]\,dx \;-\; 2a\left(\sup_{-\infty<y<\infty}|\,u(y)\,|\right)^2,$$

$$\mathrm{I}_2 \;=\; 2(a-|b|)\left(\sup_{-\infty<y<\infty}|\,u(y)\,|\right)^2, \qquad \mathrm{I}_3 \;=\; 2|b|\left[\left(\sup_{-\infty<y<\infty}|\,u(y)\,|\right)^2 \;-\; |\,u(0)\,|^2\right].$$

$\mathrm{I}_3 \ge 0$ is obvious. Since $\mathrm{I}_1 \ge 0$ holds in the special case of Theorem 1.2 $(b = 0)$, i.e. Sobolev inequality in a usual sence, we have $(u,u)_H \ge \mathrm{I}_2$. Note that $a - |b| > 0$, $u(x) = 0$ $(-\infty < x < \infty)$ follows from $(u,u)_H = 0$.

We next consider case III. The eigenfunction

$$\varphi(x) \;=\; 2|b|^{3/2}\,H(\,|b|\,;\,|x|\,) \;=\; |b|^{1/2}\exp(\,-|b||x|\,) \qquad (-\infty < x < \infty)$$

satisfies the following relations:

$$\varphi'(x) \;=\; -\,|b|\,\mathrm{sgn}(x)\,\varphi(x), \qquad \varphi''(x) \;=\; |b|^2\varphi(x) \qquad (-\infty < x < \infty, \quad x \neq 0),$$
$$\varphi'(-0) \;-\; \varphi'(+0) \;-\; 2\,|b|\,\varphi(0) \;=\; 0.$$

Using the above relations, we have

$$\begin{aligned}
&(\varphi,\varphi)_H \;=\; \int_{-\infty}^{\infty} \Big[(\,\varphi'(x)\,)^2 \,+\, |b|^2\,\varphi^2(x)\Big]\,dx \;-\; 2|b|\,\varphi^2(0) \;=\\
&\left\{\int_{-\infty}^{0} + \int_{0}^{\infty}\right\} \left[\Big(\varphi(x)\,\varphi'(x)\Big)' \,+\, \varphi(x)\Big(-\,\varphi''(x) + |b|^2\varphi(x)\Big)\right]dx \;-\; 2|b|\,\varphi^2(0) \;=\\
&\varphi(0)\Big[\varphi'(-0) - \varphi'(+0) - 2|b|\varphi(0)\Big] \;=\; 0
\end{aligned} \tag{4.1}$$

and for any $u(x) \in H$

$$\begin{aligned}
&(u,\varphi)_H \;=\; \int_{-\infty}^{\infty} \Big[u'(x)\,\varphi'(x) \,+\, |b|^2\,u(x)\,\varphi(x)\Big]\,dx \;-\; 2|b|\,u(0)\,\varphi(0) \;=\\
&\left\{\int_{-\infty}^{0} + \int_{0}^{\infty}\right\} \left[\Big(u(x)\,\varphi'(x)\Big)' \,+\, u(x)\Big(-\,\varphi''(x) + |b|^2\varphi(x)\Big)\right]dx \;-\; 2|b|\,u(0)\,\varphi(0) \;=\\
&u(0)\Big[\varphi'(-0) - \varphi'(+0) - 2|b|\varphi(0)\Big] \;=\; 0.
\end{aligned} \tag{4.2}$$

In order to prove that $(u,u)_H = 0$ implies $u(x) = 0$ $(-\infty < x < \infty)$, we introduce the function $v(x)$ defined by $v(x) = u(x) - \alpha\varphi(x)$, $\alpha = u(0)/\varphi(0)$. Under the assumption $(u,u)_H = 0$, we have

$$(v,v)_H \;=\; (\,u - \alpha\,\varphi,\;\; u - \alpha\,\varphi\,)_H \;=\; (u,u)_H \;-\; 2\mathrm{Re}\,\overline{\alpha}\,(u,\varphi)_H \;+\; |\alpha|^2(\varphi,\varphi)_H \;=\; 0$$

from (4.1) and (4.2). On the other hand, we have

$$(v,v)_H \;=\; \int_{-\infty}^{\infty} \Big[\,|\,v'(x)\,|^2 \;+\; |b|^2\,|\,v(x)\,|^2\,\Big]\,dx$$

from the definition and $v(0) = 0$. Hence, the relation $v(x) = 0$ or equivalently $u(x) = \alpha\varphi(x)$ $(-\infty < x < \infty)$ holds. From the orthogonality relation $\int_{-\infty}^{\infty} u(x)\varphi(x)dx = 0$, we conclude that $u(x) = 0$ $(-\infty < x < \infty)$.

Finally, we consider case IV. First, we have

$$\begin{aligned}
&(\varphi,\varphi)_H \;=\; \int_{-\infty}^{\infty} \Big[(\,\varphi'(x)\,)^2 \,+\, a^2\varphi^2(x)\Big]\,dx \;-\; 2|b|\,\varphi^2(0) \;=\\
&\left\{\int_{-\infty}^{0} + \int_{0}^{\infty}\right\} \left[\Big(\varphi(x)\,\varphi'(x)\Big)' \,+\, \varphi(x)\Big(-\,\varphi''(x) + a^2\varphi(x)\Big)\right]dx \;-\; 2|b|\,\varphi^2(0) \;=\\
&\varphi(0)\Big[\varphi'(-0) - \varphi'(+0) - 2|b|\varphi(0)\Big] \;-\; (|b|^2 - a^2)\int_{-\infty}^{\infty} \varphi^2(x)\,dx \;=\; -\,(|b|^2 - a^2)
\end{aligned} \tag{4.3}$$

and for any $u(x) \in H$

$$
\begin{aligned}
&(u, \varphi)_H \;=\; \int_{-\infty}^{\infty} \Big[u'(x)\,\varphi'(x) \;+\; a^2\,u(x)\,\varphi(x)\Big]\,dx \;-\; 2|b|\,u(0)\,\varphi(0) \;= \\
&\left\{\int_{-\infty}^{0} + \int_{0}^{\infty}\right\} \bigg[\Big(u(x)\,\varphi'(x)\Big)' \;+\; u(x)\Big(-\;\varphi''(x) + a^2\varphi(x)\Big)\bigg]dx \;-\; 2|b|\,u(0)\,\varphi(0) \;= \\
&u(0)\Big[\varphi'(-0) - \varphi'(+0) - 2|b|\,\varphi(0)\Big] \;+\; (a^2 - |b|^2)\int_{-\infty}^{\infty} u(x)\,\varphi(x)\,dx \;=\; 0. \qquad (4.4)
\end{aligned}
$$

We show that $(u,u)_H = 0$ implies $u(x) = 0$ $(-\infty < x < \infty)$. As in case III, we introduce the function $v(x)$ defined by $v(x) = u(x) - \alpha\varphi(x),\;\; \alpha = u(0)/\varphi(0)$. We have

$$
(v,v)_H \;=\; (u,u)_H \;-\; 2\mathrm{Re}\,\overline{\alpha}\,(u,\varphi)_H \;+\; |\alpha|^2(\varphi,\varphi)_H \;=\; (u,u)_H \;-\; |\alpha|^2\,(|b|^2 - a^2).
$$

On the other hand, from $v(0) = 0$, we have

$$
(v,v)_H \;=\; \int_{-\infty}^{\infty}\left[\,|\,v'(x)\,|^2 \;+\; a^2\;|\,v(x)\,|^2\right]dx \;\geq\; a^2\int_{-\infty}^{\infty}|\,v(x)\,|^2\,dx.
$$

It follows that

$$
(u,u)_H \;=\; (v,v)_H \;+\; |\alpha|^2\,(\,|b|^2 - a^2\,) \;\geq\; a^2\int_{-\infty}^{\infty}|\,v(x)\,|^2\,dx \;+\; |\alpha|^2\,(|b|^2 - a^2).
$$

If $(u,u)_H = 0$ then we have $v(x) = 0,\;\, u(x) = \alpha\varphi(x)$ $(-\infty < x < \infty)$. From the orthogonality relation $\int_{-\infty}^{\infty} u(x)\varphi(x)dx = 0$, we have $\alpha = 0$ and therefore $u(x) = 0$ $(-\infty < x < \infty)$.

This completes the proofD ■

From Lemma 4.1, it is shown that H is Hilbert space with inner product $(\cdot,\cdot)_H$.

Theorem 4.1 (1) *Green function $G(x,y)$ is a reproducing kernel for Hilbert space H with inner product $(\cdot,\cdot)_H$. That is to say, for any function $u(x) \in H$, we have the following reproducing relation:*

$$
u(y) \;=\; (\,u(x),\,G(x,y)\,)_H \qquad (-\infty < y < \infty). \tag{4.5}
$$

(2) $G(y,y) \;=\; (\,G(x,y),\,G(x,y)\,)_H \;=\; \|\,G(x,y)\,\|_H^2 \qquad (-\infty < y < \infty).$ (4.6)

Proof of Theorem 4.1 The left-hand side of (4.5) is calculated as

$$
\begin{aligned}
&(\, u(x),\, G(x,y)\,)_H \;= \\
&\int_{-\infty}^{\infty} \Big[u'(x)\,\partial_x \,+\, a^2\,u(x) \Big] G(x,y)\,dx \,+\, 2b\,u(0)\,G(0,y) \;= \\
&\left\{ \begin{array}{ll} \displaystyle\int_{-\infty}^{0} + \int_{0}^{y} + \int_{y}^{\infty} & (0<y<\infty) \\ \displaystyle\int_{-\infty}^{y} + \int_{y}^{0} + \int_{0}^{\infty} & (-\infty<y<0) \end{array} \right\} \Big[\, \partial_x \Big(u(x)\,\partial_x G(x,y) \Big) + \\
&u(x) \Big(-\,\partial_x^2 \,+\, a^2 \Big) G(x,y) \,\Big] dx \,+\, 2b\,u(0)\,G(0,y) \;= \\
&u(y)\, \Big[\partial_x G(x,y) \Big|_{x=y-0} \,-\, \partial_x G(x,y) \Big|_{x=y+0} \Big] \,+ \\
&u(0)\, \Big[\partial_x G(x,y) \Big|_{x=-0} \,-\, \partial_x G(x,y) \Big|_{x=+0} \,+\, 2b\,G(0,y) \Big] \,- \\
&\left\{ \begin{array}{ll} 0 & (\mathrm{I,\ II}) \\ \displaystyle\Big(\int_{-\infty}^{\infty} u(x)\,\varphi(x)\,dx \Big)\,\varphi(y) & (\mathrm{III,\ IV}) \end{array} \right\} \;=\; u(y)
\end{aligned}
$$

where we have used (2.16), (2.18), and (2.20) in Theorem 2.2. We have (1). (2) follows from (1) by putting $u(x) = G(x,y)$ in (4.5). This shows Theorem 4.1. ■

5 Proof of Theorem 1.2

In this section, we prove Theorem 1.2. We prepare the following lemma concerning $C(a,b)$.

Lemma 5.1 *The best constant $C(a,b)$ is given by*

$$
C(a,b) \;=\; \sup_{-\infty<y<\infty} G(y,y) \;=\; \left\{ \begin{array}{ll} G(y,y)\Big|_{y=\pm\infty} \;=\; \dfrac{1}{2a} & (\mathrm{I,\ III,\ IV}) \\ G(0,0) \;=\; \dfrac{1}{2(a-|b|)} & (\mathrm{II}). \end{array} \right. \tag{5.1}
$$

Proof of Lemma 5.1 In cases I and II, we have

$$
G(y,y) \;=\; H(a;0) \,-\, \frac{2ab}{a+b}\,H^2(\,a\,;\,|y|\,) \;=\; \frac{1}{2a}\left[1 \,-\, \frac{b}{a+b}\,\exp(\,-2a|y|\,) \right] \qquad (-\infty<y<\infty). \tag{5.2}
$$

Since $a+b>0$, $G(y,y)$ attains its supremum at $y=\pm\infty$ if $b \geq 0$ and $y=0$ if $b<0$.

In case III, we observe the behavior of

$$
G(y,y) \;=\; \frac{1}{2a}\left[1 \,-\, \Big(2a|y| \,+\, \frac{1}{2} \Big) \exp(\,-2a|y|\,) \right] \qquad (-\infty<y<\infty). \tag{5.3}
$$

It is sufficient to investigate

$$
g(y) \;=\; 1 \,-\, (y+1/2)\exp(-y) \qquad (0<y<\infty).
$$

Since we have

$$g'(y) \;=\; (y-1/2)\,\exp(-y)\;\begin{cases} <\;0 & (0<y<1/2) \\ =\;0 & (y=1/2) \\ >\;0 & (1/2<y<\infty), \end{cases}$$

$$g(0)\;=\;1/2, \qquad g(+\infty)\;=\;1,$$

$g(y)$ attains its minimum $g(1/2) \;=\; 1\;-\;\exp(-1/2)\;>\;0$ at $y=1/2$. Hence, we have

$$\sup_{0<y<\infty} g(y) \;=\; g(+\infty)\;=\;1.$$

Finally, we consider case IV. In order to observe the behavior of

$$G(y,y)\;=\;\frac{1}{2a}\left[1\;-\;\frac{|b|}{|b|-a}\;\exp(\,-2a|y|\,)\;+\;\frac{2a|b|}{|b|^2-a^2}\;\exp(\,-2|b||y|\,)\right]$$
$$(-\infty<y<\infty), \tag{5.4}$$

it is sufficient to investigate

$$g(y)\;=\;1\;-\;\frac{|b|}{|b|-a}\;\exp(\,-2ay\,)\;+\;\frac{2a|b|}{|b|^2-a^2}\;\exp(\,-2|b|y\,) \qquad (0<y<\infty).$$

Taking the derivative of $g(y)$, we have

$$g'(y)\;=\;\frac{2a|b|}{|b|-a}\;\left[\exp(\,-2ay\,)\;-\;\frac{2|b|}{|b|+a}\;\exp(\,-2|b|y\,)\right] \qquad (0<y<\infty).$$

Since the equation $g'(y)=0$ $(0<y<\infty)$ has only one solution $y=y_0$, we have

$$g'(y)\;\begin{cases} <\;0 & (0<y<y_0) \\ =\;0 & (y=y_0) \\ >\;0 & (y_0<y<\infty). \end{cases}$$

$g(y)$ attains its minimum $g(y_0)$ at $y=y_0$. Setting

$$\exp(\,-2ay_0\,)\;=\;\frac{2|b|}{|b|+a}\;\exp(\,-2|b|y_0\,)\;=\;c_0,$$

we have

$$g(y_0)\;=\;1\;-\;c_0\;=\;1\;-\;\exp(\,-2ay_0\,)\;>\;0.$$

Since $g(0)=a\big/(|b|+a)<g(\infty)=1$, we conclude that

$$\sup_{0<y<\infty} g(y)\;=\;g(+\infty)\;=\;1$$

which completes the proof. ■

Proof of Theorem 1.2 Applying Schwarz inequality to (4.5) and using (4.6), we have

$$|\,u(y)\,|^2\;\le\;\|\,u\,\|_H^2\,\|\,G(x,y)\,\|_H^2\;=\;G(y,y)\,\|\,u\,\|_H^2 \qquad (-\infty<y<\infty). \tag{5.5}$$

Noting that Lemma 5.1, we obtain Sobolev inequality

$$\Big(\sup_{-\infty<y<\infty} |\, u(y) \,| \Big)^2 \le C(a,b) \, \| \, u \, \|_H^2. \tag{5.6}$$

First, we treat case II, in which $C(a,b) = G(0,0)$. Setting $u(x) = G(x,0) \in H$ in (5.6), we have

$$\Big(\sup_{-\infty<y<\infty} |\, G(y,0) \,| \Big)^2 \le C(a,b) \, \| \, G(x,0) \, \|_H^2 = C^2(a,b).$$

Together with a trivial inequality

$$C^2(a,b) = G^2(0,0) \le \Big(\sup_{-\infty<y<\infty} |\, G(y,0) \,| \Big)^2,$$

we have

$$\Big(\sup_{-\infty<y<\infty} |\, G(y,0) \,| \Big)^2 = C(a,b) \, \| \, G(x,0) \, \|_H^2. \tag{5.7}$$

Next, we treat cases I, III, and IV. For any y_0 satisfying $-\infty < y_0 < \infty$, we have

$$\Big(\sup_{-\infty<y<\infty} |\, G(y,y_0) \,| \Big)^2 \le C(a,b) \, \| \, G(x,y_0) \, \|_H^2 = G(y_0,y_0) \, C(a,b).$$

Together with a trivial inequality

$$G^2(y_0,y_0) \le \Big(\sup_{-\infty<y<\infty} |\, G(y,y_0) \,| \Big)^2,$$

we have

$$G^2(y_0,y_0) \le \Big(\sup_{-\infty<y<\infty} |\, G(y,y_0) \,| \Big)^2 \le C(a,b) \, \| \, G(x,y_0) \, \|_H^2 = G(y_0,y_0) \, C(a,b)$$

and so

$$0 \le C(a,b) \, \| \, G(x,y_0) \, \|_H^2 - \Big(\sup_{-\infty<y<\infty} |\, G(y,y_0) \,| \Big)^2 \le$$
$$G(y_0,y_0) \Big(C(a,b) - G(y_0,y_0) \Big) \xrightarrow[y_0 \to +\infty]{} 0 \tag{5.8}$$

which completes the proof. ■

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