# ON COMMUTATIVE BE-ALGEBRAS

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Abstract. In this paper we investigate the relationship between BE-algebras, implicative algebras, and J-algebras. Moreover, we define commutative BE-algebras and state that these algebras are equivalent to the commutative dual BCK-algebras.

## 1. Introduction

In 1967 J. C. Abbot introduced in [1] the concept of implication algebras as algebras connected with a propositional calculus. In [5] K. Iséki introduced a wide class of abstract algebras: BCK-algebras. Recently, R. A. Borzooei and S. Khosravi Shoar ([2]) showed that the implication algebras are equivalent to the dual implicative BCK-algebras. W. H. Cornish ([4]) introduced the condition (J) and proved the BCK-algebras satisfying (J) form a variety. In [7], as a generalization of a BCK-algebra, H. S. Kim and Y. H. Kim introduced the notion of a BE-algebra.

In this paper we show that any implication algebra is a BE-algebra and that every BE-algebra satisfies (J). Moreover, we define commutative BE-algebras and state that these algebras are equivalent to the commutative dual BCK-algebras.

## 2. Preliminaries

**Definition 2.1.** ([7]) An algebra $(X;*,1)$ of type $(2,0)$ is called a *BE-algebra* if for all $x,y,z \in X$ the following identities hold:
(BE1) $x*x=1$,
(BE2) $x*1=1$,
(BE3) $1*x=x$,
(BE4) $x*(y*z)=y*(x*z)$.

**Lemma 2.2.** ([7]) *If* $(X;*,1)$ *is a* BE-*algebra, then* $x*(y*x)=1$ *for any* $x,y \in X$.

**Definition 2.3.** ([8]) A *dual BCK-algebra* is an algebra $(X;*,1)$ of type $(2,0)$ satisfying (BE1), (BE2), and the following axioms:
(dBCK1) $x*y=y*x=1 \Longrightarrow x=y$,
(dBCK2) $(x*y)*((y*z)*(x*z))=1$,
(dBCK3) $x*((x*y)*y)=1$.

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**Lemma 2.4.** ([8], Theorem 2.5) *Let* $(X;*,1)$ *be a dual BCK-algebra and* $x,y,z\in X$. *Then:*
(a) $x*(y*z)=y*(x*z)$,
(b) $1*x=x$.

From Lemma 2.4 we have

**Proposition 2.5.** *Any dual BCK-algebra is a BE-algebra.*

**Example 2.6.** Let $\mathbb{N}$ be the set of all natural numbers and $*$ be the binary operation on $\mathbb{N}$ defined by

$$x*y=\begin{cases} y & \text{if} \quad x=1\\ 1 & \text{if} \quad x\neq 1.\end{cases}$$

It is easy to see that $(\mathbb{N};*,1)$ is a BE-algebra, but it is not a dual BCK-algebra.

**Definition 2.7.** ([1]) An algebra $(X;*)$ of type (2) is called an *implication algebra* if for all $x,y,z\in X$ the following identities hold:
(I1) $(x*y)*x=x$,
(I2) $(x*y)*y=(y*x)*x$,
(I3) $x*(y*z)=y*(x*z)$.

In any implication algebra $(X;*)$, $x*x=y*y$ for all $x,y\in X$. This was proved by W. Y. Chen and J. S. Oliveira [3]. Let 1 stand for the constant $x*x$. R. A. Borzooei and S. Khosravi Shoar proved the following result:

**Proposition 2.8.** ([2]) *If* $(X;*)$ *is an implication algebra, then* $(X;*,1)$ *is a dual BCK-algebra.*

Propositions 2.8 and 2.5 give

**Proposition 2.9.** *Any implication algebra is a BE-algebra.*

**Definition 2.10.** ([6]) An algebra $(X;*)$ consisting of a set $X$ with a binary operation $*$ on $X$ is said to be a *J-algebra* if
(J) $x*(x*(y*(y*x)))=y*(y*(x*(x*y)))$
for all $x,y\in X$.

**Proposition 2.11.** *Let* $(X;*,1)$ *be a BE-algebra. Then* $(X;*)$ *is a J-algebra.*

*Proof.* Let $x,y\in X$. By (BE4), Lemma 2.2, and (BE2) we have

$$x*(x*(y*(y*x)))=x*(y*(x*(y*x)))=x*(y*1)=x*1=1.$$

Similarly,

$$y*(y*(x*(x*y)))=y*(x*(y*(x*y)))=y*(x*1)=y*1=1.$$

Hence (J) holds, and therefore $X$ is a J-algebra. □

## 3. Commutative BE-algebras

**Definition 3.1.** Let $(X;*,1)$ be a BE-algebra or a dual BCK-algebra. We say that $X$ is *commutative* if
(C) $\quad (x*y)*y=(y*x)*x$
for all $x,y\in X$.

**Example 3.2.** Let $\mathbb{N}_0=\mathbb{N}\cup\{0\}$ and let $*$ be the binary operation of $\mathbb{N}_0$ defined by

$$x*y=\begin{cases}0 & \text{if} \quad x\geq y\\ y-x & \text{if} \quad y>x.\end{cases}$$

Observe that $(\mathbb{N}_0;*,0)$ is a commutative BE-algebra. Obviously, $x*x=0$, $x*0=0$, and $0*x=x$ for all $x\in\mathbb{N}_0$. Thus (BE1)–(BE3) hold. Let $x,y,z\in\mathbb{N}_0$. To prove (BE4) we consider two cases.
Case 1: $x+y<z$.
Then $x<z$ and $y<z$. Hence $x*z=z-x$ and $y*z=z-y$. Therefore

$$\begin{aligned}x*(y*z) &= x*(z-y)=z-y-x=(z-x)-y\\ &= y*(z-x)=y*(x*z).\end{aligned}$$

Case 2: $x+y\geq z$.
Then $x\geq z-y\geq y*z$. From this we obtain $x*(y*z)=0$. Similarly, since $y\geq z-x\geq x*z$, we conclude that $y*(x*z)=0$. Consequently, $x*(y*z)=y*(x*z)$.Thus $(\mathbb{N}_0;*,0)$ is a BE-algebra.
Now we shall prove that $(\mathbb{N}_0;*,0)$ is commutative. Without loss of generality we can assume that $x\geq y$. Then $(x*y)*y=0*y=y$ and $(y*x)*x=(x-y)*x=x-(x-y)=y$. Hence $(x*y)*y=(y*x)*x$ and we see that $(\mathbb{N}_0;*,0)$ is a commutative BE-algebra.

**Proposition 3.3.** *If $(X;*,1)$ is a commutative BE-algebra, then for all $x,y\in X$,*

$$x*y=1 \quad \textit{and} \quad y*x=1 \quad \textit{imply} \quad x=y.$$

*Proof.* Let $x,y\in X$ and suppose that $x*y=y*x=1$. Then

$$x=1*x=(y*x)*x=(x*y)*y=1*y=y.$$

□

**Theorem 3.4.** *If $(X;*,1)$ is a commutative BE-algebra, then $(X;*,1)$ is a dual BCK-algebra.*

*Proof.* Proposition 3.3 yields (dBCK1). Now let $x,y,z\in X$. Applying (BE4) and (C) we have

$$(y*z)*(x*z)=x*[(y*z)*z]=x*[(z*y)*y]=(z*y)*(x*y).$$

Hence

$$(x*y)*[(y*z)*(x*z)]=(x*y)*[(z*y)*(x*y)].$$

Lemma 2.2 now shows that $(x*y)*[(y*z)*(x*z)]=1$, and therefore (dBCK2) holds. Moreover, by (BE4) and (BE1), $x*((x*y)*y)=(x*y)*(x*y)=1$. From this we have (dBCK3), and consequently, $X$ is a dual BCK-algebra. □

By Proposition 2.5 and Theorem 3.4 we have

**Corollary 3.5.** *$(X;*,1)$ is a commutative BE-algebra if and only if it is a commutative dual BCK-algebra.*

**Definition 3.6.** Let $(X;*,1)$ be a BE-algebra. We define the binary operation " $+$ " on $X$ as the following: for any $x,y\in X$

$$x+y=(x*y)*y.$$

Clearly, $X$ is a commutative BE-algebra if and only if $x+y=y+x$ for all $x,y\in X$.

**Lemma 3.7.** *Let $(X;*,1)$ be a commutative BE-algebra. Then for all $x,y,z\in X$ :*
(a) $x*(x+y)=1$,
(b) $x*y=y*z=1\Longrightarrow x*z=1$,
(c) $x*y=1\Longrightarrow (x+z)*(y+z)=1$,
(d) $x*z=y*z=1\Longrightarrow (x+y)*z=1$.

*Proof.* (a) By Theorem 3.4, $X$ is a dual BCK-algebra. From (dBCK3) we obtain (a).

(b) Applying (dBCK2) and Lemma 2.4 (b) we have (b).

(c) To prove (c), let $x*y=1$. From (dBCK2) we deduce that $(y*z)*(x*z)=1$. Again using (dBCK2) we get $[(x*z)*z]*[(y*z)*z]=1$, i.e. $(x+z)*(y+z)=1$.

(d) To prove (d), let $x*y=y*z=1$. From (c) we conclude that $(x+y)*(y+z)=1$ and $(y+z)*(z+z)=1$. By (b), $(x+y)*(z+z)=1$, and hence $(x+y)*z=1$. □

**Proposition 3.8.** *If $(X;*,1)$ is a commutative* BE-*algebra, then $(X;+)$ is a semilattice.*

*Proof.* Obviously $x+x=x$ and $x+y=y+x$ for all $x,y\in X$. We will now prove that $+$ is associative. Let $x,y,z\in X$. From Lemma 3.7 (a) we have $x*(x+y)=1$ and $(x+y)*[(x+y)+z]=1$. Therefore

$$x*[(x+y)+z]=1. \tag{1}$$

Since $y*(x+y)=1$, Lemma 3.7 (c) shows that

$$(y+z)*[(x+y)+z]=1. \tag{2}$$

By Lemma 3.7 (d), from (1) and (2) we obtain

$$[x+(y+z)]*[(x+y)+z]=1. \tag{3}$$

Similarly,

$$[(x+y)+z]*[x+(y+z)]=1. \tag{4}$$

From (3) and (4) it follows by (dBCK1) that $(x+y)+z=x+(y+z)$. □

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