FINITELY GENERATED SEMIGROUPS HAVING PRESENTATION WITH REGULAR CONGRUENCE CLASSES

KUNITAKA SHOJI

Received October 12, 2007; revised June 28, 2008

Abstract. For a finitely generated semigroup S, there exist a finite alphabet X and a surjective homomorphism ϕ of X^+ to S. We say that S has a presentation given by X and ϕ. In this paper, we investigate finitely generated semigroups having a presentation with regular congruence classes or with finite congruence classes.

Introduction and preliminaries. Let X be a finite alphabet, X^* the set of all words over X and X^+ the set of all non-empty words over X, that is, $X^+ = X^* - \{1\}$. Under juxtaposition, X^* is the *free monoid* with a set X of free generators and X^+ is the *free semigroup* with a set X of free generators.

Let S be a finitely generated semigroup. Then there exist a finite alphabet X and a surjective homomorphism ϕ of X^+ to S. Let $\mu = \{(w, w') \in X^+ \times X^+) \mid \phi(w) = \phi(w')\}$. Then μ is a congruence on X^* and the Rees factor semigroup X^+/μ is isomorphic to S. Thus we say that S has a presentation with generators X and the congruence μ. In this case we write as $S = < X \mid \mu >$. If the congruence μ is generated by a set R of pairs of words (that is, R is a subset of $X^+ \times X^+$), then we say that S has a presentation with generators X and relators R. In this case we write as $S = < X \mid R >$. In the theory of rewriting systems, the congruence μ_R on X^+ which is the smallest congruence containing R is the *Thue* congruence defined by R. Thus, $S = < X \mid R > = < X \mid \mu_R >$.

A semigroup S has a *finite presentation* if there exist a finite set of X and a surjective homomorphism ϕ of X^+ to S and a finite set R consisting of pairs of words over X such that the Thue congruence μ_R equals the congruence $\{(w_1, w_2) \in X^+ \times X^+ \mid \phi(w_1) = \phi(w_2)\}$.

Any subset of X^* is called a *language over* X. A language L over X is called *regular* if it is accepted by a finite automaton. (See [6] for automata theory). We say that a semigroup S has a representation with *regular congruence classes* if there exist a finite set X and a surjective homomorphism ϕ of X^+ to S such that $\phi^{-1}(\phi(w))$ is a regular language for any word $w \in X^+$. While, a semigroup S has a representation with *finite congruence classes* if there exist a finite set X and a surjective homomorphism ϕ of X^+ to S such that $\phi^{-1}(\phi(w))$ is a finite set for any word $w \in X^+$. In 1971, A. V. Anisimov proved that a group G is a finite group if and only if there exists a surjective homomorphism ϕ of X^* to G such that $\phi^{-1}(\phi(1))$ is a regular language. This is just the case that a group G has a representation with regular congruence classes. On the other hand, from the results in word problems for semigroups obtained by J.H. Remmers or P.A. Cummings and R.Z. Goldstein, we know that any finitely presented semigroup satisfying either the condition $C(3)$ or the two conditions $C(2)$ and $T(4)$ has a presentation with finite congruence classes. (Refer to [2], [4], [5] and [8] for the conditions $C(p)$, $T(q)$.) In this paper we study finitely generated semigroups having a presentation with regular congruence classes or with finite congruence classes.

All the undefined terms are referred to [4].

2000 *Mathematics Subject Classification.* 20M05, 20M35.
Key words and phrases. semigroups, presentations, regular languages.

1 Presentations of semigroups with regular congruence classes Let X be a finite alphabet and L a language over X which is contained in X^+. Define a relation σ_L of the free semigroup X^+ by $w\sigma_L w'$ ($w, w' \in X^+$) if and only if the set $\{(x,y) \in X^+ \times X^+ \mid xwy \in L\}$ equals the set $\{(x,y) \in X^+ \times X^+ \mid xw'y \in L\}$. Then σ_L is a congruence on X^+. It is called the *syntactic* congruence of X^+ with respect to L. The Rees factor semigroup X^+/σ_L of X^+ modulo the congruence σ_L is called the *syntactic* semigroup of L denoted by $Syn(L)$.

The following result is well known .

Result 1([7]) *Let L be a language over X which is contained in X^+. Then L is regular if and only if $Syn(L)$ is a finite semigroup.*

Theorem 1 *Let S be a finitely generated semigroup. Then there exist languages $\{L_m\}_{m\in S}$ over a finite alphabet X, which are contained in X^+, such that S is isomorphic to a subdirect product of syntactic semigroups of L_m ($m \in S$).*

Proof. Since S is finitely generated, there exist a finite set X and a surjective homomorphism ϕ of X^+ to S. For each $m \in S$, let $L_m = \phi^{-1}(m)$. Then it is eay to see that $\phi(w) = \phi(w')$ ($w, w' \in X^+$) implies $w\sigma_{L_m}w'$. Letting $\sigma_\phi = \{(w,w') \in X^+ \times X^+ \mid \phi(w) = \phi(w')\}$, we have that $\sigma_\phi \subseteq \bigcap_{m\in S} \sigma_{L_m}$. Next we will show that $\sigma_\phi \supseteq \bigcap_{m\in S} \sigma_{L_m}$. To do so, let $w, w' \in X^+$ with $\phi(w) \neq \phi(w')$ and $m' = \phi(w)$. Then $w \in L_{m'}$ but $w' \notin L_{m'}$. Thus, $(w,w') \notin \bigcap_{m\in S} \sigma_{L_m}$. Therefore, $\sigma_\phi \supseteq \bigcap_{m\in S} \sigma_{L_m}$. So we have $\sigma_\phi = \bigcap_{m\in M} \sigma_{L_m}$. Consequently, X^+/σ_ϕ is a subdirect product of $Syn(L_m)$ ($m \in S$). Since $S \cong X^+/\sigma_\phi$, the theorem follows. □

A semigroup S is called *residually finite* if for each pair of elements $m, m' \in S$, there exists a congruence μ on S such that the factor semigroup S/μ is finite and $(m,m') \notin \mu$.

Theorem 2 *If a semigroup S has a presentation with regular congruence classes, then S is residually finite.*

Proof. This follows from Result 1 and Theorem 1. □

We say that a semigroup S has *decidable word* problem if there exists a finite presentation $< X \mid \mu_S >$ of S and for any pair of words $w, w' \in X^*$, there exists an algorithm to decide whether or not $(w,w') \in \mu_S$.

The following result follows from Theorem 2 and T.Evans [3].

Theorem 3 *The word problem is decidable for semigroups having a presentation with regular congruence classes.*

The following result is a characterization of semigroups that have a presentation with regular congruence classes.

Let S be a semigroup. For any $s \in S$, let $\sigma_s = \{(a,b) \in S \times S \mid xay = s \text{ if and only if } xby = s\ (x, y \in S^1)\}$. Then σ_s is a congruence on S and is called the *syntactic* congruence of S at s.

Theorem 4 *A finitely generated semigroup S has a presentation with regular congruence classes if and only if for any $s \in S$, S/σ_s is a finite semigroup.*

Proof. Let X be a finite set such that there exists a surjective homomorphism $\phi : X^+ \to S$. For $s \in S$, let $L_s = \phi^{-1}(s)$. We will show that (*) for $w, w' \in X^+$, $w\sigma_{L_s}w'$ if and only if $\phi(w)\sigma_s\phi(w')$. Actually, suppose that $w\sigma_{L_s}w'$. Then if $a\phi(w)b = s$ $(a, b \in S)$ then $pwq \in L_s$ where $p, q \in X^+$, $\phi(p) = a, \phi(q) = b$, and hence $pw'q \in L_s$. So $a\phi(w')b = s$. Similarly, if $a\phi(w')b = s$ $(a, b \in S)$ then $a\phi(w)b = s$. Thus, $\phi(w)\sigma_s\phi(w')$. Conversely, suppose that $\phi(w)\sigma_s\phi(w')$. Then it is easy to see that $w\sigma_{L_s}w'$. Now it follows from the above property (*) that the surjective homomorphism $\phi : X^+ \to S$ induces an isomorphism $Syn(L_s) \to S/\sigma_s$. Consequently it follows from Result 1 that for each $s \in S$, L_s is a regular language if and only if S/σ_s is finite. The theorem is proved. □

The following result follows immediately from Theorem 4.

Theorem 5 *For a finitely generated semigroup S, it does not depend on presentations of S that S has a presentation with regular congruence classes.*

In the remaining part of this section, we will study finiteness of some kinds of semigroups having a presentation with regular congruence classes.

Theorem 6 *Let S be a semigroup having a presentation with regular congruence classes. Then any subgroup of S is finite.*

Proof. Let e be an idempotent of S and G a subgroup of S containing e. Let $g, h \in G$ with $g\sigma_e h$. Then $egg^{-1} = e$ and hence $ehg^{-1} = e$, which leads to $g = h$. Hence G is embedded into S/σ_e. Since by Theorem 5 S/σ_e is finite, G is finite. □

As a consequence of Theorem 1 and Theorem 6, we have

Theorem 7 (Anisimov[1]) (1) *For every finite group G, there exists a regular language L of X^+ such that G is isomorphic to $Syn(L)$.*

(2) *If a group G has a presentation with regular congruence classes, then G is finite.*

Proof. Let G be a finitely generated group. Then there exist a finite set X and a surjective homomorphism $\phi : (X \cup X^{-1})^+ \to G$ such that $\phi(X)$ is a set of generators for the group G and $\phi(x^{-1})$ is an inverse of $\phi(x)$ for each $x \in X$.

(1) : Let $L = \phi^{-1}(1)$. Then in the same way used in the proof of Theorem 1, we can prove that G is isomorphic to $Syn(L)$. By Result we have that G is finite if and only if L is a regular language.

(2) : This is an immediate consequence of Theorem 6. □

Theorem 8 *Let S be a semigroup having a presentation with regular congruence classes. If S is a completely* $(0-)$ *simple semigroup, then S is finite.*

Proof. Assume that S is a completely 0-simple semigroup having a presentation with regular congruence classes. Then by Theorem 6, any subgroup of S is finite. Also, By Lemma 1.3.3 of [**?**], S is both a 0-disjoint union of 0-minimal right ideals and a 0-disjoint union of 0-minimal left ideals. Since S is finitely generated, S contains only finitely many 0-minimal right ideals and 0-minimal left ideals. By Theorem 1.3.2 of [4] (Rees-Suschkewitsch Theorem), S is finite. □

By Theorem 4, a finite semigroup S has a presentation with regular congruence classes. However a semigroup which has a presentation with regular congruence classes is not necessarily finite.

Example 1 Let X denote a finite alphabet and $w_1, \cdots, w_r$ nonempty words over X. Let $I = X^*w_1X^* \cup \cdots \cup X^*w_rX^*$. Then I is an ideal of the free semigroup X^+. Then the Rees factor semigroup X^+/I module I is an infinite semigroup having a presentation with regular congruence classes.

Example 2 A residually finite semigroup S is not always a semigroup having a presentation with regular congruence classes.

Actually, finitely generated free groups are residually finite semigroups but do not have any presentation with regular congruence classes.

2 Presentations of semigroups with finite congruence classes.

Let X be a finite alphabet. For each word w over X, the length of w, denoted by $|w|$, is the number of occurrences of letters in w.

The following result is concerned with properties of semigroups having a presentation with finite congruence classes.

Theorem 9 *Let S be a semigroup having a presentation with finite congruence classes. Then*

(1) *S is an infinite semigroup.*

(2) *S has no idempotents.*

Proof. Let ϕ be a surjective homomorphism of X^+ to S.

(1) : If S is finite, then there exists $s \in S$ such that $\phi^{-1}(s)$ is an infinite set, since X^+ is infinite.

(2) : Actually, if $\phi(w) = (\phi(w))^2$ $(w \in X^+)$ then $w^n \in \phi^{-1}(\phi(w))$ for any $n \in \mathbb{N}$. This contradicts that $\phi^{-1}(\phi(w))$ is a finite set. □

The following result is a characterization of semigroups that have a presentation with finite congruence classes.

Theorem 10 *Let S be a finitely generated semigroup.*

Then S has a presentation with finite congruence classes if and only if the following are satisfied :

(1) *S has no idempotent.*

(2) *For any $s \in S$, S/σ_s is a finite nilpotent semigroup with a zero element* 0.

Proof. Let X be a finite set and ϕ a surjective homomorphism of X^+ to S.

(Necessity) : (1) This follows from (2) of Theorem 9.

(2) Let $s \in S$. Then there exists $n = \max\{|w| \mid w \in \phi^{-1}(s)\}$, since $|\phi^{-1}(s)|$ is finite. Let $t \in X^+$ with $|t| = n+1$. Then $X^*tX^* \cap \phi^{-1}(s)$ is empty and so are $X^*xtyX^* \cap \phi^{-1}(s)$ for all $x, y \in X^+$. Hence $\sigma_s(\phi(t))$ is a zero element 0 of S/σ_s. Thus, $\sigma_s(S^{n+1}) = \{0\}$.

(Sufficiency) : Suppose that there exists an element s of S such that $\phi^{-1}(s)$ is an infinite set. Since by Theorem 4 and Theorem 5 $\phi^{-1}(s)$ is regular, by the pumping lemma (Lemma 3.1 of [6]), there exists $x, y \in X^*$ and $w \in X^+$ such that $xw^iy \in \phi^{-1}(s)$ for all $i \geq 1$. By the condition (2), there exists a positive integer k such that $\sigma_s(\phi(w^k))$ is a zero element 0 of S/σ_s. Let $a = \phi(xw), b = \phi(w), c = \phi(wy)$. Then $s = abc = ab^kc$ and $\sigma_s(b^k) = 0$. Thus, $\sigma_s(s) = \sigma_s(ab^kc) = \sigma_s(a)\sigma_s(b^k)\sigma_c(a) = 0$. So, $\sigma_s(s^2) = \sigma_s(s)$. Since σ_s consists only of s, we have $s^2 = s$. This contradicts the condition (1). Therefore, $\phi^{-1}(s)$ is finite for all $s \in S$. □

The following result follows immediately from Theorem 10.

Theorem 11 *For a finitely generated semigroup S, it does not depend on presentations of S that S has a presentaion with finite congruence classes.*

Theorem 12 *Let S be a semigroup with a finite presentation with a finite set X of generators and relators consisting of pairs of words of the same length. Then S has a presentation with finite congruence classes.*

Proof. Let μ_R be the Thue congruence on X^+ defined by R. For each $w \in X^+$, if $|w'| = n$ and $w\mu_Rw'$ ($w' \in X^+$), then $|w'| = n$. Thus each of the congruence classes μ_Rw ($w \in X^*$) is finite. □

In particular, we have

Example 3 The finitely generated free commutative semigroup has a presentation with finite congruence classes. Actually, the finitely generated free commutative semigroup is presented by generators $\{x_1, x_2, \cdots, x_r\}$ and relators $\{(x_ix_j, x_jx_i) \mid 1 \leq i < j \leq r\}$.

Theorem 13 *Any finitely generated subsemigroup of the free semigroup has a presentation with finite congruence classes.*

Proof. Let S be a subsemigroup generated by a finite number of words $w_1, w_2, \cdots, w_r$ of X^+. Clearly S has no idempotent. For any $s \in S$, if $|w| > |s|$ then $\{(x, y) \in S^1 \times S^1 \mid xwy = s\}$ is an empty set. Hence S/σ_s is a finite nilpotent semigroup. So S has a presentation with finite congruence classes. □

References

[1] A.V. Anisimov, *Groups languages*, Kybernetika **4**(1971), 18-24.

[2] P.A. Cummings and R.Z. Goldstein, *Solvable word problems in semigroups*, Semigroup Forum **50**(1995), 243-246.

[3] T. Evans, *Word problems*, Bull. Amer. Math. Soc. **84** (5)(1978), 789-802.

[4] P.M. Higgins, *Techniques of semigroup theory*, Oxford University Press, New York, 1992.

[5] P. Hill, S.J. Pride and A.D. Vella, *On the $T(q)$-conditions of small cancellation theory*, Israel J. Math. **52**(1985), 293-304.

[6] J. E. Hopcroft and J. D. Ullman, *Introduction to Automata theory, Languages, and Computation*, Addison-Wesley Publishing, 1979.

[7] J.E. Pin, *Varieties of formal languages*, North Oxford Academic Publishers, London, 1986.

[8] J.H. Remmers, *On the geometry of semigroup presentations*, Adv. in Math. **36**(1980), 283-296.

Authors' adresses:

Kunitaka Shoji, Department of Mathematics, Shimane University, Matsue, Shimane, 690-8504, Japan

E-mail : ksho@riko.shimane-u.ac.jp