SEARCH PROBLEM WITH TWO LEVELS OF EXAMINATION COSTS

Kensaku KIKUTA

Received July 11, 2008; revised September 17, 2008

Abstract. There is an (immobile) object in a node except for a specified node, with a priori probabilities. A seeker starts at the specified node and examines each node until he finds an object, traveling along edges. Associated with an examination of a node is the examination cost, and associated with a movement from a node to a node is a traveling cost. A strategy for the seeker is an ordering of nodes in which the seeker examines each node. The purpose of the seeker is to find a strategy which minimizes the expected cost. Necessary conditions for a strategy to be optimal are presented. Special cases are solved.

1 Introduction and preliminaries. An optimization problem studied in this paper is a search problem on a finite and connected graph: There is an (immobile) object in a node except for a specified node, with a priori probabilities. A seeker starts at the specified node and examines each node until he finds an object, traveling along edges. Associated with an examination of a node is the examination cost, and associated with a movement from a node to a node is a traveling cost. A strategy for the seeker is an ordering of nodes in which the seeker examines each node. The purpose of the seeker is to find a strategy which minimizes the expected cost for finding the object.

[Gluss 1961] studied this problem and found a solution approximately when the graph is linear and the seeker is at a terminal node at first. [Kikuta 1990] studied this problem when the graph is a rooted tree with two branches and the seeker is at the root at first. [Lössner and Wegener 1982] studies a more general problem and got sufficient conditions in which critical quantities are given for finding a node to be examined in the next step. Our problem treats a special case of its general model and this paper intends to analyze properties of optimal strategies in more detail, which depends on special structure of the underlying network. In [Alpern and Gal 2003], a game version of this problem is commented. [Ruckle 1983] introduces many search games on graphs.

It is difficult to find an exact analytical solution for this problem. On the other hand, imagine a search for the traces of a lost ship in the sea. In some regions, they could search only by patrol boats, and in other regions they must use helicopters as well as patrol boats. It costs much when they must use helicopters and boats. When they commit helicopters, extra set-up cost is required. In this paper we assume that the nodes are classified into two groups depending on the examination costs, and then the edges are also classified into three groups depending on the examination costs and the traveling costs. Necessary conditions are presented for a strategy to be optimal. Properties of optimal strategies are induced from the necessary conditions.

2006 *Mathematics Subject Classification.* 90B40, 90B15.

Key words and phrases. search problem, minimization, expected cost, examination cost, traveling cost, sequencing.

Here we formulate a problem on a finite and connected graph. From Section 2 on we treat a special case of this problem. Let (N, E) be a finite, connected and undirected graph where $N = \{0, 1, \ldots, |N| - 1\}, |N| \geq 3$, is the set of nodes and $E \subseteq N \times N$ is the set of edges. The node 0 is specified. A path between i_0 and i_s is an ordered $s+1$-tuple $\pi = (i_0, i_1, \ldots, i_s)$ such that $(i_{r-1}, i_r) \in E$ for $r = 1, \ldots, s$. Each edge $(i, j) \in E$ is associated with a positive real number $d(i, j) > 0$, called a traveling cost of $(i, j) \in E$. We assume $d(i, j) = d(j, i)$ for all $(i, j) \in E$ and $d(i, i) = 0$ for all $i \in N$. Furthermore, we assume that $d(i, j) + d(j, k) \geq d(i, k)$ whenever $(i, j), (j, k), (i, k) \in E$. If $i, j \in N$ and $(i, j) \notin E$, we define $d(i, j)$ by the minimum of the traveling costs of the paths between i and j.

First we state a search problem on (N, E). There is an (immobile) object in a node except for the node 0, with a priori probabilities $p_i, i \in N \setminus \{0\}$. A seeker starts at the node 0 and examines each node until he finds an object, traveling along edges. He finds an object certainly (with probability 1) if he examines the right node. Associated with an examination of $i \in N \setminus \{0\}$ is the examination cost c_i, and associated with a movement from a node $i \in N$ to a node $j \in N$ is a traveling cost $d(i, j)$. A strategy for the seeker is a permutation σ on N with $\sigma(0) = 0$, which means that the seeker examines each node in the order of $\sigma(1), \ldots, \sigma(|N| - 1)$, starting at the node 0. Σ is the set of all permutations on N such that $\sigma(0) = 0$. For $i \in N \setminus \{0\}$ and $\sigma \in \Sigma$, $f(i, \sigma)$ is the cost of finding the object at the node i when the seeker takes a strateg y σ:

$$f(i, \sigma) = \sum_{x=0}^{\sigma^{-1}(i)-1} \{d(\sigma(x+1), \sigma(x)) + c_{\sigma(x+1)}\}.$$

For $\sigma \in \Sigma$, $f(\sigma)$ is the expected cost of finding the object, starting at the node 0:

$$f(\sigma) = \sum_{i \in N \setminus \{0\}} p_i f(i, \sigma) = \sum_{i \in N \setminus \{0\}} p_{\sigma(i)} f(\sigma(i), \sigma).$$

A strategy $\sigma^* \in \Sigma$ is said to be optimal if

$$f(\sigma^*) = \min_{\sigma \in \Sigma} f(\sigma).$$

The purpose of the seeker is to find an optimal strategy.

2 Model. As we know from the literature, it is difficult to solve the problem mentioned in Section 1. In this paper we try to solve a problem when depending on the examination cost, the nodes are classified into two groups except for the node 0. That is, $H = \{1, \ldots, m\}$ is the set of nodes with examination cost $c_i = h, i \in H$ and $L = \{m+1, \ldots, m+n\}$ is the set of nodes with examination cost $c_i = \ell, i \in L$. Thus, $N = H \cup L \cup \{0\}$ and $H \cap L = \emptyset$. We assume

$$h > \ell > 0.$$

Secondly, we assume that the graph is complete, $E = N \times N$. So the seeker does not visit any node where he visited before. The seeker would not pass through any node without examination. That is, he examines certainly that node when he reaches there. This means that at each movement of the seeker from a node i to another node j, it costs a traveling cost $d(i,j)$ and an examination cost c_j. We let

$$d(i,j) = \begin{cases} s, & \text{if } i \in H, j \in L \text{ or } i \in L, j \in H; \\ t_H, & \text{if } i,j \in H; \\ t_L, & \text{if } i,j \in L; \\ t_0, & \text{if } i = 0 \text{ and } j \in N \setminus \{0\} \text{ or } j = 0 \text{ and } i \in N \setminus \{0\}; \\ 0, & \text{if } i = j. \end{cases}$$

For a movement between different groups, the seeker must pay extra set-up cost $s - t_H$ or $s - t_L$. Thus we assume $s \geq 0, t_H \geq 0, t_L \geq 0, t_0 \geq 0$ and

$$s \geq \max\{t_H, t_L\}.$$

In our model, we assume that the seeker should not return to the node 0. So, the value of $d(i,j)$ when $i = 0$ (or $j = 0$) is paid exactly once at the beginning of the search[1]. For simplicity we let $t_0 = 0$. Without loss of generality we assume

$$p_1 \geq \ldots \geq p_m > 0 \text{ and } p_{m+1} \geq \ldots \geq p_{m+n} > 0.$$

By these assumptions on a priori probabilities, we see a property of a strategy.

Lemma 2.1. When he considers nodes in the same group, the seeker must examine a node first with the highest a priori probability.

Proof: For $i,j \in H$ suppose $\sigma^{-1}(i) < \sigma^{-1}(j)$. Define τ by $\tau^{-1}(i) = \sigma^{-1}(j)$, $\tau^{-1}(j) = \sigma^{-1}(i)$ and $\tau^{-1}(k) = \sigma^{-1}(k)$ for $k \neq i,j$. We see that $f(\sigma) > f(\tau)$ if and only if $p_j > p_i$. By definition it is easy to see that for $i,j \in H$,

$$f(j,\sigma) - f(i,\sigma) = f(i,\tau) - f(j,\tau), \text{ and } f(k,\sigma) = f(k,\tau), \forall k \neq i,j.$$

So

$$f(\sigma) - f(\tau) = (p_j - p_i)\{f(j,\sigma) - f(i,\sigma)\}.$$

Since $f(j,\sigma) - f(i,\sigma) > 0$ we have the first half of the lemma. Next, for $m+i, m+j \in L$ suppose $\sigma^{-1}(m+i) < \sigma^{-1}(m+j)$. Define τ by $\tau^{-1}(m+i) = \sigma^{-1}(m+j)$, $\tau^{-1}(m+j) = \sigma^{-1}(m+i)$ and $\tau^{-1}(k) = \sigma^{-1}(k)$ for $k \neq m+i, m+j$. We see that $f(\sigma) > f(\tau)$ if and only if $p_{m+j} > p_{m+i}$. This is shown by the next relation:

$$f(\sigma) - f(\tau) = (p_{m+j} - p_{m+i})\{f(m+j,\sigma) - f(m+i,\sigma)\}.$$

□

By this lemma we can consider a restricted set of permutations: Σ^* is the set of permutations σ on N such that $\sigma(0) = 0$ and $\sigma^{-1}(i) < \sigma^{-1}(j)$ if $i < j \leq m$ and $\sigma^{-1}(m+i) < \sigma^{-1}(m+j)$ if $i < j \leq n$. We express a permutation $\sigma \in \Sigma^*$ as $\sigma = [\sigma(1)\cdots\sigma(m+n)]$. If $\sigma(i) \in H, \sigma(j) \in H$ and $i < j$ then we have $p_{\sigma(i)} \geq p_{\sigma(j)}$. In the same way, if $\sigma(i) \in L, \sigma(j) \in L$ and $i < j$ then we have $p_{\sigma(i)} \geq p_{\sigma(j)}$. In Section 4 we treat specified permutations $\sigma^H \in \Sigma^*$ and $\sigma^L \in \Sigma^*$,where $\sigma^H(a) \in H$ for $1 \leq a \leq m$ and $\sigma^L(a) \in L$ for $1 \leq a \leq n$, that is, $\sigma^H = [1,\cdots,m,m+1,\cdots,m+n]$ and $\sigma^L = [m+1,\cdots,m+n,1,\cdots,m]$.

[1] The node 0 would be more important when the underlying graph is not complete.

3 Necessary Conditions for Optimal Strategies. In this section we give necessary conditions for a strategy to be optimal.

Theorem 3.1. Let $\sigma \in \Sigma^*$ be an optimal strategy such that for $1 \leq i \leq m-1$ and $0 \leq j \leq n-y$,

$$\sigma(a) = i \in H, \sigma(a+y+1) = i+1 \in H, \sigma(a+b) = m+j+b \in L, \text{ for } 1 \leq b \leq y.$$

Then

$$\frac{\sum_{b=1}^{y} p_{m+j+b}}{2s - t_H - t_L + y(t_L+\ell)} > \frac{p_{i+1}}{t_H+h},$$
$$\frac{p_i}{t_H+h} > \frac{\sum_{b=1}^{y} p_{m+j+b}}{2s - t_H - t_L + y(t_L+\ell)}, \text{ if } a \geq 2 \text{ and } \sigma(a-1) \in H,$$
$$\frac{p_i}{2s - t_L + h} > \frac{\sum_{b=1}^{y} p_{m+j+b}}{y(t_L+\ell)}, \text{ if } a \geq 2 \text{ and } \sigma(a-1) \in L,$$
$$\frac{p_i}{s+h} > \frac{\sum_{b=1}^{y} p_{m+j+b}}{s - t_L + y(t_L+\ell)}, \text{ if } a = 1.$$

Next let $\sigma \in \Sigma^*$ be an optimal strategy such that for $1 \leq i \leq n-1$ and $0 \leq j \leq m-x$,

$$\sigma(a) = m+i \in L, \sigma(a+x+1) = m+i+1 \in L, \sigma(a+b) = j+b \in H, \text{ for } 1 \leq b \leq x.$$

Then

$$\frac{\sum_{b=1}^{x} p_{j+b}}{2s - t_H - t_L + x(t_H+h)} > \frac{p_{m+i+1}}{t_L+\ell},$$
$$\frac{p_{m+i}}{t_L+\ell} > \frac{\sum_{b=1}^{x} p_{j+b}}{2s - t_H - t_L + x(t_H+h)}, \text{ if } a \geq 2 \text{ and } \sigma(a-1) \in L,$$
$$\frac{p_{m+i}}{2s - t_H + \ell} > \frac{\sum_{b=1}^{x} p_{j+b}}{x(t_H+h)}, \text{ if } a \geq 2 \text{ and } \sigma(a-1) \in H,$$
$$\frac{p_{m+i}}{s+\ell} > \frac{\sum_{b=1}^{x} p_{j+b}}{s - t_H + x(t_H+h)}, \text{ if } a = 1.$$

To prove this theorem we need two lemmas.

Lemma 3.1A. Let $\sigma \in \Sigma^*$ be a strategy such that for $1 \leq i \leq m-1$ and $0 \leq j \leq n-y$,

$$\sigma(a) = i \in H, \sigma(a+y+1) = i+1 \in H, \sigma(a+b) = m+j+b \in L, \text{ for } 1 \leq b \leq y.$$

Let $\sigma' \in \Sigma^*$ be a strategy such that $\sigma'(b) = \sigma(b)$ for $1 \leq b \leq a-1$ and $a+y+2 \leq b \leq m+n$ and

$$\sigma'(a) = i \in H, \sigma'(a+1) = i+1 \in H, \sigma'(a+b) = m+j+b-1 \in L, \text{ for } 2 \leq b \leq y+1.$$

Then $f(\sigma) < f(\sigma')$ if and only if

$$\frac{p_{i+1}}{t_H+h} + \alpha < \frac{\sum_{b=1}^{y} p_{m+j+b}}{2s - t_H - t_L + y(t_L+\ell)},$$

where $\alpha = 0$ if $\sigma(a+y+2) \in H$ and

$$\alpha = \frac{2s - t_H - t_L}{2s - t_H - t_L + y(t_L+\ell)} \times \frac{\sum_{b=a+y+2}^{m+n} p_{\sigma(b)}}{t_H+h}, \text{ if } \sigma(a+y+2) \in L.$$

Let $\sigma'' \in \Sigma^*$ be a strategy such that $\sigma''(b) = \sigma(b)$ for $1 \le b \le a-1$ and $a+y+2 \le b \le m+n$ and

$$\sigma''(a+y) = i \in H, \sigma''(a+y+1) = i+1 \in H, \sigma''(a+b) = m+j+b+1 \in L, \text{ for } 0 \le b \le y-1.$$

Then $f(\sigma) < f(\sigma'')$ if and only if

$$\frac{\sum_{b=1}^{y} p_{m+j+b}}{2s - t_H - t_L + y(t_L+\ell)} < \frac{p_i}{t_H + h}, \text{ if } \sigma(a-1) \in H,$$

and

$$\frac{\sum_{b=1}^{y} p_{m+j+b}}{y(t_L+\ell)} + \frac{2s - t_H - t_L}{2s - t_L + h} \times \frac{\sum_{b=a+y+1}^{m+n} p_{\sigma(b)}}{y(t_L+\ell)} < \frac{p_i}{2s - t_L + h}, \text{ if } \sigma(a-1) \in L,$$

and

$$\frac{\sum_{b=1}^{y} p_{m+j+b}}{s - t_L + y(t_L+\ell)} + \frac{s - t_H}{s+h} \times \frac{\sum_{b=a+y+1}^{m+n} p_{\sigma(b)}}{s - t_L + y(t_L+\ell)} < \frac{p_i}{s+h}, \text{ if } a = 1.$$

Proof of Lemma 3.1A: First we note that for $a > 1$

$$f(\sigma(b), \sigma) = f(\sigma'(b), \sigma') = f(\sigma''(b), \sigma''), \text{ for } 1 \le b \le a-1.$$

For other terms,

$$\begin{aligned}
f(i,\sigma) &= f(\sigma(a-1),\sigma) + d(\sigma(a-1),\sigma(a)) + h,\\
f(i+1,\sigma) &= f(\sigma(a-1),\sigma) + d(\sigma(a-1),\sigma(a)) + h + s + \ell + (y-1)(t_L+\ell) + s + h,\\
f(m+j+b,\sigma) &= f(\sigma(a-1),\sigma) + d(\sigma(a-1),\sigma(a)) + h + s + \ell + (b-1)(t_L+\ell), \text{ for } 1 \le b \le y,\\
f(i,\sigma') &= f(i,\sigma),\\
f(i+1,\sigma') &= f(\sigma(a-1),\sigma) + d(\sigma'(a-1),\sigma'(a)) + h + t_H + h,\\
f(m+j+b,\sigma') &= f(\sigma(a-1),\sigma) + d(\sigma'(a-1),\sigma'(a)) + h + t_H\\
&\quad + h + s + \ell + (b-1)(t_L+\ell), \text{ for } 1 \le b \le y.\\
f(i,\sigma'') &= f(\sigma(a-1),\sigma) + d(\sigma''(a-1),\sigma''(a)) + \ell + (y-1)(t_L+\ell) + s + h,\\
f(i+1,\sigma'') &= f(\sigma(a-1),\sigma) + d(\sigma''(a-1),\sigma''(a)) + \ell + (y-1)(t_L+\ell) + s + h + t_H + h,\\
f(m+j+b,\sigma'') &= f(\sigma(a-1),\sigma) + d(\sigma''(a-1),\sigma''(a)) + \ell + (b-1)(t_L+\ell), \text{ for } 1 \le b \le y.
\end{aligned}$$

For $b \ge a+y+2$,

$$\begin{aligned}
f(\sigma(b),\sigma) - f(\sigma'(b),\sigma') &= f(\sigma(a+y+1),\sigma) - f(\sigma'(a+y+1),\sigma')\\
&\quad + d(\sigma(a+y+1),\sigma(a+y+2)) - d(\sigma'(a+y+1),\sigma'(a+y+2))\\
&= f(i+1,\sigma) - f(m+j+y,\sigma') + \begin{cases} s - t_L, & \text{if } \sigma(a+y+2) \in L,\\ t_H - s, & \text{if } \sigma(a+y+2) \in H,\end{cases}\\
&= s - t_H + \begin{cases} s - t_L, & \text{if } \sigma(a+y+2) \in L,\\ t_H - s, & \text{if } \sigma(a+y+2) \in H,\end{cases}\\
&= \begin{cases} 2s - t_H - t_L, & \text{if } \sigma(a+y+2) \in L,\\ 0, & \text{if } \sigma(a+y+2) \in H,\end{cases}
\end{aligned}$$

So, noting that $d(\sigma(a-1),\sigma(a)) = d(\sigma'(a-1),\sigma'(a))$, it holds $f(\sigma) < f(\sigma')$ if and only if

$$p_i f(i,\sigma) + p_{i+1} f(i+1,\sigma) + \sum_{b=1}^{y} p_{m+j+b} f(m+j+b,\sigma) + \sum_{b \geq a+y+2} p_{\sigma(b)} f(\sigma(b),\sigma)$$
$$< p_i f(i,\sigma') + p_{i+1} f(i+1,\sigma') + \sum_{b=1}^{y} p_{m+j+b} f(m+j+b,\sigma') + \sum_{b \geq a+y+2} p_{\sigma'(b)} f(\sigma'(b),\sigma')$$
$$\Longleftrightarrow$$
$$p_{i+1}\{2s - t_H - t_L + y(t_L+\ell)\} + \alpha\{2s - t_H - t_L + y(t_L+\ell)\}(t_H+h) < \sum_{b=1}^{y} p_{m+j+b}(t_H+h).$$

This implies the first half. On the other hand, $f(\sigma) < f(\sigma'')$ if and only if

$$p_i f(i,\sigma) + p_{i+1} f(i+1,\sigma) + \sum_{b=1}^{y} p_{m+j+b} f(m+j+b,\sigma) + \sum_{b \geq a+y+2} p_{\sigma(b)} f(\sigma(b),\sigma)$$
$$< p_i f(i,\sigma'') + p_{i+1} f(i+1,\sigma'') + \sum_{b=1}^{y} p_{m+j+b} f(m+j+b,\sigma'') + \sum_{b \geq a+y+2} p_{\sigma''(b)} f(\sigma''(b),\sigma'')$$

Here, we note that

$$d(\sigma(a-1),\sigma(a)) = \begin{cases} t_H, & \text{if } \sigma(a-1) \in H, \\ s, & \text{if } \sigma(a-1) \in L, \end{cases}$$
$$d(\sigma''(a-1),\sigma''(a)) = \begin{cases} s, & \text{if } \sigma''(a-1) \in H, \\ t_L, & \text{if } \sigma''(a-1) \in L. \end{cases}$$

Furthermore,

$$f(i,\sigma) - f(i,\sigma'') = \begin{cases} t_H + t_L - 2s - y(t_L+\ell), & \text{if } \sigma(a-1) \in H, \\ -y(t_L+\ell), & \text{if } \sigma(a-1) \in L, \end{cases}$$
$$f(i+1,\sigma) - f(i+1,\sigma'') = \begin{cases} 0, & \text{if } \sigma(a-1) \in H, \\ 2s - t_L - t_H, & \text{if } \sigma(a-1) \in L. \end{cases}$$

For $1 \leq b \leq y$,

$$f(m+j+b,\sigma) - f(m+j+b,\sigma'') = \begin{cases} h + t_H, & \text{if } \sigma(a-1) \in H, \\ 2s - t_L + h, & \text{if } \sigma(a-1) \in L. \end{cases}$$

For $b \geq a+y+2$,

$$f(\sigma(b),\sigma) - f(\sigma''(b),\sigma'') = f(\sigma(a+y+1),\sigma) - f(\sigma''(a+y+1),\sigma'').$$

So we have the desired result when $a > 1$.
Let $a = 1$. We let $f(\sigma(a-1),\sigma) = 0, d(\sigma(a-1),\sigma(a)) = 0, d(\sigma'(a-1),\sigma'(a)) = 0$ and $d(\sigma''(a-1),\sigma''(a)) = 0$. Then

$$f(i,\sigma) - f(i,\sigma'') = -s + t_L - y(t_L+\ell),$$
$$f(i+1,\sigma) - f(i+1,\sigma'') = s - t_H,$$
$$f(m+j+b,\sigma) - f(m+j+b,\sigma'') = s + h, \text{ for } 1 \leq b \leq y.$$

So,

$$(s-t_H)p_{i+1}+(s+h)\sum_{b=1}^{y}p_{m+j+b}+(s-t_H)\sum_{b\geq a+y+2}p_{\sigma(b)}\leq p_i\{s-t_L+y(t_L+\ell)\}.$$

From this we have the desired result. □

By transposing H and L, h and ℓ, and x and y, we have the next lemma. The proof is similar to that of Lemma 3.1A, and we omit it.

Lemma 3.1B. Let $\sigma\in\Sigma^*$ be a strategy such that for $1\leq i\leq n-1$ and $0\leq j\leq m-x$,

$$\sigma(a)=m+i\in L,\sigma(a+x+1)=m+i+1\in L,\sigma(a+b)=j+b\in H,\ \text{for } 1\leq b\leq x.$$

Let $\sigma'\in\Sigma^*$ be a strategy such that

$$\sigma'(a)=m+i\in L,\sigma'(a+1)=m+i+1\in L,\sigma'(a+b)=j+b-1\in H,\ \text{for } 2\leq b\leq x+1.$$

Then $f(\sigma)<f(\sigma')$ if and only if

$$\frac{p_{m+i+1}}{t_L+\ell}+\beta<\frac{\sum_{b=1}^{x}p_{j+b}}{2s-t_H-t_L+x(t_H+h)},$$

where $\beta=0$ if $\sigma(a+x+2)\in L$ and

$$\beta=\frac{2s-t_H-t_L}{2s-t_H-t_L+x(t_H+h)}\times\frac{\sum_{b=a+x+2}^{m+n}p_{\sigma(b)}}{t_L+\ell},\ \text{if } \sigma(a+x+2)\in H.$$

Let $\sigma''\in\Sigma^*$ be a strategy such that

$$\sigma''(a+x)=m+i\in L,\sigma''(a+x+1)=m+i+1\in L,\sigma''(a+b)=j+b+1\in H,\ \text{for } 0\leq b\leq x-1.$$

Then $f(\sigma)<f(\sigma'')$ if and only if

$$\frac{\sum_{b=1}^{x}p_{j+b}}{2s-t_H-t_L+x(t_H+h)}<\frac{p_{m+i}}{t_L+\ell},\ \text{if } \sigma(a-1)\in L,$$

and

$$\frac{\sum_{b=1}^{x}p_{j+b}}{x(t_H+h)}+\frac{2s-t_H-t_L}{2s-t_H+\ell}\times\frac{\sum_{b=a+x+1}^{m+n}p_{\sigma(b)}}{x(t_H+h)}<\frac{p_{m+i}}{2s-t_H+\ell},\ \text{if } \sigma(a-1)\in H,$$

and

$$\frac{\sum_{b=1}^{x}p_{j+b}}{s-t_H+x(t_H+h)}+\frac{s-t_L}{s+\ell}\times\frac{\sum_{b=a+x+1}^{m+n}p_{\sigma(b)}}{s-t_H+x(t_H+h)}<\frac{p_i}{s+\ell},\ \text{if } a=1.$$

Proof of Theorem 3.1. Noting that $s-\max\{t_H,t_L\}\geq 0$, $\alpha\geq 0$ and $\beta\geq 0$, apply Lemmas 3.1A and 3.1B repeatedly. □

The next corollary states that a priori probability must decrease whenever a switch occurs.

Corollary 3.2. Assume $\sigma \in \Sigma^*$ is an optimal strategy. Suppose $\sigma(i) \in H$, $\sigma(i+1) \in L$ and $\sigma(i) < m$. Then $p_{\sigma(i)} > p_{\sigma(i)+1}$. Similarly if $\sigma(i) \in L$, $\sigma(i+1) \in H$ and $\sigma(i) < n$, then $p_{\sigma(i)} > p_{\sigma(i)+1}$.

Proof: Suppose $\sigma(i) \in H$, $\sigma(i+1) \in L$ and $\sigma(i) < m$. There exists y such that $\sigma(i+b) \in L$ for $1 \leq b \leq y$. Since $\sigma(i) < m$, we have $i + y < m + n$, and then $\sigma(i + y + 1) \in H$. By the first half of Theorem 3.1, we see

$$\frac{p_{\sigma(i)}}{t_H + h} > \frac{p_{\sigma(i+y+1)}}{t_H + h},$$

noting $2s - t_H - t_L > 0$. Since $\sigma(i + y + 1) = \sigma(i) + 1$, we have the first half. The second half follows similarly. □

4 Observations. In this section we solve the problem when parameters of the model have relations mutually.

4.1 An object is uniformly distributed. Assume $p_1 = \cdots = p_m = p$ and $p_{m+1} = \cdots = p_{m+n} = q$. So

$$mp + nq = 1, p > 0, q > 0.$$

From Corollary 3.2 we see that either σ^H or σ^L is optimal.

$$\begin{aligned} f(\sigma^H) &= \sum_{k=1}^{m} p_k k(t_H + h) - p_1 t_H + \sum_{k=1}^{n} p_{m+k}\{(m-1)t_H + mh + s - t_L + k(t_L + \ell)\} \\ &= p(t_H + h)\frac{m(m+1)}{2} - pt_H + q(t_L + \ell)\frac{n(n+1)}{2} + qn\{(m-1)t_H + mh + s - t_L\}. \end{aligned}$$

On the other hand,

$$\begin{aligned} f(\sigma^L) &= \sum_{k=1}^{n} p_{m+k} k(t_L + \ell) - p_{m+1} t_L + \sum_{k=1}^{m} p_k\{(n-1)t_L + n\ell + s - t_H + k(t_H + h)\} \\ &= p(t_H + h)\frac{m(m+1)}{2} + q(t_L + \ell)\frac{n(n+1)}{2} - qt_L + pm\{(n-1)t_L + n\ell + s - t_H\}. \end{aligned}$$

Hence

$$f(\sigma^H) < f(\sigma^L) \Longleftrightarrow \frac{mp}{s - t_L - t_H + \frac{t_L}{n} + m(t_H + h)} > \frac{nq}{s - t_H - t_L + \frac{t_H}{m} + n(t_L + \ell)}.$$

Roughly speaking, the seeker must examine the nodes in H first if and only if the probability density for H is greater than the probability density for L.

4.2 The traveling costs are the same. We assume $s = t_H = t_L$. For $\sigma \in \Sigma^*$, suppose $\sigma(i) \in H$ and $\sigma(i+1) \in L$. Define $\tau \in \Sigma^*$ by

$$\tau(j) = \begin{cases} \sigma(j), & \text{if } j \neq i, i+1; \\ \sigma(i), & \text{if } j = i+1; \\ \sigma(i+1), & \text{if } j = i. \end{cases}$$

Noting that $s = t_H = t_L$, we have

$$f(\sigma) < f(\tau) \iff \frac{p_{\sigma(i)}}{s+h} > \frac{p_{\sigma(i+1)}}{s+\ell}.$$

When $\sigma(i) \in L$ and $\sigma(i+1) \in H$, in a similar way we have

$$f(\sigma) < f(\tau) \iff \frac{p_{\sigma(i)}}{s+\ell} > \frac{p_{\sigma(i+1)}}{s+h}.$$

By the assumption on a priori probabilities, we know

$$\frac{p_1}{s+h} \geq \cdots \geq \frac{p_m}{s+h}, \text{ and } \frac{p_{m+1}}{s+\ell} \geq \cdots \geq \frac{p_{m+n}}{s+\ell}.$$

So, if $\sigma \in \Sigma^*$ is optimal, we must have

$$\frac{p_{\sigma(i)}}{s+c_{\sigma(i)}} \geq \frac{p_{\sigma(i+1)}}{s+c_{\sigma(i+1)}}, \quad 1 \leq \forall i \leq m+n-1.$$

4.3 The traveling costs are different. We assume the difference $s - \max\{t_H, t_L\}$ is very large. By the necessary conditions in Theorem 3.1, we see an optimal strategy must be either σ^H or σ^L. Then

$$f(\sigma^H) = \sum_{k=1}^{m} p_k k(t_H+h) - p_1 t_H + \sum_{k=1}^{n} p_{m+k}\{(m-1)t_H + mh + s - t_L + k(t_L+\ell)\}$$
$$= (t_H+h)\sum_{k=1}^{m} p_k k - p_1 t_H + (t_L+\ell)\sum_{k=1}^{n} k p_{m+k} + \{(m-1)t_H + mh + s - t_L\}\sum_{k=1}^{n} p_{m+k}.$$

On the other hand,

$$f(\sigma^L) = \sum_{k=1}^{n} p_{m+k} k(t_L+\ell) - p_{m+1} t_L + \sum_{k=1}^{m} p_k\{(n-1)t_L + n\ell + s - t_H + k(t_H+h)\}$$
$$= (t_L+\ell)\sum_{k=1}^{n} p_{m+k} k - p_{m+1} t_L + (t_H+h)\sum_{k=1}^{m} p_k k + \{(n-1)t_L + n\ell + s - t_H\}\sum_{k=1}^{m} p_k.$$

Hence

$$f(\sigma^H) < f(\sigma^L) \iff \frac{\sum_{k=1}^{m} p_k}{s - t_H - t_L + m(t_H+h)} + \frac{p_1 t_H}{\gamma} > \frac{\sum_{k=1}^{n} p_{m+k}}{s - t_H - t_L + n(t_L+\ell)} + \frac{p_{m+1} t_L}{\gamma},$$

where $\gamma = \{s - t_H - t_L + m(t_H+h)\}\{s - t_H - t_L + n(t_L+\ell)\}$. The seeker must examine the nodes in H first if and only if the probability density for H is greater than the probability density for L.

4.4 The examination cost for each node in H is large. Assume the examination cost h for each node in H is very large. By the necessary condition in Theorem 3.1, we see that σ^L is optimal.

5 A numerical example In this section we see difficulties in solving the problem in general, by using a numerical example with $m = n = 2$. For simplicity assume $t_H = t_L = t$.

We can calculate $f(\sigma)$ for every $\sigma \in \Sigma^* = \{1234, 1324, 1342, 3412, 3142, 3124\}$.

$$\begin{aligned}
f(1234) &= p_1 h + p_2(t + 2h) + p_3(s + t + 2h + \ell) + p_4(s + 2t + 2h + 2\ell)\\
f(1324) &= p_1 h + p_2(2s + 2h + \ell) + p_3(s + h + \ell) + p_4(3s + 2h + 2\ell)\\
f(1342) &= p_1 h + p_2(t + 2s + 2h + 2\ell) + p_3(s + h + \ell) + p_4(t + s + h + 2\ell)\\
f(3412) &= p_1(t + s + h + 2\ell) + p_2(2t + s + 2h + 2\ell) + p_3\ell + p_4(t + 2\ell)\\
f(3142) &= p_1(s + h + \ell) + p_2(3s + 2h + 2\ell) + p_3\ell + p_4(2s + 2\ell + h)\\
f(3124) &= p_1(s + h + \ell) + p_2(t + s + 2h + \ell) + p_3\ell + p_4(t + 2s + 2h + 2\ell)
\end{aligned}$$

Let

$$\ell = 1, t = 1, s = 2, p_1 = \frac{2p}{3}, p_2 = \frac{p}{3}, p_3 = \frac{2(1-p)}{3}, p_4 = \frac{1-p}{3}, \text{ and } 0 \leq p \leq 1.$$

Then

$$f(1234) = 2h + \frac{14}{3} - \frac{p}{3}(13 + 2h), f(1324) = \frac{4h}{3} + \frac{14}{3} - 3p, f(1342) = h + \frac{11}{3} + \frac{p}{3}(h - 4),$$

$$f(3412) = \frac{5}{3} + \frac{p}{3}(11 + 4h), f(3142) = \frac{8}{3} + \frac{h}{3} + p(2 + h), f(3124) = 3 + \frac{2h}{3} + \frac{p}{3}(1 + 2h).$$

Let

$$D^H \equiv \min\{f(1234), f(1324), f(1342)\} \text{ and } D^L \equiv \min\{f(3412), f(3142), f(3124)\}.$$

By an assumption of the model, we consider only the case : $h \geq 1 = \ell$. Then

$$D^H = \begin{cases} f(1234), & \text{if } h < \frac{3p-1}{1-p};\\ f(1342), & \text{if } h > \frac{3p-1}{1-p}.\end{cases}$$

$$D^L = \begin{cases} f(3412), & \text{if } h > \frac{5p-2}{1-p};\\ f(3124), & \text{if } h < \frac{5p-2}{1-p}.\end{cases}$$

Noting that

$$f(1342) < f(3124) \Longleftrightarrow h < \frac{5p-2}{1-p},$$

we have

$$\begin{cases} [1,2,3,4], & \text{if } \frac{3p-1}{1-p} > h;\\ [1,3,4,2], & \text{if } \frac{5p-2}{1-p} > h > \frac{3p-1}{1-p};\\ [3,4,1,2], & \text{if } h > \frac{5p-2}{1-p}.\end{cases}$$

The next diagram shows optimal strategies in the (p, h)-plane. The seeker must examine a node in H first if and only if $h < \frac{5p-2}{1-p}$. That is, if h is large, then the examination cost for nodes in $H = \{1, 2\}$ is large, and the seeker must examine nodes in L first. If p is large, then the a priori probability for nodes in H is large, and the seeker must examine nodes in H first. The order $[1, 3, 4, 2]$ is an intermediate solution between $[1, 2, 3, 4]$ and $[3, 4, 1, 2]$.

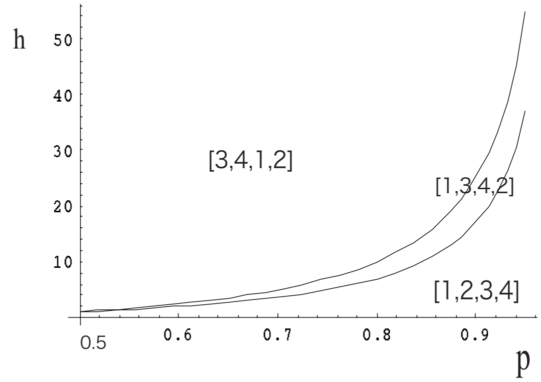

Figure 1: Optimal sequences

6 Remark on alternative formulation of the problem. Let's consider a model: Depending on the examination cost, the nodes are classified into two groups except for the node 0. That is, $H = \{1, \ldots, m\}$ is the set of nodes with examination cost $\overline{c}_i = \overline{h}, i \in H$ and $L = \{m+1, \ldots, m+n\}$ is the set of nodes with examination cost $\overline{c}_i = \overline{\ell}, i \in L$. Thus, $N = H \cup L \cup \{0\}$ and $H \cap L = \emptyset$. We assume

$$\overline{h} > \overline{\ell} > 0.$$

Secondly, we assume that the graph is complete, $E = N \times N$. This means that at each movement of the seeker from a node i to another node j, it costs a traveling cost $\overline{d}(i,j)$ and an examination cost $\overline{c}_j$. In this formulation we do not assume $\overline{d}(i,j) = \overline{d}(j,i)$. We let

$$\overline{d}(i,j) = \begin{cases} s_L, & \text{if } i \in H, j \in L; \\ s_H, & \text{if } i \in L, j \in H; \\ 0, & \text{if } i, j \in H; \\ 0, & \text{if } i, j \in L; \\ t_H^0, & \text{if } i = 0 \text{ and } j \in H; \\ t_L^0, & \text{if } i = 0 \text{ and } j \in L; \\ 0, & \text{if } i = j. \end{cases}$$

For a movement between different groups, the seeker must pay extra set-up cost s_H or s_L. In each group, the seeker must pay only the examination cost. We assume that the seeker should not return to the node 0. In this formulation, the seeker considers only the examination costs in each group. This formulation is transformed into the previous formulation by relations

$$\overline{h} = t_H + h, \overline{\ell} = t_L + \ell, s_H = s - t_H, s_L = s - t_L, t_H^0 = -t_H, t_L^0 = -t_L.$$

References

[1] S. Alpern and S. Gal, *The theory of search games and rendezvous*, Kluwer's INTERNATIONAL SEARIES, Boston, (2003), esp. pp.95-97.

[2] B. Gluss, *Approximately optimal one-dimensional search policies in which search costs vary through time*, Naval Res.Logist.Quart.**8** (1961) 277-283.

[3] K. Kikuta, *A one-dimensional search with traveling cost*, J.Oper.Res.Soc.Japan. **33** (1990), 262-276.

[4] U. Lössner and I. Wegener, *Discrete sequential search with positive switch cost*, Math.Oper.Res. **7** (1982), 426-440.

[5] W. H. Ruckle, *Geometric games and Their Applications*, Pitman, Boston. (1983).

Acknowledgements

The author wishes to thank an Associate Editor and an anonymous referee for helpful comments. This work is partially supported by Grant-in-Aid for Scientic Research from the Ministry of Education, Science and Culture of Japan, No. 20510139.

Authors' adresses:

Kensaku Kikuta, School of Business Administration, University of Hyogo, Gakuen-nishi 8-2-1,Nishi-ku,Kobe 651-2197 Japan, e-mail:kikuta@biz.u-hyogo.ac.jp