LYAPUNOV DECOMPOSITION OF MEASURES ON EFFECT ALGEBRAS

Anna Avallone and Paolo Vitolo

University of Basilicata, Potenza, Italy

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Abstract. We prove that every closed exhaustive vector-valued modular measure on a lattice ordered effect algebra L can be decomposed into the sum of a Lyapunov exhaustive modular measure (i.e. its restriction to every interval of L has convex range) and an "anti-Lyapunov" exhaustive modular measure.

This result extends a Kluvanek-Knowles decomposition theorem for measures on Boolean algebras.

1. Introduction.

In 1974 I. Kluvanek and G. Knowles (see [K-K]) proved a decomposition theorem for a closed σ-additive measure μ on a σ-algebra with values in a quasi-complete locally convex linear space. Precisely, μ can be expressed as the sum of a Lyapunov vector measure and an anti-Lyapunov vector measure.

The decomposition theorem of [K-K] is based on a characterization of Lyapunov measures given in [K-R] and in [K]. In [A-B_1] a similar characterization has been proved for modular measures on D-lattices (i.e. lattice ordered effect algebras), extending a result of [D-W] for measures on σ-algebras. Then a natural question which arises is if for modular measures on D-lattices a Kluvanek-Knowles type decomposition theorem also holds.

In this paper we give a positive answer to this question.

Precisely, we prove (see Theorem (3.16)) that, if X is a Hausdorff locally convex linear space, every closed exhaustive X-valued modular measure on a D-lattice can be decomposed into the sum of a Lyapunov exhaustive modular measure and an "anti-Lyapunov" exhaustive modular measure.

We recall that effect algebras have been introduced by D.J. Foulis and M.K. Bennett in 1994 (see [B-F]) for modelling unsharp measurement in a quantum mechanical system. They are a generalization of many structures which arise in quantum physics (see [B-C]) and in Mathematical Economics (see [B-K], [G-M] and [E-Z]), in particular of orthomodular lattices in non-commutative measure theory and MV-algebras in fuzzy measure theory. After 1994, there have been a great number of papers concerning effect algebras. We refer to [D-P] for a bibliography.

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2. Preliminaries.

We will fix some notations.

Definition (2.1). *Let $(L, \leq)$ be a partial ordered set (a poset for short). A partial binary operation $\ominus$ on L such that $b \ominus a$ is defined if and only if $a \leq b$ is called a difference on $(L, \leq)$ if the following conditions are satisfied:*

(1) *If $a \leq b$, then $b \ominus a \leq b$ and $b \ominus (b \ominus a) = a$.*
(2) *If $a \leq b \leq c$, then $c \ominus b \leq c \ominus a$ and $(c \ominus a) \ominus (c \ominus b) = b \ominus a$.*

Definition (2.2). *Let $(L, \leq \ominus)$ be a poset with difference. If L has greatest and smallest elements 1 and 0, respectively, the structure $(L, \leq, \ominus)$ is called a difference poset (D-poset for short), or a difference lattice (D-lattice for short) if L is a lattice.*

An alternative structure to a D-poset is that of an effect algebra introduced by Foulis and Bennett in [B-K]. These two structures, D-posets and effect algebras, are equivalent as shown in [D-P, Theorem 1.3.4].

We recall that a D-lattice is complete (σ-complete) if every set (countable set) has a supremum and an infimum.

We write $a_\alpha \uparrow a$ (respectively, $a_\alpha \downarrow a$) whenever (a_α) is an increasing net in L and $a = \sup_\alpha a_\alpha$ (respectively, (a_α) is a decreasing net in L and $a = \inf_\alpha a_\alpha$).

If $a, b \in L$, we set $a \triangle b = (a \vee b) \ominus (a \wedge b)$. If $a \leq b$, we set $[a, b] = \{c \in L : a \leq c \leq b\}$. Moreover we set $\Delta = \{(a, b) \in L \times L : a = b\}$.

If $a \in L$, we set $a^\perp = 1 \ominus a$. By (1) of (2.1), we have $(a^\perp)^\perp = a$ for every $a \in L$. It is easy to see that, if L is a D-lattice, then $(a \vee b)^\perp = a^\perp \wedge b^\perp$.

We say that a and b are *orthogonal* if $a \leq b^\perp$ (or, equivalently, if $b \leq a^\perp$), and we write $a \perp b$. If $a \perp b$, we set $a \oplus b = (a^\perp \ominus b)^\perp$. Thus $a \oplus b$ exists and equals c if and only if $b \ominus c$ exists and equals a. This sum is commutative and associative.

If $a_1, \cdots, a_n$ are in L, we inductively define $a_1 \oplus \cdots \oplus a_n = (a_1 \oplus \cdots \oplus a_{n-1}) \oplus a_n$ if the right-side exists. The definition is independent on any permutation of the elements. We say that a finite family $(a_1, \cdots, a_n)$ is *orthogonal* if $a_1 \oplus \cdots \oplus a_n$ exists. We say that a family (a_α) is *orthogonal* if every finite subfamily is orthogonal. If (a_α) is orthogonal, we define $\bigoplus_{\alpha \in A} a_\alpha = \sup\{\bigoplus_{\alpha \in F} a_\alpha : F \subseteq A \text{ finite}\}$.

We need the following result of [D-P] (see 1.1.2 and 1.1.6).

Proposition (2.3).

(1) *If $a \leq b$ and $b \leq c$, then $b \ominus a \leq c \ominus a$ and $(c \ominus a) \ominus (b \ominus a) = c \ominus b$.*
(2) *If $a \perp b$ and $b \leq c$, then $a \oplus b \leq a \oplus c$ and $(a \oplus c) \ominus (a \oplus b) = c \ominus b$.*

An element c in a D-poset is said to be *central* if, for every $a \in L$, both $a \wedge c$ and $a \wedge c^\perp$ exist and $a = (a \wedge c) \vee (a \wedge c^\perp)$. By [A-V] (Lemma 5.1), if L is a D-lattice, $c \in L$ is central if and only if, for each $a \in L$, $a = (a \wedge c) \oplus (a \wedge c^\perp)$. The set $C(L)$ of all central elements of L is called *centre* of L and is a Boolean algebra, as proved in [D-P, 1.9.14].

A subset I of L is said to be a *D-ideal* if the following conditions are satisfied:

(1) For every $a, b \in I$ with $a \perp b$, $a \oplus b \in I$.
(2) For every $a \in I$ and $c \in L$, $(a \vee c) \ominus c \in I$.

We will need the following result of [A-V] (see 4.4 and 5.3).

Theorem (2.4). *If I is a D-ideal and $\sup I$ exists, then it is central.*

A *D-congruence* on a D-lattice L is a lattice congruence N which satisfies the following condition: if $(a, b) \in N$, $(c, d) \in N$, $c \leq a$ and $d \leq b$, then $(a \ominus c, b \ominus d) \in N$.

If $(G, +)$ is an Abelian group and L is a D-lattice, a function $\mu : L \to G$ is said to be *modular* if, for every $a, b \in L$, $\mu(a \vee b) + \mu(a \wedge b) = \mu(a) + \mu(b)$ and it said to be a *measure* if, for every $a, b \in L$, with $a \perp b$, $\mu(a \oplus b) = \mu(a) + \mu(b)$. It is easy to see that μ is a measure if and only if, for every $a, b \in L$, with $a \leq b$, $\mu(b \ominus a) = \mu(b) - \mu(a)$.

If G is a topological Abelian group, by 4.2 of [A-B_2], every modular measure $\mu : L \to G$ generates a *D-uniformity* $\mathcal{U}(\mu)$, i.e. a uniformity on L which makes $\vee, \wedge, \ominus$ and $\oplus$ uniformly continuous.

A measure μ is said to be σ*-additive* if, for every orthogonal sequence (a_n) in L such that $a = \bigoplus_n a_n$ exists, $\mu(a) = \sum_{n \in N} \mu(a_n)$. Moreover μ is said to be *completely additive* if, for every orthogonal family $(a_\alpha)_{\alpha \in A}$ in L such that $a = \bigoplus_\alpha a_\alpha$ exists, the family $(\mu(a_\alpha) : \alpha \in A)$ is summable and $\mu(a) = \sum_\alpha \mu(a_\alpha)$. We say that μ is σ*-order continuous* (σ-o.c. for short) if $a_n \uparrow a$ implies that $(\mu(a_n))$ converges to $\mu(a)$ and *order-continuous* (o.c. for short) if $a_\alpha \uparrow a$ implies that $(\mu(a_\alpha))$ converges to $\mu(a)$. By [A-B_2, 2.4], a measure μ is σ-additive if and only if it is σ-o.c. We say that μ is *exhaustive* if, for every orthogonal sequence (a_n) in L, the sequence $(\mu(a_n))$ converges to 0. By 2.3 of [A], a modular measure μ is exhaustive if and only if μ is exhaustive in the sense of [A-B_1] (i.e. every monotone sequence in L is Cauchy in $\mathcal{U}(\mu)$).

Throughout this paper, X is a Hausdorff locally convex linear space and L is a D-lattice.

3. Lyapunov decomposition theorem.

Let $\mu : L \to X$ be an exhaustive modular measure.

Set

$$I(\mu) = \{a \in L : \mu([0, a]) = \{0\}\}$$

and

$$N(\mu) = \{(a, b) \in L : \forall\ c \leq a \triangle b,\ \mu(c) = 0\}.$$

By 3.1 of [W], 4.3 of [A-B_2] and 4.5 of [A-V_2], $N(\mu)$ is a D-congruence, $I(\mu)$ is a D-ideal and the quotient $\hat{L} = L/N(\mu)$ is a D-lattice. Moreover the function $\hat{\mu} : \hat{L} \to X$ defined as $\hat{\mu}(\hat{a}) = \mu(a)$ for $a \in \hat{a} \in \hat{L}$ clearly is a modular measure, too.

We say that μ is *closed* if $\hat{L}$ is complete with respect to the uniformity $\mathcal{U}(\hat{\mu})$ generated by $\hat{\mu}$.

We need the following result of [A-B_1] (see 4.2).

Lemma (3.1).

(1) *μ is closed iff $\hat{\mu}$ is o.c. and $(\hat{L}, \leq)$ is complete.*
(2) *If μ is o.c., then μ is completely additive.*
(3) *If X is metrizable, then μ is closed.*

Definition (3.2). *We say that μ is semiconvex with respect to $h \in L$ if, for every $a \leq h$, there exists $b \leq a$ such that $\mu(b) = 2\mu(c)$.*

Definition (3.3). *We say that μ is pseudo-injective with respect to $h \in L$ if, for every $b, c \notin I(\mu)$ with $b \perp c$ and $b \oplus c \leq h$, $\mu(b) \neq \mu(c)$.*

Definition (3.4). *We say that μ is pseudo non-injective with respect to $h \in L$ if, for every $a \leq h$ with $a \notin I(\mu)$, μ is not pseudo-injective with respect to a.*

Definition (3.5). *We say that μ is Lyapunov with respect to $h \in L$ if, for every $a \leq h$, $\mu([0, a])$ is convex.*

Definition (3.6). *We say that μ is anti-Lyapunov with respect to $h \in L$ if, for every $a \leq h$ with $a \notin I(\mu)$, μ is not Lyapunov with respect to a.*

Observe that μ is Lyapunov (or anti-Lyapunov, or pseudo non-injective or semiconvex, respectively) with respect to $h \in L$ if and only if, for every $k \leq h$, μ is Lyapunov (or anti-Lyapunov, or pseudo non-injective or semiconvex, respectively) with respect to k.

If μ is Lyapunov (anti-Lyapunov, respectively) with respect to 1 (and therefore with respect to any element of L), we say that μ is *Lyapunov* (*anti-Lyapunov*, respectively).

In the sequel, we need the following result.

Lemma (3.7). *Let $(b_\alpha)_{\alpha \in A}$ be a family of elements of L and suppose that the supremum $b = \sup_\alpha b_\alpha$ exists in L. The following conditions hold:*

(1) *Let $a \in L$ be such that $a \perp b$. Then $c = \sup_\alpha(a \oplus b_\alpha)$ exists in L and $c = a \oplus b$.*
(2) *Let $c \in L$ be such that $c \geq b$. Then $a = \inf_\alpha(c \ominus b_\alpha)$ exists in L and $a = c \ominus b$.*

Proof. (1) is proved in 1.8.7 of [D-P].

(2) Let $d \in L$ be such that $d \leq c \ominus b_\alpha$ for every α. Then $d \perp b_\alpha$ and $d \oplus b_\alpha \leq c$ for every α. Therefore $d \perp b$ and, by (1), $d \oplus b = \sup_\alpha(d \oplus b_\alpha)$. Hence we obtain that $d \oplus b \leq c$, whence $d \leq c \ominus b$. Since $c \ominus b \leq c \ominus b_\alpha$ for every α, we have that $\inf(c \ominus b_\alpha)$ exists and equals $c \ominus b$. □

¿From 4.5 of [A-B_1], the following result can be derived.

Theorem (3.8). *Let μ be closed. Then μ is pseudo non-injective with respect to $h \in L$ if and only if μ is Lyapunov with respect to h.*

Proof. By 4.5 of [A-B_1], the assertion holds for $h = 1$. Then, since $[0, h]$ is clearly a D-lattice, it is sufficient to prove that the restriction $\overline{\mu}$ of μ to $[0, h]$ is closed.

It is easy to see that we can replace L by $\hat{L} = L/N(\mu)$, since μ is closed iff $\hat{\mu}$ is closed and μ is pseudo non-injective (respectively, Lyapunov) with respect to $h \in L$ iff $\hat{\mu}$ is pseudo non-injective (respectively, Lyapunov) with respect to $\hat{h} \in \hat{L}$. Hence we can suppose $N(\mu) = \Delta$. Moreover, since μ is closed and the infimum in L of every subset of $[0, h]$ coincides with the infimum in $[0, h]$, by (3.1) it is clear that $[0, h]$ is complete and $\overline{\mu}$ is o.c. Then, again by (3.1), $\overline{\mu}$ is closed. □

Corollary (3.9). *Let μ be closed. Then:*

(1) *μ is anti-Lyapunov with respect to $h \in L$ if and only if, for every $a \leq h$ with $a \notin I(\mu)$, there exists $b \leq a$ such that $b \notin I(\mu)$ and μ is pseudo-injective with respect to b.*
(2) *If μ is pseudo-injective with respect to $h \in L$, then μ is anti-Lyapunov with respect to h.*

In a similar way as in (3.8), the following result can be derived by 4.3 of [A-B_1], but we prefer to give here an alternative proof based on transfinite induction.

Theorem (3.10). *Let L be complete and μ o.c. Then μ is pseudo non-injective with respect to $h \in L$ if and only if μ is semiconvex with respect to h.*

Proof. $\Leftarrow$ Let $h \in L$ and $a \notin I(\mu)$ with $a \leq h$. We can suppose that $\mu(a) \neq 0$, otherwise we replace a by an element $r \leq a$ with $\mu(r) \neq 0$. By assumption, we can find $b \leq a$ such that $\mu(a) = 2\mu(b)$. Set $c = a \ominus b$. Then $\mu(c) = \mu(a) - \mu(b) = \mu(b)$, $b, c \notin I(\mu)$, $b \perp c$ and $b \oplus c = a$. Hence μ is pseudo non-injective with respect to h.

$\Rightarrow$ Suppose that μ is not semiconvex with respect to h. Then we can find $a \leq h$ such that, for every $b \leq a$, $2\mu(b) \neq \mu(a)$. It follows that $a \notin I(\mu)$.

We construct four sequences by transfinite induction.

Set $\lambda = |L|$ and let χ be a cardinal greater then λ. We prove that, for every ordinal $\beta < \chi$, there exist $a_\beta, c_\beta, d_\beta$ and r_β such that $(a_\beta)_{\beta<\chi}, (c_\beta)_{\beta<\chi}$ and $(d_\beta)_{\beta<\chi}$ are strictly increasing, $(r_\beta)_{\beta<\chi}$ is strictly decreasing, and the following properties hold:

(1) $c_\beta \perp d_\beta$ and $c_\beta \oplus d_\beta = a_\beta$.
(2) $a_\beta \perp r_\beta$ and $a_\beta \oplus r_\beta = a$.
(3) $\mu(c_\beta) = \mu(d_\beta)$.

From (1), (2) and (3) it follows that $c_\beta \le a$, $d_\beta \le a$ and $2\mu(c_\beta) = \mu(c_\beta) + \mu(d_\beta) = \mu(c_\beta \oplus d_\beta) = \mu(a_\beta) = \mu(a \ominus r_\beta) = \mu(a) - \mu(r_\beta)$.

Let $\beta = 0$. Since μ is pseudo non-injective and $a \le h$, we can find $c_0, d_0 \notin I(\mu)$ such that $c_0 \perp d_0$, $c_0 \oplus d_0 \le a$ and $\mu(c_0) = \mu(d_0)$. Set $a_0 = c_0 \oplus d_0$ and $r_0 = a \ominus a_0$. Then the assertion is true for $\beta = 0$. Now suppose by induction that (1), (2) and (3) are true for every β less than an ordinal $\alpha > 0$ and that $(a_\beta)_{\beta<\alpha}$, $(c_\beta)_{\beta<\alpha}$ and $(d_\beta)_{\beta<\alpha}$, are strictly increasing, while $(r_\beta)_{\beta<\alpha}$ is strictly decreasing. We construct $c_\alpha, d_\alpha, a_\alpha$ and r_α.

We distinguish two cases:

(i) α is a limit ordinal.
(ii) α is a successor ordinal.

(i) In this case, we set

$$c_\alpha = \sup\{c_\beta : \beta < \alpha\}, d_\alpha = \sup\{d_\beta : \beta < \alpha\}.$$

Since $c_\beta \perp d_\beta$ for every $\beta < \alpha$, we have also $c_\alpha \perp d_\alpha$. Set $a_\alpha = c_\alpha \oplus d_\alpha$. Applying (1) of (3.7), we have

$$a_\alpha = c_\alpha \oplus \sup_{\gamma<\alpha} d_\gamma = \sup_{\gamma<\alpha}(c_\alpha \oplus d_\gamma) =$$

$$= \sup_{\gamma<\alpha}(\sup_{\beta<\alpha}(c_\beta \oplus d_\gamma)) = \sup_{\beta<\alpha,\gamma<\alpha}(c_\beta \oplus d_\gamma) = \sup_{\beta<\alpha}(c_\beta \oplus d_\beta) = \sup_{\beta<\alpha} a_\beta.$$

Therefore we have $a_\alpha \le a$. Set $r_\alpha = a \ominus a_\alpha$. From (2) of (3.7), we have

$$r_\alpha = \inf\{r_\beta : \beta < \alpha\}.$$

Since $c_\beta \uparrow c_\alpha$ and μ is o.c., $\mu(c_\alpha) = \lim \mu(c_\beta) = \lim \mu(d_\beta) = \mu(d_\alpha)$. Moreover $c_\alpha > c_\beta$, $d_\alpha > d_\beta$, $a_\alpha > a_\beta$ for every $\beta < \alpha$ and $r_\alpha < r_\beta$ for every $\beta < \alpha$ by the inductive assumption.

(ii) In this case, there exists an ordinal γ such that $\alpha = \gamma + 1$. Then we know $a_\gamma, c_\gamma, d_\gamma$ and r_γ and we have to construct a_α, c_α and d_α greater then a_γ, c_γ and d_γ, respectively, and $r_\alpha < r_\gamma$.

Since μ is not semiconvex, we have $2\mu(c_\gamma) \ne \mu(a)$. Then, from $2\mu(c_\gamma) = \mu(a) - \mu(r_\gamma)$, we obtain $\mu(r_\gamma) \ne 0$. Therefore $r_\gamma \notin I(\mu)$. Since μ is pseudo non-injective, we can find $h_\gamma, k_\gamma \notin I(\mu)$ such that $h_\gamma \perp k_\gamma$, $h_\gamma \oplus k_\gamma \le r_\gamma$ and $\mu(h_\gamma) = \mu(k_\gamma)$. Note that, since r_γ is orthogonal to a_γ and $c_\gamma, d_\gamma \le a_\gamma$, then r_γ is also orthogonal to c_γ and d_γ. Since $h_\gamma \le r_\gamma$ and $k_\gamma \le r_\gamma$, we have that h_γ and k_γ are orthogonal to c_γ and d_γ. Set

$$c_\alpha = c_\gamma \oplus h_\gamma, d_\alpha = d_\gamma \oplus k_\gamma.$$

Note that $c_\alpha > c_\gamma$ and $d_\alpha > d_\gamma$ since $h_\gamma, k_\gamma \notin I(\mu)$. Since r_γ is orthogonal to a_γ and $h_\gamma \oplus k_\gamma \le r_\gamma$, we have $h_\gamma \oplus k_\gamma \perp a_\gamma$. Hence there exists

$$(h_\gamma \oplus k_\gamma) \oplus a_\gamma = (h_\gamma \oplus k_\gamma) \oplus (c_\gamma \oplus d_\gamma) =$$

$$= (c_\gamma \oplus h_\gamma) \oplus (d_\gamma \oplus k_\gamma) = c_\alpha \oplus d_\alpha.$$

Set $a_\alpha = c_\alpha \oplus d_\alpha$. Since $a = r_\gamma \oplus a_\gamma \geq a_\alpha$, $r_\alpha = a \ominus a_\alpha$ exists. Since $c_\alpha > c_\gamma$ and $d_\alpha > d_\gamma$, we have $a_\alpha > a_\gamma$ and then $r_\alpha < r_\gamma$. Moreover

$$\mu(c_\alpha) = \mu(c_\gamma \oplus h_\gamma) = \mu(c_\gamma) + \mu(h_\gamma) = \mu(b_\gamma) + \mu(k_\gamma) = \mu(b_\gamma \oplus k_\gamma) = \mu(d_\alpha).$$

This completes the construction of the four sequences.

Now set $A = \{a_\alpha : \alpha \in \chi\}$. Since $(a_\alpha)_{\alpha<\chi}$ is strictly increasing, we have $|A| = \chi$, which is impossible since $\chi > \lambda = |L|$. □

We will need the following result.

Lemma (3.11). *Suppose that L is complete. If I is a D-ideal and $h = \sup I$, then for every $a \in L$ $a \wedge h = \sup\{a \wedge b : b \in I\}$.*

Proof. Recall that by (2.4) h is central.

Let $a \in L$ and set $I_a = \{a \wedge b : b \in I\}$. Observe that $I_a = \{c \in I : c \leq a\}$. Let $r = \sup I_a$. Since $h = \sup I$, we have that $r \leq a \wedge h$. Then the assertion follows if we prove that there exists $H \subseteq I_a$ such that $\sup H = a \wedge h$.

Since h is central, from 5.1 of [A-V_1] we have $a \wedge h = a \ominus (a \wedge h^\perp)$. Set

$$H = \{(b \vee a^\perp) \ominus a^\perp : b \in I\}.$$

Therefore $H \subseteq I_a$ since, if $s = (b \vee a^\perp) \ominus a^\perp \in H$, with $b \in I$, then $s \in I$ since I is a D-ideal and $s \leq 1 \ominus (1 \ominus a) = a$. Set $t = \sup H$. By 5.2 of [A-V_1] and 2.3, recalling that h is central, we have

$$t = \sup\{(a^\perp \vee b) \ominus a^\perp : b \in J\} = (a^\perp \vee h) \ominus a^\perp = a \ominus (a \wedge h^\perp) = a \wedge h.$$

□

Now we set

$$J = \{a \in L : \mu \text{ is semiconvex with respect to } a\},$$

$$J_1 = \{a \in L : \mu \text{ is pseudo non-injective with respect to } a\}$$

and

$$J_2 = \{a \in L : \mu \text{ is anti-Lyapunov with respect to } a\}.$$

By (3.10), if L is complete and μ is o.c., then $J = J_1$.

The following is a crucial result.

Theorem (3.12). *The set J is a D-ideal.*

Proof. We have to prove that J is closed with respect to $\oplus$ and that, for every $r \in L$ and $a \in J$, $(a \vee r) \ominus r \in J$.

(i) Let $a_1, a_2 \in J$ with $a_1 \perp a_2$ and set $a = a_1 \oplus a_2$. We prove that $a \in J$.

Let $b \leq a$ and set

$$b_1 = b \wedge a_1, \; d_2 = (a_1 \vee b) \ominus a_1.$$

Since $b_1 \leq a_1$, $d_2 \leq a \ominus a_1 = a_2$ and $a_1, a_2 \in J$, we can find $c_1 \leq b_1$ and $e_2 \leq d_2$ such that

$$\mu(b_1) = 2\mu(c_1) \text{ and } \mu(d_2) = 2\mu(e_2).$$

Set

$$s_1 = (a_1 \vee b) \ominus e_2.$$

Since $s_1 \leq a_1 \vee b$ and $s_1 \geq (a_1 \vee b) \ominus d_2 = (a_1 \vee b) \ominus ((a_1 \vee b) \ominus a_1) = a_1$, we obtain $a_1 \vee b = s_1 \vee b$. Therefore we have $(s_1 \vee b) \ominus s_1 = (a_1 \vee b) \ominus ((a_1 \vee b) \ominus e_2) = e_2$. Set

$$t_2 = b \ominus (b \wedge s_1).$$

Observe that, since $b \wedge s_1 \geq b \wedge a_1 = b_1$, we have $t_2 \leq b \ominus b_1$. Then, since $c_1 \leq b_1$, we obtain that $t_2 \perp c_1$. Set

$$c = c_1 \oplus t_2.$$

¿From $c_1 \leq b_1$ and $t_2 \leq b \ominus b_1$, we obtain $c \leq b$. Moreover, since μ is modular, we have

$$\mu(t_2) = \mu(b \ominus (b \wedge s_1)) = \mu((b \vee s_1) \ominus s_1) = \mu(e_2).$$

Since μ is a modular measure, we have

$$\mu(b) = \mu((a_1 \vee b) \ominus a_1) + \mu(a_1 \wedge b) = \mu(d_2) + \mu(b_1) =$$

$$= 2\mu(e_2) + 2\mu(c_1) = 2\mu(t_2) + 2\mu(c_1) = 2\mu(c).$$

Hence $a \in J$.

(ii) Let $a \in J$ and $r \in L$. We prove that $h = (a \vee r) \ominus r \in J$.

Let $h' \leq h$. Set

$$s = (a \vee r) \ominus h'.$$

From $s \leq a \vee r$ and $s \geq (a \vee r) \ominus ((a \vee r) \ominus r) = r$, we get $a \vee r = a \vee s$. Then we have $s = (a \vee s) \ominus h'$, from which we get $h' = (a \vee s) \ominus s$. Now set

$$b = a \ominus (a \wedge s).$$

Since $b \leq a \in J$, we can find $c \leq b$ such that $\mu(b) = 2\mu(c)$. Note that, since $c \leq b$ and $b \perp a \wedge s$, $q = c \oplus (a \wedge s)$ exists. From $q \geq a \wedge s$ and $q \leq b \oplus (a \wedge s) = a$, we obtain $q \wedge s = a \wedge s$ and hence $q \ominus (q \wedge s) = c$. Now set

$$c' = (q \vee s) \ominus s.$$

Since $q \leq a$, we have $c' \leq (a \vee s) \ominus s = h'$. Moreover we have

$$\mu(h') = \mu((a \vee s) \ominus s) = \mu(a \ominus (a \wedge s)) = \mu(b) =$$

$$= 2\mu(c) = 2\mu(q \ominus (q \wedge s)) = 2\mu((q \vee s) \ominus s) = 2\mu(c').$$

Therefore $h \in J$. □

Proposition (3.13). *Suppose that μ is closed and $N(\mu) = \Delta$. Then $p = \sup J_1$ exists and is a central element of L.*

Proof. By assumption, $L = L/N(\mu)$. Then, by (3.1), L is complete. Hence p exists. Moreover, by (3.10) and (3.12) $J_1 = J$ is a D-ideal. Then, by (2.4), p is central. □

Lemma (3.14). *Suppose that μ is closed and $N(\mu) = \Delta$. Then the following conditions hold:*

(1) *If $a \notin J_1$, there exists $b \leq a$ such that $b \neq 0$ and $b \in J_2$.*
(2) *If $a \notin J_2$, there exists $b \leq a$ such that $b \neq 0$ and $b \in J_1$.*
(3) *$J_1 \cap J_2 = \{0\}$.*

Proof. (1) If $a \notin J_1$, μ is not pseudo non-injective with respect to a. Then we can find $b \leq a$ such that $b \neq 0$ and μ is pseudo-injective with respect to b. By (3.9)-(2), we obtain that $b \in J_2$.

(2) If $a \notin J_2$, we can find $b \leq a$ with $b \neq 0$ such that μ is Lyapunov with respect to b. Then, by (3.8), $b \in J_1$.

(3) If $a \in J_2$, we have that, for every $b \leq a$ with $b \neq 0$, $b \notin J_1$. In particular, if $a \neq 0$, $a \notin J_1$. □

Proposition (3.15). *Suppose that μ is closed and $N(\mu) = \Delta$. Set $p = \sup J_1$. Then:*

(1) *$a \in J_2$ if and only if $a \wedge p = 0$*
(2) *$J_2 = [0, p^\perp]$.*
(3) *$a \in J_1$ if and only if $a \wedge p^\perp = 0$.*
(4) *$J_1 = [0, p]$.*

Proof. (1) $\Leftarrow$ Suppose that $a \notin J_2$. Then, by (3.14), we can find $b \leq a$ with $b \neq 0$ and $b \in J_1$. Therefore, since $p = \sup J_1$, we have $b \leq a \wedge p = 0$, a contradiction.

$\Rightarrow$ If $a \in J_2$, we have $a \wedge b = 0$ for every $b \in J_1$ since by (3.14) $J_1 \cap J_2 = \{0\}$. By (3.11) we get $a \wedge p = \sup\{a \wedge b : b \in J_1\} = 0$.

(2) Since by (3.13) p is central, we have $a = (a \wedge p) \vee (a \wedge p^\perp)$. Then we obtain that $a \in J_2$ if and only if $a = a \wedge p^\perp$, i.e. $a \leq p^\perp$. Therefore $J_2 = [0, p^\perp]$.

(3) $\Leftarrow$ Suppose that $a \notin J_1$. Then, by (3.14) we can find $b \leq a$ such that $b \neq 0$ and $b \in J_2$. Hence, by (2), we have $b \leq a \wedge p^\perp = 0$, a contradiction.

$\Rightarrow$ If $a \in J_1$, by (2) we have that $a \wedge p^\perp \in J_1 \cap J_2$ and therefore, by (3.14)-(3), $a \wedge p^\perp = 0$.

(4) In a similar way as in (2), we obtain by (3) that $a \in J_1$ if and only if $a \leq p$. □

Notation.

For $h \in L$, denote by μ_h the function defined as

$$\mu_h(a) = \mu(a \wedge h), \ a \in L.$$

It is easy to see that, if h is central, then μ_h is a modular measure and $\mu = \mu_h + \mu_{h^\perp}$. Moreover, if μ is exhaustive (respectively, o.c.), then μ_h and $\mu_{h^\perp}$ are exhaustive (o.c., respectively), too.

Now we can prove the main result.

Theorem (3.16) (Lyapunov decomposition theorem). *Let μ be closed. Then there exists $p \in L$ such that μ_p is a Lyapunov exhaustive modular measure on L, $\mu_{p^\perp}$ is an anti-Lyapunov exhaustive modular measure on L and $\mu = \mu_p + \mu_{p^\perp}$. Moreover the equivalence class $\hat{p}$ of p in $\hat{L} = L/N(\mu)$ is a central element of $\hat{L}$ and, if $q \in L$ has the same properties as p, then $\hat{q} = \hat{p}$.*

Proof. It is easy to see that it is sufficient to prove the theorem in the case that $N(\mu) = \Delta$. Then, by (3.13), $p = \sup J_1$ is central. Therefore μ_p and $\mu_{p^\perp}$ are exhaustive modular measures and $\mu = \mu_p + \mu_{p^\perp}$. Moreover, by (3.8) and (3.15), μ is Lyapunov with respect to p and anti-Lyapunov with respect to $p^{\perp}$. It follows that μ_p is Lyapunov since, for every $a \in L$, $\mu_p([0,a]) = \mu([0, a \wedge p])$.

Now we see that $\mu_{p^{\perp}}$ is anti-Lyapunov. First observe that, since $N(\mu) = \Delta$, $I(\mu_{p^{\perp}}) = \{a \in L : \ \forall \ b \leq a, \ b \wedge p^{\perp} = 0\}$. Hence, by (3.15), $I(\mu_{p^{\perp}}) = J_1$. Now let $a \notin J_1$. Since μ is anti-Lyapunov with respect to $p^{\perp}$ and by (3.15) $a \wedge p^{\perp} \neq 0$, we can find $b \leq a \wedge p^{\perp}$ such that $\mu([0,b])$ is not convex. Therefore $\mu_{p^{\perp}}([0,b]) = \mu([0,b])$ is not convex. Then $\mu_{p^{\perp}}$ is anti-Lyapunov.

If q has the same properties as p, then $q \in J_1$ and $q^{\perp} \in J_2$, hence by (3.15) $q \leq p$ and $q^{\perp} \leq p^{\perp}$, from which $q \geq p$ and therefore $q = p$. $\square$

Remark. It is easy to see that, if we introduce the notion of convexity in a group as in [D-W], all the results of this paper also hold if X is a group which does not contain Z_2 as a semigroup.

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Address: Anna Avallone and Paolo Vitolo, Dipartimento di Matematica. Universitá della Basilicata. Contrada Macchia Romana. 85100 Potenza (Italy).

E-mail: anna.avallone@unibas.it, paolo.vitolo@unibas.it