A TRINARY RELATION ARISING FROM A MATCHED PAIR OF R-DISCRETE GROUPOIDS

Moto O'uchi

Received June 18, 2008; revised September 17, 2008

Abstract. We introduce a notion of a matched pair of r-discrete groupoids and that of a trinary relation associated with a matched pair. We construct three C^*-algebras from a trinary relation and study properties of these algebras. The above results are applied to an action of a countable discrete semidirect product group on a topological space with an invariant measure.

1 Introduction A matched pair of groups has been studied in the theory of quantum groups (cf, [4], [5]) and in the theory of operator algebras (cf. [2]). The notion of a matched pair of groups is a generalization of that of semidirect product groups. It is natural to study a matched pair of groupoids as a generalization of an action of a semidirect product group on a space. In this paper, we introduce a notion of a matched pair of r-discrete groupoids, which is a generalization of that of an action of a discrete semidirect product group. A matched pair of r-discrete groupoid is an r-discrete groupoid G and open and closed subgroupoids G_1 and G_2 which satisfy $G = G_1G_2$, $G_1 \cap G_2 = G^{(0)}$ and other conditions.

On the other hand, a notion of multiplicative unitaries was introduced by S. Baaj and G. Skandalis in [1] and a notion of pseudo-multiplicative unitaries was introduced by J. M. Vallin in [14] (see also [3]). The author has studied pseudo-multiplicative unitaries in the setting of Hilbert C^*-modules (cf. [6, 7, 8, 9]). Recently C^*-pseudo-multiplicative unitaries have been studied intensely by T. Timmermann (cf. [13]). A notion of pseudo-multiplicative unitaries can be converted naturally to a notion of maps on trinary relations satisfying pentagonal equations. The author has studied a sort of these maps in [10]. In this paper, we introduce a trinary relation $\mathcal{T}$ and construct a map $\mathcal{W} : \mathcal{T} *_q \mathcal{T} \longrightarrow \mathcal{T} *_r \mathcal{T}$ that satisfies a pentagonal equation. We use $\mathcal{W}$ to construct C^*-algebras associated with a matched pair (G_1, G_2) of r-discrete groupoids. We construct a C^*-algebra $A \simeq C^*_r(G)$ and C^*-subalgebra $A_i \simeq C^*_r(G_i)$ $(i = 1, 2)$ such that $A = \overline{\text{span}} A_1A_2 = \overline{\text{span}} A_2A_1$ and $A_1 \cap A_2 \simeq C_0(G^{(0)})$.

The paper is organized as follows: In Section 2, we introduce a notion of a matched pair (G_1, G_2) of r-discrete groupoids. In Section 3, we construct a trinary relation $\mathcal{T}$ associated with (G_1, G_2) and construct C^*-algebras A_1 and A_2 using $\mathcal{T}$ when a matched pair has an invariant system. In Section 4, we show that A_i is isomorphic to the reduced groupoid C^*-algebra $C^*_r(G_i)$ $(i = 1, 2)$ when the induced action is preserving. In Section 5, we construct a map π of $C_c(\mathcal{T})$ to $\mathcal{B}(H)$ for some Hilbert space H. Let A be the closed linear span of A_1A_2. Then we show that A is also the closed linear span of A_2A_1 and it is the closure of $\pi(C_c(\mathcal{T}))$. In Section 6, we introduce a $*$-algebraic structure on $C_c(\mathcal{T})$ using π and show that A is isomorphic to $C^*_r(G)$. In Section 7, we construct a conditional expectation $E_i : A \longrightarrow A_i$ for $i = 1, 2$ and show that $A_1 \cap A_2$ is isomorphic to $C_0(G^{(0)})$. In Section 8, we apply the above results to an action of a countable discrete semidirect product group on a space with an invariant measure.

2000 *Mathematics Subject Classification.* 46L89.

Key words and phrases. r-discrete groupoid, groupoid C^*-algebra, trinary relation, pentagonal equation, matched pair .

2 A matched pair of groupoids Let G be a second countable locally compact Hausdorff r-discrete groupoid. We denote by r_G (resp. s_G) the range (resp. source) map of G, by $G^{(0)}$ the unit space of G and by $G^{(2)}$ the set of composable pairs. For details of groupoids, we refer the reader to [11] and [12].

Definition 2.1. Let G_1 and G_2 be clopen subgroupoids of G. A pair (G_1, G_2) is called a matched pair if $G_1G_2 = G$, $G_1 \cap G_2 = G^{(0)}$ and there exist continuous maps $p_1 : G \to G_1$ and $p_2 : G \to G_2$ such that $g = p_1(g)p_2(g)$ for all $g \in G$.

Let (G_1, G_2) be a matched pair. For $i = 1, 2$, we have $G_i^{(0)} = G^{(0)}$ and set $G_{i,x} = s_G^{-1}(x) \cap G_i$ and $G_i^x = r_G^{-1}(x) \cap G_i$ for $x \in G^{(0)}$. Note that we have $r_G(g) = r_G(p_1(g))$ and $s_G(g) = s_G(p_2(g))$ for $g \in G$. For $(g_2, g_1) \in G^{(2)} \cap (G_2 \times G_1)$, set $g_2 \triangleright g_1 = p_1(g_2g_1)$ and $g_2 \triangleleft g_1 = p_2(g_2g_1)$.

Lemma 2.2. (1) *For $g_2 \in G_2$, $g_1 \in G_1^{s_G(g_2)}$ and $h_2 \in G_{2,r_G(g_2)}$, the following equations hold:*

$$g_2^{-1} \triangleright (g_2 \triangleright g_1) = g_1, \quad (g_2^{-1}h_2^{-1}) \triangleright (h_2 \triangleright (g_2 \triangleright g_1)) = g_1.$$

(2) *For $g_1 \in G_1$, $g_2 \in G_{2,r_G(g_1)}$ and $h_1 \in G_1^{s_G(g_1)}$, the following equations hold:*

$$(g_2 \triangleleft g_1) \triangleleft g_1^{-1} = g_2, \quad ((g_2 \triangleleft g_1) \triangleleft h_1) \triangleleft (h_1^{-1}g_1^{-1}) = g_2.$$

Proof. (1) Since we have $g_2^{-1}p_1(g_2g_1)p_2(g_2g_1) = g_2^{-1}(g_2g_1) = g_1$, we have

$$p_1(g_2^{-1}p_1(g_2g_1)) = p_1(g_1p_2(g_2g_1)^{-1}) = g_1.$$

Therefore the first statement of (1) follows.

Set $\tilde{g}_1 = g_2 \triangleright g_1$. It follows from the above argument that we have $g_2^{-1}\tilde{g}_1 = g_1p_2(g_2g_1)^{-1}$. Since we have

$$(g_2^{-1}h_2^{-1})p_1(h_2\tilde{g}_1)p_2(h_2\tilde{g}_1) = g_2^{-1}\tilde{g}_1 = g_1p_2(g_2g_1)^{-1},$$

we have

$$(g_2^{-1}h_2^{-1})p_1(h_2\tilde{g}_1) = g_1p_2(g_2g_1)^{-1}p_2(h_2\tilde{g}_1)^{-1}.$$

Thus we have $p_1((g_2^{-1}h_2^{-1})p_1(h_2\tilde{g}_1)) = g_1$. Therefore the second statement of (1) follows.

We can prove the statements of (2) similarly. □

The following proposition is an immediate consequence of the above lemma.

Proposition 2.3. (1) *For every $g_2 \in G_2$, the map $g_1 \in G_1^{s_G(g_2)} \mapsto g_2 \triangleright g_1 \in G_1^{r_G(g_2)}$ is a bijection.*

(2) *For every $g_1 \in G_1$, the map $g_2 \in G_{2,r_G(g_1)} \mapsto g_2 \triangleleft g_1 \in G_{2,s_G(g_1)}$ is a bijection.*

3 A trinary relation associated with a matched pair Let (G_1, G_2) be a matched pair. Set $\mathcal{T} = \{(g_1, g_2) \in G_1 \times G_2; s_G(g_1) = s_G(g_2)\}$. Define maps q, r, $s : \mathcal{T} \to G^{(0)}$ by $q(g_1, g_2) = r_G(g_1)$, $r(g_1, g_2) = r_G(g_2)$ and $s(g_1, g_2) = s_G(g_1) = s_G(g_2)$ respectively. We denote by $\mathcal{T} *_q \mathcal{T}$ the fibered product $\{(u, v) \in \mathcal{T}^2; s(u) = q(v)\}$. Define the fibered product $\mathcal{T} *_r \mathcal{T}$ similarly. We define a continuous map $\mathcal{W} : \mathcal{T} *_q \mathcal{T} \to \mathcal{T} *_r \mathcal{T}$ by

$$\mathcal{W}((g_1, g_2), (h_1, h_2)) = ((p_1(g_2h_1), h_2p_2(g_2h_1)^{-1}), (g_1h_1, p_2(g_2h_1))$$

for $((g_1, g_2), (h_1, h_2)) \in \mathcal{T} *_q \mathcal{T}$. Then $\mathcal{W}$ is a homeomorphism whose inverse is given by

$$\mathcal{W}^{-1}((g_1, g_2), (h_1, h_2)) = ((h_1p_1(h_2^{-1}g_1^{-1}), p_2(h_2^{-1}g_1^{-1})^{-1}), (p_1(h_2^{-1}g_1^{-1})^{-1}, g_2h_2))$$

for $((g_1, g_2), (h_1, h_2)) \in \mathcal{T} *_r \mathcal{T}$. We call $(\mathcal{T}, \mathcal{W})$ a trinary relation associated with (G_1, G_2).

If $\mathcal{W}(u, v) = (u', v')$, then we have $q(u) = q(v')$, $r(u) = q(u')$, $r(v) = r(u')$ and $s(v) = s(v')$. We denote by $\mathcal{T} *_q \mathcal{T} *_q \mathcal{T}$ the fibered product $\{(u, v, w) \in \mathcal{T}^3; s(u) = q(v), s(v) = q(w)\}$. Define the fibered products $\mathcal{T} *_r \mathcal{T} *_q \mathcal{T}$, $\mathcal{T} *_q \mathcal{T} *_r \mathcal{T}$ and $\mathcal{T} *_r \mathcal{T} *_r \mathcal{T}$ similarly. We also denote by $(\mathcal{T} \times \mathcal{T}) * \mathcal{T}$ the fibered product $\{(u, v, w) \in \mathcal{T}^3; s(u) = q(w), s(v) = r(w)\}$. Then we can define a map $\mathcal{W} *_q I : \mathcal{T} *_q \mathcal{T} *_q \mathcal{T} \to \mathcal{T} *_r \mathcal{T} *_q \mathcal{T}$ by $(\mathcal{W} *_q I)(u, v, w) = (\mathcal{W}(u, v), w)$. Similarly we can define the following maps; $I *_r \mathcal{W} : \mathcal{T} *_r \mathcal{T} *_q \mathcal{T} \to \mathcal{T} *_q \mathcal{T} *_r \mathcal{T}$, $\mathcal{W} *_r I : \mathcal{T} *_q \mathcal{T} *_r \mathcal{T} \to \mathcal{T} *_r \mathcal{T} *_r \mathcal{T}$ and $I *_q \mathcal{W} : \mathcal{T} *_q \mathcal{T} *_q \mathcal{T} \to (\mathcal{T} \times \mathcal{T}) * \mathcal{T}$. We can also define a map $\mathcal{W}_{(13)} : (\mathcal{T} \times \mathcal{T}) * \mathcal{T} \to \mathcal{T} *_r \mathcal{T} *_r \mathcal{T}$ by $\mathcal{W}_{(13)}(u, v, w) = (v, \mathcal{W}(u, w))$.

Theorem 3.1. *The homeomorphism $\mathcal{W}$ satisfies the following pentagonal equation;*

$$(\mathcal{W} *_r I)(I *_r \mathcal{W})(\mathcal{W} *_q I) = \mathcal{W}_{(13)}(I *_q \mathcal{W}). \tag{PE}$$

Proof. For $(u, v, w) \in \mathcal{T} *_q \mathcal{T} *_q \mathcal{T}$, put

$$\begin{aligned}(\mathcal{W} *_r I)(I *_r \mathcal{W})(\mathcal{W} *_q I)(u, v, w) &= (u', v', w')\\ \mathcal{W}_{(13)}(I *_q \mathcal{W})(u, v, w) &= (u'', v'', w'').\end{aligned}$$

If $u = (f_1, f_2)$, $v = (g_1, g_2), w = (h_1, h_2)$, the first coordinate u'_1 of u' is

$$p_1(g_2 p_2(f_2 g_1)^{-1} p_1(p_2(f_2 g_1)h_1))$$

and the first coordinate u''_1 of u'' is $p_1(g_2 h_1)$. We have

$$\begin{aligned}g_2 p_2(f_2 g_1)^{-1} p_1(p_2(f_2 g_1)h_1) &= g_2 h_1 p_2(p_2(f_2 g_1)h_1)^{-1},\\ p_1(g_2 h_1 p_2(p_2(f_2 g_1)h_1)^{-1}) &= p_1(g_2 h_1).\end{aligned}$$

Therefore we have $u'_1 = u''_1$. Similarly we have $u'_2 = u''_2$ and conclude that $u' = u''$. Similarly we have $v' = v''$ and $w' = w''$. ☐

The following map κ plays a role of an involution on $\mathcal{T}$.

Lemma 3.2. *Define maps κ, κ_1, $\kappa_2 : \mathcal{T} \to \mathcal{T}$ by $\kappa(g_1, g_2) = (g_2 \triangleright g_1^{-1}, (g_2 \triangleleft g_1^{-1})^{-1})$, $\kappa_1(g_1, g_2) = (g_1^{-1}, g_2 \triangleleft g_1^{-1})$ and $\kappa_2(g_1, g_2) = ((g_2 \triangleright g_1^{-1})^{-1}, g_2^{-1})$ respectively. Then κ^2, κ_1^2 and κ_2^2 are the identity maps, in particular, κ, κ_1 and κ_2 are homeomorphisms.*

Proof. Since we have, for $i = 1, 2$ and $(g_1, g_2) \in \mathcal{T}$,

$$p_i((p_1(g_2 g_1^{-1}) p_2(g_2 g_1^{-1}))^{-1}) = p_i(g_1 g_2^{-1}),$$

we have $\kappa^2(g_1, g_2) = (g_1, g_2)$. It follows from Lemma 2.2 that κ_1^2 and κ_2^2 are the identity maps. ☐

Let $\{\tilde{\lambda}_x; x \in G^{(0)}\}$ be a right Haar system on G such that $\tilde{\lambda}_x$ is a counting measure on G_x for every $x \in G^{(0)}$. For $i = 1, 2$, we denote by $\{\tilde{\lambda}_{i,x}; x \in G^{(0)}\}$ the right Haar system on G_i which is the restriction of $\{\tilde{\lambda}_x\}$ to G_i. We denote by $C_c(\mathcal{T})$ the set of complex valued continuous functions on $\mathcal{T}$ with compact supports. Define a measure λ_x on $\mathcal{T}$ by

$$\int_{\mathcal{T}} \xi(u)\, d\lambda_x(u) = \iint_{G_1 \times G_2} \xi(g_1, g_2)\, d\tilde{\lambda}_{1,x}(g_1) d\tilde{\lambda}_{2,x}(g_2)$$

for $\xi \in C_c(\mathcal{T})$. Note that the support of λ_x is $\mathcal{T}_x = s^{-1}(x)$ and that the map $x \in G^{(0)} \mapsto \int_{\mathcal{T}} \xi(u)\, d\lambda_x(u)$ is continuous for every $\xi \in C_c(\mathcal{T})$. We say that $\{\lambda_x\}$ is $\mathcal{W}$-invariant if it satisfies the following equation:

$$\iint_{\mathcal{T} *_q \mathcal{T}} \xi(\mathcal{W}(u, v))\, d\lambda_{q(v)}(u) d\lambda_x(v) = \iint_{\mathcal{T} *_r \mathcal{T}} \xi(u, v)\, d\lambda_{r(v)}(u) d\lambda_x(v)$$

for every $\xi \in C_c(\mathcal{T} *_r \mathcal{T})$ and $x \in G^{(0)}$.

For ξ, $\eta \in C_c(\mathcal{T})$, define a product $\xi * \eta$ in $C_c(\mathcal{T})$ by

$$(\xi * \eta)(v) = \int_{\mathcal{T}} (\xi \otimes \eta)(\mathcal{W}^{-1}(u, v)) d\lambda_{r(v)}(u)$$

and define a product $\xi \bullet \eta$ in $C_c(\mathcal{T})$ by

$$(\xi \bullet \eta)(v) = \int_{\mathcal{T}} (\xi \otimes \eta)(\mathcal{W}(u, v)) d\lambda_{q(v)}(u).$$

Proposition 3.3. *Suppose that $\{\lambda_x\}$ is $\mathcal{W}$-invariant. The above products are associative, that is, $(\xi * \eta) * \zeta = \xi * (\eta * \zeta)$ and $(\xi \bullet \eta) \bullet \zeta = \xi \bullet (\eta \bullet \zeta)$ for ξ, η, $\zeta \in C_c(\mathcal{T})$.*

Proof. Set $\mathcal{W}^{-1}(u,v) = (\Psi_1(u,v), \Psi_2(u,v))$. Then we have, for $w \in \mathcal{T}$,

$$\begin{aligned}
&((\xi * \eta) * \zeta)(w) \\
&= \iint (\xi \otimes \eta \otimes \zeta)(\mathcal{W}^{-1}(u, \Psi_1(v,w)), \Psi_2(v,w))\, d\lambda_{q(v)}(u) d\lambda_{r(w)}(v) \\
&= \iint (\xi \otimes \eta \otimes \zeta)((\mathcal{W} *_q I)^{-1}(I *_r \mathcal{W})^{-1}(u,v,w))\, d\lambda_{q(v)}(u) d\lambda_{r(w)}(v) \\
&= \iint (\xi \otimes \eta \otimes \zeta)((\mathcal{W} *_q I)^{-1}(I *_r \mathcal{W})^{-1}(\mathcal{W} *_r I)^{-1}(u,v,w))\, d\lambda_{r(v)}(u) d\lambda_{r(w)}(v).
\end{aligned}$$

The last equation follows from the invariance of $\{\lambda_x\}$. On the other hand, we have

$$\begin{aligned}
&(\xi * (\eta * \zeta))(w) \\
&= \iint (\xi \otimes \eta \otimes \zeta)(\Psi_1(v,w), \mathcal{W}^{-1}(u, \Psi_2(v,w)))\, d\lambda_{r(v)}(u) d\lambda_{r(w)}(v) \\
&= \iint (\xi \otimes \eta \otimes \zeta)((I *_q \mathcal{W})^{-1}(\Psi_1(v,w), u, \Psi_2(v,w)))\, d\lambda_{r(v)}(u) d\lambda_{r(w)}(v) \\
&= \iint (\xi \otimes \eta \otimes \zeta)((I *_q \mathcal{W})^{-1}\mathcal{W}_{(13)}^{-1}(u,v,w))\, d\lambda_{r(v)}(u) d\lambda_{r(w)}(v).
\end{aligned}$$

Since $\mathcal{W}$ satisfies (PE), we have $(\xi * \eta) * \zeta = \xi * (\eta * \zeta)$.

Set $\mathcal{W}(u,v) = (\Phi_1(u,v), \Phi_2(u,v))$. Then we have, for $w \in \mathcal{T}$,

$$\begin{aligned}
&((\xi \bullet \eta) \bullet \zeta)(w) \\
&= \iint (\xi \otimes \eta \otimes \zeta)(\mathcal{W}(u, \Phi_1(v,w)), \Phi_2(v,w))\, d\lambda_{r(v)}(u) d\lambda_{q(w)}(v) \\
&= \iint (\xi \otimes \eta \otimes \zeta)((\mathcal{W} *_r I)(I *_r \mathcal{W})(u,v,w))\, d\lambda_{r(v)}(u) d\lambda_{q(w)}(v) \\
&= \iint (\xi \otimes \eta \otimes \zeta)((\mathcal{W} *_r I)(I *_r \mathcal{W})(\mathcal{W} *_q I)(u,v,w))\, d\lambda_{q(v)}(u) d\lambda_{q(w)}(v).
\end{aligned}$$

The last equation follows from the invariance of $\{\lambda_x\}$. On the other hand, we have

$$\begin{aligned}
&(\xi \bullet (\eta \bullet \zeta))(w) \\
&= \iint (\xi \otimes \eta \otimes \zeta)(\Phi_1(v,w), \mathcal{W}(u, \Phi_2(v,w)))\, d\lambda_{q(v)}(u) d\lambda_{q(w)}(v) \\
&= \iint (\xi \otimes \eta \otimes \zeta)(\mathcal{W}_{(13)}(u, \Phi_1(v,w), \Phi_2(v,w)))\, d\lambda_{q(v)}(u) d\lambda_{q(w)}(v) \\
&= \iint (\xi \otimes \eta \otimes \zeta)(\mathcal{W}_{(13)}(I *_q \mathcal{W})(u,v,w))\, d\lambda_{q(v)}(u) d\lambda_{q(w)}(v).
\end{aligned}$$

Since $\mathcal{W}$ satisfies (PE), we have $(\xi \bullet \eta) \bullet \zeta = \xi \bullet (\eta \bullet \zeta)$. □

We denote by $\mathcal{A}_1$ the opposite algebra of $(C_c(\mathcal{T}), *)$, that is, $\mathcal{A}_1 = C_c(\mathcal{T})$ is an associative algebra over $\mathbb{C}$ whose product is defined by $\xi\eta = \eta * \xi$ and we denote by $\mathcal{A}_2$ the opposite algebra of $(C_c(\mathcal{T}), \bullet)$, that is, $\mathcal{A}_2 = C_c(\mathcal{T})$ is an associative algebra over $\mathbb{C}$ whose product is defined by $\xi\eta = \eta \bullet \xi$. Let μ be a positive regular Radon measure on $G^{(0)}$ whose support is $G^{(0)}$. For $i = 1, 2$, define a measure $\tilde{\lambda}_i$ on G_i by $\tilde{\lambda}_i = \int_{G^{(0)}} \tilde{\lambda}_{i,x}\, d\mu(x)$. We say that μ is G_i-invariant if it satisfies the following equation

$$\int_{G_i} \xi(g_i^{-1})\, d\tilde{\lambda}_i(g_i) = \int_{G_i} \xi(g_i)\, d\tilde{\lambda}_i(g_i)$$

for every $\xi \in C_c(G_i)$. Define a measure λ on $\mathcal{T}$ by $\lambda = \int_{G^{(0)}} \lambda_x \, d\mu(x)$. We denote by H the Hilbert space $L^2(\mathcal{T}, \lambda)$.

Let $\rho_1 : \mathcal{T} \to G_1$ be a Borel map such that $s_G(\rho_1(g_1, g_2)) = r_G(g_2)$. We say that ρ_1 satisfies the condition (A1) if it holds the equation

$$\text{(A1)} \qquad \int_{G_2} \xi(p_1(g_2 g_1), p_2(g_2 g_1)^{-1}) \, d\tilde{\lambda}_{2, r_G(g_1)}(g_2) = \int_{G_2} \xi(\rho_1(g_1, g_2), g_2^{-1}) \, d\tilde{\lambda}_{2, s_G(g_1)}(g_2)$$

for every $g_1 \in G_1$ and every positive Borel function ξ on $\mathcal{T}$ and we say that ρ_1 satisfies the condition (B1) if it holds the equation

$$\text{(B1)} \qquad \int_{G_1} \xi(\rho_1(g_1, g_2)) \, d\tilde{\lambda}_{1, s_G(g_2)}(g_1) = \int_{G_1} \xi(g_1) \, d\tilde{\lambda}_{1, r_G(g_2)}(g_1)$$

for every $g_2 \in G_2$ and every positive Borel function ξ on G_1. Let $\rho_2 : \mathcal{T} \to G_2$ be a Borel map such that $s_G(\rho_2(g_1, g_2)) = r_G(g_1)$. We say that ρ_2 satisfies the condition (A2) if it holds the equation

$$\text{(A2)} \qquad \int_{G_1} \xi(p_1(g_2^{-1} g_1^{-1}), p_2(g_2^{-1} g_1^{-1})^{-1}) \, d\tilde{\lambda}_{1, r_G(g_2)}(g_1) = \int_{G_1} \xi(g_1^{-1}, \rho_2(g_1, g_2)) \, d\tilde{\lambda}_{1, s_G(g_2)}(g_1)$$

for every $g_2 \in G_2$ and every positive Borel function ξ on $\mathcal{T}$ and we say that ρ_2 satisfies the equation (B2) if it holds the equation

$$\text{(B2)} \qquad \int_{G_2} \xi(\rho_2(g_1, g_2)) \, d\tilde{\lambda}_{2, s_G(g_1)}(g_2) = \int_{G_2} \xi(g_2) \, d\tilde{\lambda}_{2, r_G(g_1)}(g_2)$$

for every $g_1 \in G_1$ and every positive Borel function ξ on G_2. The existence of ρ_1 that satisfies the conditions (A1) and (B1) implies that $\{\lambda_x\}$ is $\mathcal{W}$-invariant and the existence of ρ_2 that satisfies the conditions (A2) and (B2) also implies that $\{\lambda_x\}$ is $\mathcal{W}$-invariant.

Theorem 3.4. (1) *Suppose that μ is G_1-invariant and that there exists a map ρ_2 which satisfies conditions* (A2) *and* (B2). *Then, for every $\xi \in C_c(\mathcal{T})$, there exists a positive number M such that $\|\eta * \xi\|_H \le M \|\eta\|_H$ for every $\eta \in C_c(\mathcal{T})$.*

(2) *Suppose that μ is G_2-invariant and that there exists a map ρ_1 which satisfies conditions* (A1) *and* (B1). *Then, for every $\xi \in C_c(\mathcal{T})$, there exists a positive number M such that $\|\eta \bullet \xi\|_H \le M \|\eta\|_H$ for every $\eta \in C_c(\mathcal{T})$.*

Proof. (1) For $i = 1, 2$, let K_i be a compact set in G_i such that the support of ξ is contained in $K_1 \times K_2$. We denote by χ_{K_i} the characteristic function of K_i. Set

$$\chi(g_1, g_2, h_2) = \chi_{K_1}(p_1(h_2^{-1} g_1^{-1})^{-1}) \chi_{K_2}(g_2 h_2)$$

for $h_2 \in G_2$ and $(g_1, g_2) \in \mathcal{T}_{r_G(h_2)}$. For $(h_1, h_2) \in \mathcal{T}$, set

$$F(h_1, h_2) = \int_{\mathcal{T}} |\eta(h_1 p_1(h_2^{-1} g_1^{-1}), p_2(h_2^{-1} g_1^{-1})^{-1})|^2 \chi(g_1, g_2, h_2) \, d\lambda_{r_G(h_2)}(g_1, g_2),$$

$$\tilde{\chi}(h_2) = \int_{\mathcal{T}} \chi(g_1, g_2, h_2) \, d\lambda_{r_G(h_2)}(g_1, g_2).$$

Then we have

$$\|\eta * \xi\|_H^2 \le \|\xi\|_\infty^2 \int_{\mathcal{T}} F(h_1, h_2) \tilde{\chi}(h_2) \, d\lambda(h_1, h_2).$$

Set $M_i = \sup\{\tilde{\lambda}_{i,x}(K_i);\, x \in G^{(0)}\}$. It follows from the condition (A2) that we have $\tilde{\chi}(h_2) \leq M_1 M_2$ and that we have

$$F(h_1,h_2) \leq M_2 \int_{G_1} |\eta(h_1 g_1^{-1}, \rho_2(g_1,h_2))|^2 \chi_{K_1}(g_1)\, d\tilde{\lambda}_{1,s_G(h_2)}(g_1).$$

It follows from the condition (B2) that we have, for $g_1 \in G_{1,x}$,

$$\int_{\mathcal{T}} |\eta(h_1 g_1^{-1}, \rho_2(g_1,h_2))|^2\, d\lambda_x(h_1,h_2) = \int_{\mathcal{T}} |\eta(u)|^2\, d\lambda_{r_G(g_1)}(u).$$

Set $\|\eta\|_x^2 = \int |\eta(u)|^2\, d\lambda_x(u)$ and set $M_i' = \sup\{\tilde{\lambda}_{i,x}(K_i^{-1});\, x \in G^{(0)}\}$. Since μ is G_1-invariant, we have

$$\begin{aligned}\int_{G_1} \|\eta\|_{r_G(g_1)}^2 \chi_{K_1}(g_1)\, d\tilde{\lambda}_1(g_1) &= \int_{G_1} \|\eta\|_{s_G(g_1)}^2 \chi_{K_1^{-1}}(g_1)\, d\tilde{\lambda}_1(g_1) \\ &\leq M_1' \|\eta\|_H^2.\end{aligned}$$

Therefore we have $\|\eta * \xi\|_H \leq M_1^{1/2} {M_1'}^{1/2} M_2 \|\xi\|_\infty \|\eta\|_H$.

(2) We keep the notations in the proof of (1). Set

$$\chi'(g_1,g_2,h_1) = \chi_{K_1}(g_1h_1)\chi_{K_2}(p_2(g_2h_1))$$

for $h_1 \in G_1$ and $(g_1,g_2) \in \mathcal{T}_{r_G(h_1)}$. For $(h_1,h_2) \in \mathcal{T}$, set

$$\begin{aligned}F'(h_1,h_2) &= \int_{\mathcal{T}} |\eta(p_1(g_2h_1), h_2p_2(g_2h_1)^{-1})|^2 \chi'(g_1,g_2,h_1)\, d\lambda_{r_G(h_1)}(g_1,g_2), \\ \tilde{\chi}'(h_1) &= \int_{\mathcal{T}} \chi'(g_1,g_2,h_1)\, d\lambda_{r_G(h_1)}(g_1,g_2).\end{aligned}$$

Then we have

$$\|\eta \bullet \xi\|_H^2 \leq \|\xi\|_\infty^2 \int_{\mathcal{T}} F'(h_1,h_2)\tilde{\chi}'(h_1)\, d\lambda(h_1,h_2).$$

It follows from the condition (A1) that we have $\tilde{\chi}'(h_1) \leq M_1M_2$ and that we have

$$F'(h_1,h_2) \leq M_1 \int_{G_2} |\eta(\rho_1(h_1,g_2), h_2g_2^{-1})|^2 \chi_{K_2}(g_2)\, d\tilde{\lambda}_{2,s_G(h_1)}(g_2).$$

It follows from the condition (B1) that we have, for $g_2 \in G_{2,x}$,

$$\int_{\mathcal{T}} |\eta(\rho_1(h_1,g_2), h_2g_2^{-1})|^2\, d\lambda_x(h_1,h_2) = \int_{\mathcal{T}} |\eta(u)|^2\, d\lambda_{r_G(g_2)}(u).$$

Since μ is G_2-invariant, we have

$$\begin{aligned}\int_{G_2} \|\eta\|_{r_G(g_2)}^2 \chi_{K_2}(g_2)\, d\tilde{\lambda}_2(g_2) &= \int_{G_2} \|\eta\|_{s_G(g_2)}^2 \chi_{K_2^{-1}}(g_2)\, d\tilde{\lambda}_2(g_2) \\ &\leq M_2' \|\eta\|_H^2.\end{aligned}$$

Therefore we have $\|\eta \bullet \xi\|_H \leq M_1 M_2^{1/2} {M_2'}^{1/2} \|\xi\|_\infty \|\eta\|_H$. □

A triplet (ρ_1, ρ_2, μ) is called an invariant system for (G_1, G_2) if ρ_i satisfies conditions (Ai) and (Bi) for $i = 1, 2$ and μ is G_1- and G_2-invariant. Let (G_1, G_2) be a matched pair with an invariant system (ρ_1, ρ_2, μ). It follows from Theorem 3.4 (1) that there exists a homomorphism $\pi_1 : \mathcal{A}_1 \to \mathcal{B}(H)$ as algebras over $\mathbb{C}$ such that $\pi_1(\xi)\eta = \eta * \xi$ for ξ, $\eta \in C_c(\mathcal{T})$. We denote by A_1 the C^*-subalgebra of $\mathcal{B}(H)$ generated by $\pi_1(\mathcal{A}_1)$. It follows from Theorem 3.4 (2) that there exists a homomorphism $\pi_2 : \mathcal{A}_2 \to \mathcal{B}(H)$ as algebras over $\mathbb{C}$ such that $\pi_2(\xi)\eta = \eta \bullet \xi$ for ξ, $\eta \in C_c(\mathcal{T})$. We denote by A_2 the C^*-subalgebra of $\mathcal{B}(H)$ generated by $\pi_2(\mathcal{A}_2)$.

4 Preserving actions induced by a matched pair

Definition 4.1. Let (G_1, G_2) be a matched pair with an invariant system (ρ_1, ρ_2, μ). Then the induced action $\triangleright$ (resp. $\triangleleft$) of (G_1, G_2) is said to be preserving if $\rho_1(g_1, g_2) = (g_2 \triangleright g_1^{-1})^{-1}$ (resp. $\rho_2(g_1, g_2) = g_2 \triangleleft g_1^{-1}$) for every $(g_1, g_2) \in \mathcal{T}$.

If $\triangleright$ (resp. $\triangleleft$) is preserving, then ρ_1 (resp. ρ_2) always satisfies (B1) (resp. (B2)).

For $i = 1, 2$, $C_c(G_i)$ is a $*$-algebra with the following product and involution;

$$(ab)(g) = \int_{G_i} a(gh^{-1})b(h)\, d\tilde{\lambda}_{i,s_G(g)}(h),$$
$$a^*(g) = \overline{a(g^{-1})}$$

for $a, b \in C_c(G_i)$ and $g \in G_i$. For $x \in G^{(0)}$, set $\tilde{H}_{i,x} = L^2(G_{i,x}, \tilde{\lambda}_{i,x})$. Define a $*$-representation $\tilde{\pi}_{i,x} : C_c(G_i) \to \mathcal{B}(\tilde{H}_{i,x})$ by

$$(\tilde{\pi}_{i,x}(a)\zeta)(g) = \int_{G_i} a(gh^{-1})\zeta(h)\, d\tilde{\lambda}_{i,x}(h)$$

for $a \in C_c(G_i)$, $\zeta \in \tilde{H}_{i,x}$ and $g \in G_{i,x}$. Define the reduced norm $\|a\|$ by $\|a\| = \sup\{\|\tilde{\pi}_{i,x}(a)\|\, ;\, x \in G^{(0)}\}$. The reduced groupoid C^*-algebra $C_r^*(G_i)$ is the completion of $C_c(G_i)$ by the reduced norm. We can extend $\tilde{\pi}_{i,x}$ to the $*$-representation of $C_r^*(G_i)$ on $\tilde{H}_{i,x}$, which we denote again by $\tilde{\pi}_{i,x}$.

In this section, we will prove the following theorem.

Theorem 4.2. *Let (G_1, G_2) be a matched pair with an invariant system (ρ_1, ρ_2, μ) and suppose that the actions $\triangleright$ and $\triangleleft$ are preserving. Then, A_1 and A_2 are $*$-isomorphic to the reduced groupoid C^*-algebras $C_r^*(G_1)$ and $C_r^*(G_2)$ respectively.*

Proof. Since μ is G_1- and G_2-invariant, we have

$$\begin{aligned}\int_{\mathcal{T}} \xi(u)\, d\lambda(u) &= \int_{G^{(0)}}\int_{G_1}\int_{G_2} \xi(g_1^{-1}, g_2)\, d\tilde{\lambda}_{2,r_G(g_1)}(g_2)\tilde{\lambda}_{1,x}(g_1)d\mu(x) \\ &= \int_{G^{(0)}}\int_{G_2}\int_{G_1} \xi(g_1, g_2^{-1})\, d\tilde{\lambda}_{1,r_G(g_2)}(g_1)\tilde{\lambda}_{2,x}(g_2)d\mu(x)\end{aligned}$$

for $\xi \in C_c(\mathcal{T})$. Then we can define unitary operators T_1 and T_2 in $\mathcal{B}(H)$ by

$$(T_1\xi)(g_1, g_2) = \xi \circ \kappa_1(g_1, g_2) = \xi(g_1^{-1}, \rho_2(g_1, g_2))$$

and

$$(T_2\xi)(g_1, g_2) = \xi \circ \kappa_2(g_1, g_2) = \xi(\rho_1(g_1, g_2), g_2^{-1})$$

for $\xi \in H$ and $(g_1, g_2) \in \mathcal{T}$ respectively. It follows from Lemma 3.2 that we have $T_i^2 = I$ for $i = 1, 2$. Thus we have $T_i^* = T_i$.

For $x \in G^{(0)}$, set $H_x = L^2(\mathcal{T}, \lambda_x)$. Note that we have $H_x = \tilde{H}_{1,x} \otimes \tilde{H}_{2,x}$ and $H = \int^{\oplus} H_x\, d\mu(x)$. Define a $*$-representation $\tilde{\pi}_{1,x} \otimes \iota : C_r^*(G_1) \to \mathcal{B}(H_x)$ by $(\tilde{\pi}_{1,x} \otimes \iota)(a) = \tilde{\pi}_{1,x}(a) \otimes I$ for $a \in C_r^*(G_1)$ and define a $*$-representation $\tilde{\pi}_1 : C_r^*(G_1) \to \mathcal{B}(H)$ by $\tilde{\pi}_1 = \int^{\oplus} (\tilde{\pi}_{1,x} \otimes \iota)\, d\mu(x)$. Similarly define $*$-representations $\iota \otimes \tilde{\pi}_{2,x} : C_r^*(G_2) \to \mathcal{B}(H_x)$ and $\tilde{\pi}_2 : C_r^*(G_2) \to \mathcal{B}(H)$. Since the support of μ is $G^{(0)}$, $\tilde{\pi}_1$ and $\tilde{\pi}_2$ are faithful. Define a linear map $\varphi_1 : C_c(\mathcal{T}) \to C_c(G_1)$ by

$$\varphi_1(\xi)(g_1) = \int \xi(g_1^{-1}, g_2)\, d\tilde{\lambda}_{2,r_G(g_1)}(g_2)$$

for $\xi \in C_c(\mathcal{T})$ and $g_1 \in G_1$ and define a linear map $\varphi_2 : C_c(\mathcal{T}) \to C_c(G_2)$ by

$$\varphi_2(\xi)(g_2) = \int \xi(g_1, g_2^{-1})\, d\tilde{\lambda}_{1,r_G(g_2)}(g_1)$$

for $\xi \in C_c(\mathcal{T})$ and $g_2 \in G_2$. Using the conditions (A1) and (A2), we have, for $\xi \in C_c(\mathcal{T})$, $\eta \in H$ and $(g_1, g_2) \in \mathcal{T}$,

$$\begin{aligned}&(\pi_1(\xi)\eta)(g_1, g_2)\\&= \int_{G_1\times G_2} \xi(h_1, h_2)\eta(g_1h_1^{-1}, \rho_2(h_1, g_2))\, d\tilde{\lambda}_{1,s_G(g_2)}(h_1)d\tilde{\lambda}_{2,s_G(g_2)}(h_2),\\&(\pi_2(\xi)\eta)(g_1, g_2)\\&= \int_{G_1\times G_2} \xi(h_1, h_2)\eta(\rho_1(g_1, h_2), g_2h_2^{-1})\, d\tilde{\lambda}_{1,s_G(g_1)}(h_1)d\tilde{\lambda}_{2,s_G(g_1)}(h_2).\end{aligned}$$

It follows from Lemma 2.2 that we have $T_i\pi_i(\xi)T_i = \tilde{\pi}_i(\varphi_i(\xi))$ for $i = 1, 2$ and $\xi \in C_c(\mathcal{T})$. Therefore $T_iA_iT_i$ is contained in $\tilde{\pi}_i(C^*_r(G_i))$ for $i = 1, 2$.

We denote by $\chi_{G^{(0)}}$ the characteristic function of $G^{(0)}$ in G. Since G is r-discrete, $\chi_{G^{(0)}}$ is a continuous function on G. For $f_i \in C_c(G_i)$, define an element $\psi_1(f_1)$ (resp. $\psi_2(f_2)$) of $C_c(\mathcal{T})$ by $\psi_1(f_1)(g_1, g_2) = f_1(g_1^{-1})\chi_{G^{(0)}}(g_2)$ (resp. $\psi_2(f_2)(g_1, g_2) = \chi_{G^{(0)}}(g_1)f_2(g_2^{-1})$). We have $\varphi_i(\psi_i(f_i)) = f_i$. Therefore we have $T_i\pi_i(\psi_i(f_i))T_i = \tilde{\pi}_i(f_i)$. This implies that $\tilde{\pi}_i(C^*_r(G_i))$ is contained in $T_iA_iT_i$ for $i = 1, 2$. □

Corollary 4.3. *For $i = 1$, 2, A_i is the closure of the set of elements $\pi_i(\psi_i(f))$ with $f \in C_c(G_i)$.*

5 C^*-algebras arising from $C_c(\mathcal{T})$ Let (G_1, G_2) be a matched pair with an invariant system (ρ_1, ρ_2, μ). Moreover, suppose that the actions $\triangleright$ and $\triangleleft$ are preserving. In this section, we define a map $\pi : C_c(\mathcal{T}) \to \mathcal{B}(H)$ and show that the closure of $\pi(C_c(\mathcal{T}))$ is a C^*-algebra.

For $\xi \in C_c(\mathcal{T})$, define $\psi(\xi) \in C_c(\mathcal{T} *_r \mathcal{T})$ by

$$\psi(\xi)((g_1, g_2), (h_1, h_2)) = \xi(g_1, h_2^{-1})\chi_{G^{(0)}}(h_1)\chi_{G^{(0)}}(g_2).$$

For ξ, $\eta \in C_c(\mathcal{T})$, define $\pi(\xi)\eta \in C_c(\mathcal{T})$ by

$$(\pi(\xi)\eta)(w) = \int_{\mathcal{T}}\int_{\mathcal{T}} (\eta \otimes \psi(\xi))((\mathcal{W} *_r I)^{-1}\mathcal{W}_{(13)}(u, v, w))\, d\lambda_{r(w)}(v)d\lambda_{q(w)}(u)$$

for $w \in \mathcal{T}$. Then we will show the following proposition.

Proposition 5.1. *For every $\xi \in C_c(\mathcal{T})$, there exists a positive number M such that $\|\pi(\xi)\eta\|_H \le M\|\eta\|_H$ for every $\eta \in C_c(\mathcal{T})$.*

The above proposition implies that we can extend $\pi(\xi)$ to a bounded linear operator on H, which we denote again by $\pi(\xi)$. Therefore we have a linear map $\pi : C_c(\mathcal{T}) \to \mathcal{B}(H)$. From the proof of theorem 4.2, we have the following lemma.

Lemma 5.2. *For $f_i \in C_c(G_i)$ $(i = 1, 2)$, $\eta \in H$ and $(g_1, g_2) \in \mathcal{T}$, the following equations hold;*

$$\begin{aligned}(\pi_1(\psi_1(f_1))\eta)(g_1, g_2) &= \int_{G_1} f_1(h_1^{-1})\eta(g_1h_1^{-1}, p_2(g_2h_1^{-1}))\, d\tilde{\lambda}_{1,s_G(g_2)}(h_1),\\(\pi_2(\psi_2(f_2))\eta)(g_1, g_2) &= \int_{G_2} f_2(h_2^{-1})\eta(p_1(h_2g_1^{-1})^{-1}, g_2h_2^{-1})\, d\tilde{\lambda}_{2,s_G(g_1)}(h_2).\end{aligned}$$

For $(g_1, g_2) \in \mathcal{T}$, $h_1 \in G_{1,s_G(g_2)}$ and $h_2 \in G_{2,r_G(h_1)}$, set

$$\theta_1(g_1, g_2; h_1, h_2) = (p_1(h_2h_1g_1^{-1})^{-1}, p_2(g_2(h_2h_1)^{-1})) \in \mathcal{T},$$

and for $(g_1, g_2) \in \mathcal{T}$, $h_2 \in G_{2,s_G(g_1)}$ and $h_1 \in G_{2,r_G(h_2)}$, set

$$\theta_2(g_1, g_2; h_1, h_2) = (p_1(h_1h_2g_1^{-1})^{-1}, p_2(g_2(h_1h_2)^{-1})) \in \mathcal{T}.$$

Proof of Proposition 5.1. By using the conditions (A1) and (A2) and the fact that the induced actions are preserving, we have

$$(5.1)\qquad \begin{aligned}&(\pi(\xi)\eta)(g_1, g_2)\\ &= \int_{G_2}\int_{G_1} \xi(h_1, h_2^{-1})\eta(\theta_2(g_1, g_2; h_1, h_2))\, d\tilde{\lambda}_{1,r_G(h_2)}(h_1)d\tilde{\lambda}_{2,s_G(g_1)}(h_2).\end{aligned}$$

For $i = 1$, 2, let K_i be a compact subset of G_i such that the support of ξ is contained in $K_1 \times K_2$. We can define $\pi_i(\psi_i(\chi_{K_i})) \in \mathcal{B}(H)$ by a similar formula to that in Lemma 5.2. Then we have $\|\pi_i(\psi(\chi_{K_i}))\| \le M_i^{1/2} M_i'^{1/2}$, where $M_i = \sup\{\tilde{\lambda}_{i,x}(K_i);\, x \in G^{(0)}\}$ and $M_i' = \sup\{\tilde{\lambda}_{i,x}(K_i^{-1});\, x \in G^{(0)}\}$. Since we have, for $u \in \mathcal{T}$,

$$|(\pi(\xi)\eta)(u)| \le \|\xi\|_\infty (\pi_2(\psi(\chi_{K_2}))\pi_1(\psi_1(\chi_{K_1^{-1}}))|\eta|)(u),$$

we have

$$\|\pi(\xi)\eta\|_H \le \|\xi\|_\infty \|\pi_1(\psi_1(\chi_{K_1^{-1}}))\| \|\pi_2(\psi_2(\chi_{K_2}))\| \|\eta\|_H.$$

□

For $f_i \in C_c(G_i)$ $(i = 1, 2)$, define $\check{f}_i \in C_c(G_i)$ by $\check{f}_i(g_i) = f_i(g_i^{-1})$. We denote by $f_1 \otimes f_2$ the restriction of $f_1 \otimes f_2 \in C_c(G_1 \times G_2)$ to $\mathcal{T}$ by abuse of notation. Recall that $\kappa : \mathcal{T} \to \mathcal{T}$ is the homeomorphism introduced in Lemma 3.2. Then we have the following proposition.

Proposition 5.3. *For $f_i \in C_c(G_i)$ $(i = 1, 2)$, the following equations hold;*

$$\begin{aligned}\pi_2(\psi_2(f_2))\pi_1(\psi_1(f_1)) &= \pi(\check{f}_1 \otimes f_2),\\ \pi_1(\psi_1(f_1))\pi_2(\psi_2(f_2)) &= \pi((f_1 \otimes \check{f}_2) \circ \kappa).\end{aligned}$$

Proof. By Lemma 5.2, we have

$$\begin{aligned}&(\pi_1(\psi_1(f_1))\pi_2(\psi_2(f_2))\eta)(g_1, g_2)\\ &= \int_{G_1}\int_{G_2} f_1(h_1^{-1})f_2(h_2^{-1})\eta(\theta_1(g_1, g_2; h_1, h_2))\, d\tilde{\lambda}_{2,r_G(h_1)}(h_2)d\tilde{\lambda}_{1,s_G(g_2)}(h_1),\\ &(\pi_2(\psi_2(f_2))\pi_1(\psi_1(f_1))\eta)(g_1, g_2)\\ &= \int_{G_2}\int_{G_1} f_1(h_1^{-1})f_2(h_2^{-1})\eta(\theta_2(g_1, g_2; h_1, h_2))\, d\tilde{\lambda}_{1,r_G(h_2)}(h_1)d\tilde{\lambda}_{2,s_G(g_1)}(h_2).\end{aligned}$$

Note that we have $(h_2^{-1} \triangleright h_1^{-1})^{-1} h_2^{-1} h_1^{-1} = h_2^{-1} \triangleleft h_1^{-1}$. By using the conditions (A1) and (B1) and the fact that $\triangleright$ is preserving, we have

$$\begin{aligned}&(\pi_1(\psi_1(f_1))\pi_2(\psi_2(f_2))\eta)(g_1, g_2)\\ &= \iint f_1(h_2^{-1} \triangleright h_1^{-1})f_2(h_2^{-1} \triangleleft h_1^{-1})\eta(\theta_2(g_1, g_2; h_1, h_2))\, d\tilde{\lambda}_{1,r_G(h_2)}(h_1)d\tilde{\lambda}_{2,s_G(g_1)}(h_2).\end{aligned}$$

Using (5.1), we have the equations in the proposition. □

We denote by A_1A_2 the set of elements a_1a_2 with $a_i \in A_i$ $(i = 1, 2)$ and by $\overline{\mathrm{span}}\, A_1A_2$ the closed linear span of A_1A_2. Set $A = \overline{\mathrm{span}}\, A_1A_2$.

Theorem 5.4. *The closed linear space A is a C^*-algebra.*

The above theorem is an immediate consequence of the following proposition.

Proposition 5.5. $\overline{\mathrm{span}}\, A_1A_2 = \overline{\mathrm{span}}\, A_2A_1$.

Proof. For $f_i \in C_c(G_i)$ $(i=1,2)$, $F = (f_1 \otimes \check{f}_2) \circ \kappa$ is an element of $C_c(\mathcal{T})$. For every $\varepsilon > 0$, there exist $f_{ij} \in C_c(G_i)$ $(i = 1, 2,\ j = 1, \cdots n)$ such that $\|F - \sum f_{1,j} \otimes f_{2,j}\|_\infty < \varepsilon$ and the supports of F and $\sum f_{1,j} \otimes f_{2,j}$ are contained in some compact set K of $G_1 \times G_2$. It follows from Proposition 5.3 and the proof of Proposition 5.1 that we have

$$\|\pi_1(\psi_1(f_1))\pi_2(\psi_2(f_2)) - \sum_{j=1}^{n} \pi_2(\psi_2(f_{2,j}))\pi_1(\psi_1(\check{f}_{1,j}))\| \le \varepsilon M,$$

where M is a constant that depends only on K. This implies that $\overline{\operatorname{span}}\, A_1A_2 \subset \overline{\operatorname{span}}\, A_2A_1$. By taking adjoint, we have the reverse inclusion. □

It is easy to show that A_i is contained in $\overline{A_1A_2}$ $(i = 1, 2)$. In particular A_i is a C^*-subalgebra of A. By Proposition 5.3 we have the following corollary.

Corollary 5.6. *The C^*-algebra A is the closure of $\pi(C_c(\mathcal{T}))$.*

6 A $*$-algebraic structure for $C_c(\mathcal{T})$ In this section, we prove that A is isomorphic to the groupoid C^*-algebra $C^*_r(G)$. To prove this fact, we introduce a $*$-algebraic structure for $C_c(\mathcal{T})$.

Let $\mathcal{S}$ be a closed subset of $(G_1 \times G_2)^2$ consisting of elements (g_1, g_2, h_1, h_2) such that $s_G(g_1) = s_G(g_2)$, $r_G(g_2) = s_G(h_2)$ and $s_G(h_1) = r_G(h_2)$. Let $\mathcal{S}'$ be a closed subset of $\mathcal{T}^2$ consisting of elements (u, v) such that $q(u) = r(v)$. Define a homeomorphism $\alpha : \mathcal{S} \to \mathcal{S}'$ by

$$\alpha(g_1, g_2, h_1, h_2) = (h_1, h_2^{-1}, p_1(g_1g_2^{-1}(h_1h_2)^{-1}), p_2(g_1g_2^{-1}(h_1h_2)^{-1})^{-1}).$$

The inverse of α is given by

$$\alpha^{-1}(g_1, g_2, h_1, h_2) = (p_1(h_1h_2^{-1}g_1g_2^{-1}), p_2(h_1h_2^{-1}g_1g_2^{-1})^{-1}, g_1, g_2^{-1}).$$

For ξ, $\eta \in C_c(\mathcal{T})$, define an element $\xi\eta \in C_c(\mathcal{T})$ by

$$(\xi\eta)(g_1, g_2) = \iint (\xi \otimes \eta)(\alpha(g_1, g_2, h_1, h_2))\, d\tilde{\lambda}_{1,r_G(h_2)}(h_1) d\tilde{\lambda}_{2,r_G(g_2)}(h_2).$$

For $\xi \in C_c(\mathcal{T})$, define an element $\xi^* \in C_c(\mathcal{T})$ by $\xi^* = \bar{\xi} \circ \kappa$. We will show that $C_c(\mathcal{T})$ is a $*$-algebra with respect to the product $\xi\eta$ and the involution ξ^* defined above.

Lemma 6.1. *For ξ, $\eta \in C_c(\mathcal{T})$, the following equation holds; $\pi(\xi)\pi(\eta) = \pi(\xi\eta)$.*

Proof. For $f_i \in C_c(G_i)$ $(i = 1, 2)$ and $\zeta \in C_c(\mathcal{T})$, define $f_i\zeta \in C_c(\mathcal{T})$ by

$$(f_1\zeta)(g_1, g_2) = \int_{G_1} f_1(h_1^{-1})\zeta(h_1g_1, g_2)\, d\tilde{\lambda}_{1,r_G(g_1)}(h_1),$$
$$(f_2\zeta)(g_1, g_2) = \int_{G_2} f_2(h_2^{-1})\zeta(g_1, h_2g_2)\, d\tilde{\lambda}_{2,r_G(g_2)}(h_2).$$

Then we have $\pi_1(\psi_1(f_1))\pi(\zeta) = \pi((f_1(\zeta \circ \kappa)) \circ \kappa)$ and $\pi_2(\psi_2(f_2))\pi(\zeta) = \pi(f_2\zeta)$. It follows from Proposition 5.3 that we have

$$\pi(f_1 \otimes f_2)\pi(\eta) = \pi(f_2(\check{f}_1(\eta \circ \kappa)) \circ \kappa) = \pi((f_1 \otimes f_2)\eta).$$

For ξ_1, $\xi_2 \in C_c(\mathcal{T})$, we have $\|\xi_1\xi_2\|_\infty \le M\|\xi_1\|_\infty\|\xi_2\|_\infty$, where M is a constant that depends only on the supports of ξ_1 and ξ_2. For every $\varepsilon > 0$, there exist $f_{ij} \in C_c(G_i)$ $(i = 1, 2, j = 1, \cdots, n)$ such that $\|\xi - \sum f_{1j} \otimes f_{2j}\|_\infty < \varepsilon$ and the supports of ξ and $\sum f_{1j} \otimes f_{2,j}$ are contained in some compact set K of $G_1 \times G_2$. It follows from the proof of Proposition 5.1 that we have

$$\begin{aligned}
&\|\pi(\xi)\pi(\eta) - \pi(\xi\eta)\|_\infty \\
&\le \|\pi(\xi)\pi(\eta) - \sum \pi(f_{1j} \otimes f_{2,j})\pi(\eta)\|_\infty + \|\sum \pi((f_{1j} \otimes f_{2,j})\eta) - \pi(\xi\eta)\|_\infty \\
&\le \varepsilon M'\|\eta\|_\infty,
\end{aligned}$$

where M' depends only on K and the support of η. This implies that $\pi(\xi)\pi(\eta) = \pi(\xi\eta)$. □

Lemma 6.2. *For $\xi \in C_c(\mathcal{T})$, the following equation holds; $\pi(\xi)^* = \pi(\xi^*)$.*

Proof. Let $f_i \in C_c(G_i)$, $(i = 1, 2)$. In the proof of Theorem 4.2, we show that $T_i\pi(\psi_i(f_i))T_i = \tilde{\pi}_i(f_i)$. Since $\tilde{\pi}_i$ is a $*$-representation, we have $\pi(\psi(f_i))^* = \pi(\psi_i(f_i^*))$. It follows from Proposition 5.3 that we have $\pi(f_1 \otimes f_2)^* = \pi((f_1 \otimes f_2)^*)$. As in the proof of Lemma 6.1, we can show that $\pi(\xi)^* = \pi(\xi^*)$ for every $\xi \in C_c(\mathcal{T})$. □

Proposition 6.3. *The map $\pi : C_c(\mathcal{T}) \to A$ is injective.*

Proof. Let $\xi \in C_c(\mathcal{T})$. Suppose that $\pi(\xi) = 0$. It follows from Lemmas 6.1 and 6.2 that we have $\pi(\xi^*\xi) = \pi(\xi)^*\pi(\xi) = 0$. Take $\eta \in C_c(\mathcal{T})$ whose support is contained in $G^{(0)} \times G^{(0)}$. Then we have $(\pi(\xi^*\xi)\eta)(x, x) = (\xi^*\xi)(x, x)\eta(x, x)$ for every $x \in G^{(0)}$. Therefore we have $(\xi^*\xi)(x, x) = 0$ for $x \in G^{(0)}$. Since we have

$$(\xi^*\xi)(x, x) = \iint |\xi \circ \kappa(h_1, h_2^{-1})|^2 \, d\tilde{\lambda}_{1,r_G(h_2)}(h_1) d\tilde{\lambda}_{2,x}(h_2),$$

we have $\xi \circ \kappa = 0$. This implies that $\xi = 0$ □

Theorem 6.4. *The set $C_c(\mathcal{T})$ is a $*$-algebra with respect to the product $\xi\eta$ and the involution ξ^* and π becomes an injective $*$-homomorphism.*

Proof. The statement follows from Lemmas 6.1 and 6.2 and Proposition 6.3. □

Theorem 6.5. *The C^*-algebra A is isomorphic to $C_r^*(G)$.*

Proof. For $i = 1, 2$, define a Hilbert space $\tilde{H}_i$ by $\tilde{H}_i = \int^{\oplus} \tilde{H}_{i,x} \, d\mu(x)$ and define a faithful $*$-representation $\tilde{\pi}_{(i)} : C_r^*(G_i) \to \mathcal{B}(\tilde{H}_i)$ by $\tilde{\pi}_{(i)} = \int^{\oplus} \tilde{\pi}_{i,x} \, d\mu(x)$. Define a measure $\tilde{\lambda}$ on G by $\tilde{\lambda} = \int \tilde{\lambda}_x \, d\mu(x)$ and define a Hilbert space K by $K = L^2(G, \tilde{\lambda})$. We denote by $\pi_G : C_r^*(G) \to \mathcal{B}(K)$ a faithful representation such that $\pi_G(f)\eta = f\eta$ for $f, \eta \in C_c(G)$, where $f\eta$ is a convolution product in $C_c(G)$. Note that we have

$$\int_G f(g) \, d\tilde{\lambda}_x(g) = \int_{G_2} \int_{G_1} f(g_1 g_2) \, d\tilde{\lambda}_{1,r_G(g_2)}(g_1) d\tilde{\lambda}_{2,x}(g_2)$$

for $f \in C_c(G)$ and $x \in G^{(0)}$. Since μ is G_1- and G_2-invariant, we can show that μ is G-invariant using the conditions (A1) and (B1).

Define a homeomorphism $\omega : G \to \mathcal{T}$ by $\omega(g) = (p_1(g^{-1}), p_2(g^{-1})^{-1})$. The inverse of ω is given by $\omega^{-1}(g_1, g_2) = g_2 g_1^{-1}$. Define a map $\omega_* : C_c(G) \to C_c(\mathcal{T})$ by $\omega_*(f) = f \circ \omega^{-1}$. Then ω_* is a $*$-isomorphism. Define a unitary operator $\tilde{\omega} \in \mathcal{B}(K, H)$ by $\tilde{\omega}(\eta) = \eta \circ \omega^{-1}$. Since we have $(\pi(\xi)(\eta \circ \kappa_2)) \circ \kappa_2 = \xi\eta$ for $\xi, \eta \in C_c(\mathcal{T})$, we have $(T_2\tilde{\omega})\pi_G(f)(T_2\tilde{\omega})^{-1} = \pi(\omega_*(f))$ for $f \in C_c(G)$. Then the theorem follows from Corollary 5.6. □

7 Conditional expectations Define $P_1 \in \mathcal{B}(H, \tilde{H}_1)$ (resp. $P_2 \in \mathcal{B}(H, \tilde{H}_2)$) by $P_1(\xi)(g_1) = \xi(g_1, s_G(g_1))$ (resp. $P_2(\xi)(g_2) = \xi(s_G(g_2), g_2)$) for $\xi \in H$ and $g_1 \in G_1$ (resp. $g_2 \in G_2$). Note that we have $(P_1^*\eta_1)(g_1, g_2) = \eta_1(g_1)\chi_{G^{(0)}}(g_2)$ and $(P_2^*\eta_2)(g_1, g_2) = \chi_{G^{(0)}}(g_1)\eta_2(g_2)$. For $i = 1, 2$, recall that T_i is a unitary operator defined in the proof of Theorem 4.2 and that $\tilde{\pi}_{(i)}$ is the faithful $*$-representation of $C_r^*(G_i)$ defined in the proof of Theorem 6.5. Note that we have $T_iA_iT_i = \tilde{\pi}_i(C_r^*(G_i))$ by the proof of Theorem 4.2. Then we have the following lemma.

Lemma 7.1. *The image of $\tilde{\pi}_{(i)}$ is $P_iT_iAT_iP_i^*$ for $i = 1, 2$.*

Proof. For $i = 1, 2$, Define a linear map $\epsilon_i : C_c(\mathcal{T}) \to C_c(G_i)$ by $\epsilon_1(\xi)(g_1) = \xi(g_1, s_G(g_1))$ and by $\epsilon_2(\xi)(g_2) = \xi(s_G(g_2), g_2)$ respectively. Then we have

$$P_1T_1\pi(\xi)T_1P_1^* = \tilde{\pi}_{(1)}(\epsilon_1(\xi)\check{}) \quad \text{and} \quad P_2T_2\pi(\xi)T_2P_2^* = \tilde{\pi}_{(2)}(\epsilon_2(\xi))$$

for $\xi \in C_c(\mathcal{T})$. These imply that $P_iT_iAT_iP_i^* \subset \tilde{\pi}_{(i)}(C_r^*(G_i))$ for $i = 1, 2$. Since we have $\epsilon_1(\psi_1(f_1))\check{} = f_1$ for $f_1 \in C_c(G_1)$ and $\epsilon_2(\psi_2(\check{f}_2)) = f_2$ for $f_2 \in C_c(G_2)$, the reverse inclusions hold. □

Define a $*$-isomorphism $\iota_i : A_i \to C_r^*(G_i)$ by $\iota_i(a) = \tilde{\pi}_i^{-1}(T_i a T_i)$ for $a \in A_i$ and define a map $E_i : A \to A_i$ by $E_i(a) = \iota_i^{-1} \circ \tilde{\pi}_{(i)}^{-1}(P_i T_i a T_i P_i^*)$ for $a \in A$.

Theorem 7.2. *For $i = 1, 2$, E_i is a faithful conditional expectation.*

Proof. For f_i, $f_i' \in C_c(G_i)$ $(i = 1, 2)$, set $a_i = \pi_i(\psi_i(f_i))$ and $a_i' = \pi_i(\psi_i(f_i'))$. Then we have

$$P_1 T_1 a_2 a_1 a_1' T_1 P_1^* = P_1 T_1 a_2 a_1 T_1 P_1^* \tilde{\pi}_{(1)}(f_1'),$$
$$P_2 T_2 a_2' a_2 a_1 T_2 P_2^* = \tilde{\pi}_{(2)}(f_2') P_2 T_2 a_2 a_1 T_2 P_2^*.$$

Since we have $T_i a_i' T_i = \tilde{\pi}_i(f_i')$ by the proof of Theorem 4.2, we have $E_1(a_2 a_1 a_1') = E_1(a_2 a_1) a_1'$ and $E_2(a_2' a_2 a_1) = a_2' E_2(a_2 a_1)$. This implies that $E_1(a a_1) = E_1(a) a_1$ and $E_2(a_2 a) = a_2 E_2(a)$ for every $a \in A$ and $a_i \in A_i$. Since we have $E_i(a^*) = E_i(a)^*$ $(i = 1, 2)$, we have $E_1(a_1 a) = a_1 E_1(a)$ and $E_2(a a_2) = E_2(a) a_2$ for every $a \in A$ and $a_i \in A_i$. It is easy to show that $E_i(a_i) = a_i$ for $a_i \in A_i$ and that $E_i(a^* a) \geq 0$.

We show that E_i is faithful. Note that elements of $C_r^*(G)$ can be viewed as elements of $C_0(G)$ ([12], Proposition 4.2) and that the restriction map $\tilde{E} : C_r^*(G) \to C_0(G^{(0)})$ is a faithful conditional expectation (cf. [12], Proposition 4.8). It follows from the proof of Theorem 6.5 that we can define a $*$-isomorphism $\iota : A \to C_r^*(G)$ by $\iota(a) = \pi_G^{-1}((\tilde{\kappa}_2 \tilde{\omega})^{-1} a (\tilde{\kappa}_2 \tilde{\omega}))$. Then we have $\tilde{E}\iota = \iota_1 E_1 E_2 = \iota_2 E_2 E_1$. This implies that E_i is faithful. □

Since we have $E_1(A_2) = \iota_1^{-1}(C_0(G^{(0)}))$ and $E_2(A_1) = \iota_2^{-1}(C_0(G^{(0)}))$, we have the following Corollary.

Corollary 7.3. $A_1 \cap A_2 = \iota_1^{-1}(C_0(G^{(0)})) = \iota_2^{-1}(C_0(G^{(0)}))$.

8 An action of a semi-direct product group Let Γ_1 and Γ_2 be countable discrete groups and let $\sigma : \Gamma_2 \to \mathrm{Aut}(\Gamma_1)$ be a homomorphism. We denote by Γ the semidirect product group $\Gamma_1 \times_\sigma \Gamma_2$. Then Γ_1 and Γ_2 are subgroups of Γ and we have $\Gamma = \Gamma_1 \Gamma_2$ and $\Gamma_1 \cap \Gamma_2 = \{e\}$. Note that we have $\gamma_2 \gamma_1 = \sigma_{\gamma_2}(\gamma_1)\gamma_2$. Let X be a second countable locally compact Hausdorff space and let $\alpha : \Gamma \to \mathrm{Homeo}(X)$ be an action of Γ on X by homeomorphisms. We set $\alpha_\gamma(x) = \gamma \cdot x$ for $\gamma \in \Gamma$ and $x \in X$. We denote by G the r-discrete groupoid $\Gamma \times X$. The source (rep. range) map is defined by $s_G(\gamma, x) = x$ (resp. $r_G(\gamma, x) = \gamma \cdot x$) and the product and inverse are defined by $(\gamma', \gamma \cdot x)(\gamma, x) = (\gamma' \gamma, x)$ and $(\gamma, x)^{-1} = (\gamma^{-1}, \gamma \cdot x)$ respectively. Let $G_1 = \Gamma_1 \times X$ and $G_2 = \Gamma_2 \times X$ be clopen subgroupoids of G. Then (G_1, G_2) is a matched pair in the sense of Definition 2.1. For $g_1 = (\gamma_1, x) \in G_1$ and $g_2 = (\gamma_2, \gamma_1 \cdot x) \in G_2$, we have $g_2 \triangleright g_1 = (\sigma_{\gamma_2}(\gamma_1), \gamma_2 \cdot x)$ and $g_2 \triangleleft g_1 = (\gamma_2, x)$. We identify $((\gamma_1, x), (\gamma_2, x)) \in \mathcal{T}$ with (γ_1, γ_2, x) and identify $\mathcal{T}$ with $\Gamma_1 \times \Gamma_2 \times X$. The map $\mathcal{W}$ is given by

$$\mathcal{W}((\gamma_1, \gamma_2, \gamma_1' \cdot x), (\gamma_1', \gamma_2', x)) = ((\sigma_{\gamma_2}(\gamma_1'), \gamma_2' \gamma_2^{-1}, \gamma_2 \cdot x), (\gamma_1 \gamma_1', \gamma_2, x))$$

and the inverse is given by

$$\mathcal{W}^{-1}((\gamma_1, \gamma_2, \gamma_2' \cdot x), (\gamma_1', \gamma_2', x)) = ((\gamma_1' \gamma^{-1}, \gamma_2', \gamma \cdot x), (\gamma, \gamma_2 \gamma_2', x)),$$

where $\gamma = \sigma_{\gamma_2'}^{-1}(\gamma_1)$. Define $\rho_1 : \mathcal{T} \to G_1$ by $\rho_1(\gamma_1, \gamma_2, x) = (\sigma_{\gamma_2}(\gamma_1), \gamma_2 \cdot x)$ and define $\rho_2 : \mathcal{T} \to G_2$ by $\rho_2(\gamma_1, \gamma_2, x) = (\gamma_2, \gamma_1 \cdot x)$. Let μ be a positive regular Radon measure on $G^{(0)}$ whose support is X. We assume that μ is invariant under the action α. Then (ρ_1, ρ_2, μ) is an invariant system for (G_1, G_2). Moreover the induced actions $\triangleright$ and $\triangleleft$ are preserving.

The representations π_1, π_2 and π satisfy the following equations: for $\xi, \eta \in C_c(\mathcal{T})$. With respect to the $*$-algebraic structure for $C_c(\mathcal{T})$ introduced in Section 6, the product satisfies the following equations;

$$(\xi\eta)(\gamma_1, \gamma_2, x) = \sum_{\gamma_1' \in \Gamma_1} \sum_{\gamma_2' \in \Gamma_2} \xi(\gamma_1', \gamma_2 {\gamma_2'}^{-1}, \gamma_2' \cdot x) \eta(\gamma_1 \gamma^{-1}, \gamma_2', \gamma \cdot x),$$

where $\gamma = \sigma_{\gamma_2'}^{-1}(\gamma_1')$ and the involution satisfies the following equations;

$$\xi^*(\gamma_1, \gamma_2, x) = \overline{\xi(\sigma_{\gamma_2}(\gamma_1^{-1}), \gamma_2^{-1}, (\gamma_2 \gamma_1) \cdot x)}.$$

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Department of Mathematics and Information Sciences, Graduate School of Science, Osaka Prefecture University, Sakai City, Osaka 599-8531, Japan.
E-mail : ouchi@mi.s.osakafu-u.ac.jp