OPERATOR INEQUALITIES OF ANDO-HIAI TYPE AND THEIR APPLICATIONS

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Dedicated to the memory of Professor Masahiro Nakamura
with deep sorrow and the greatest respect

Abstract. In the discussion of the equivalence between Furuta inequality and Ando-Hiai inequality, the key is the inequality that if $A \geq B > 0$, then

$$A^{-r} \natural_{\frac{r}{p+r}} B^p \leq I \quad \text{for} \quad p, r \geq 1.$$

Here $\sharp_\alpha$ is the α-geometric mean in the sense of Kubo-Ando. In this note, we assume that $A^{-r} \natural_{\frac{r}{p+r}} B^p \leq I$ for some $p, r \geq 1$ instead of $A \geq B > 0$. Then we show that

$$A^{-r} \natural_{\frac{\delta+r}{p+r}} B^p \leq A^{-r} \natural_{\frac{\delta+r}{\mu+r}} B^\mu \quad \textit{for} \ \ 0 \leq \delta \leq \mu \leq p,$$

and for each $t \in [0, r]$

$$A^{-r} \natural_{\frac{\delta+r}{p+r}} B^p \leq A^{-t} \natural_{\frac{\delta+t}{p+t}} B^p \quad \textit{for} \ -t \leq \delta \leq p.$$

As an application, we discuss recent development of grand Furuta inequality due to Furuta himself.

1. Introduction. A real-valued continuous function on $[0, \infty)$ is called operator monotone if it is order-preserving, i.e.,

$$A \geq B \geq 0 \quad \Longrightarrow \quad f(A) \geq f(B).$$

The Löwner-Heinz inequality says that the function t^α is operator monotone for $\alpha \in [0, 1]$, cf. [18] and [12]. It induces the α-geometric operator mean defined for $\alpha \in [0, 1]$ as

$$A \sharp_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}$$

if $A > 0$, i.e., A is invertible, by the Kubo-Ando theory [17]. Incidentally, we use $\natural_s$ for $s \notin [0, 1]$ instead of $\sharp_s$ because it is not an operator mean.

Now, one of the most interesting operator inequalities related to the α-geometric mean is the Ando-Hiai inequality [1], say (AH). So we cite it first:

Ando-Hiai inequality. *For $A, B > 0$,*

$$A \sharp_\alpha B \leq 1 \ \Rightarrow \ A^r \sharp_\alpha B^r \leq 1 \ \textit{for} \ r \geq 1. \tag{1}$$

By the way, one of the motivation of the Ando-Hiai inequality might be the Furuta inequality, see [2], [7], [9], [10], [14], [15] and [19]:

Furuta inequality. *If $A \geq B > 0$, then for each $r \geq 0$*

$$(A^{\frac{r}{2}} B^p A^{\frac{r}{2}})^{\frac{1}{q}} \leq A^{\frac{p+r}{q}}$$

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holds for $p \geq 0$ *and* $q \geq 1$ *satisfying* $(1+r)q \geq p+r$.

Recently, we discussed in [7] the equivalence between the Ando-Hiai inequality and the Furuta inequality, whose key is that if $A \geq B > 0$, then

$$A^{-r} \natural_{\frac{r}{p+r}} B^p \leq I \quad \text{for} \quad p, r \geq 1.$$

It is a "usual order" version of the chaotic Furuta inequality [3]:

$\log A \geq \log B$ *if and only if*

$$A^{-r} \natural_{\frac{r}{p+r}} B^p \leq I \quad \textit{for} \quad p, r \geq 0.$$

To clarify the difference between the usual order and the chaotic order, i.e., $\log A \geq \log B$, we point out that the essense of the Furuta inequality is as follows:

(FI) *If* $A \geq B > 0$, *then*

$$A^{-r} \natural_{\frac{1+r}{p+r}} B^p \leq A \quad \textit{for} \quad p \geq 1 \textit{ and } r \geq 0. \tag{2}$$

2. Recent development of grand Furuta inequality. For reader's convenience, we cite the grand Furuta inequality (GFI). It was established by Furuta [11] and interpolates Ando-Hiai and Furuta inequalities.

(GFI) *If* $A \geq B > 0$ *and* $t \in [0, 1]$, *then*

$$[A^{\frac{r}{2}}(A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^sA^{\frac{r}{2}}]^{\frac{1-t+r}{(p-t)s+r}} \leq A^{1-t+r}$$

holds for $r \geq t$ *and* $p, s \geq 1$.

We note that (GFI) for $t=1,\ r=s$ (resp. $t=0,\ s=1$) is just (AH) (resp. (FI)).

For fixed $A > 0,\ B \geq 0,\ t \in [0,\ 1]$ and $p \geq 1$, we define an operator function $F(\lambda,\ \mu)$ by

$$F(\lambda,\ \mu) = A^{-\lambda} \natural_{\frac{1-t+\lambda}{(p-t)\mu+\lambda}} (A^{-\frac{t}{2}}B^pA^{-\frac{t}{2}})^{\mu} \tag{3}$$

for $\lambda \geq t-1$ and $\mu \geq \frac{1-t}{p-t}$.

As another simultaneous extension of (FI) and (AH), we presented the following inequality in [8] recently.

Theorem A. *If* $A \geq B \geq 0$ *and* $A > 0$, *then* $F(r,s) \leq F(0,1)$, *i.e.,*

$$A^{-r+t} \natural_{\frac{1-t+r}{(p-t)s+r}} (A^t \natural_s B^p) \leq A^t \natural_{\frac{1-t}{p-t}} B^p$$

holds for $r \geq t$ *and* $s \geq 1$.

If we take $t=0$ and $s=1$ in Theorem A, then we have the satellite form of (FI), due to Kamei [14], cf. Kamei-Nakamura [16]:

(SF) *If* $\log A \geq \log B$ *for* $A,\ B > 0$, *then*

$$A^{-r} \natural_{\frac{1+r}{p+r}} B^p \leq B \quad \textit{for} \quad p \geq 1 \textit{ and } r \geq 0, \tag{4}$$

which means that Theorem A interpolates (AH) with (SF) instead of (FI). Motivated by Theorem A, Furuta [13] showed the following inequality including both (GFI) and Theorem A very recently.

Theorem 1. (Furuta) *Let* $F(\lambda,\ \mu)$ *be as in above for fixed* $A > 0,\ B \geq 0,\ t \in [0,\ 1]$ *and* $p \geq 1$. *If* $A \geq B \geq 0$, *then*

$$F(r,\ w) \geq F(r,\ 1) \geq F(r,\ s) \geq F(r,\ s') \tag{i}$$

holds for $r \geq t,\ w \in [\frac{1-t}{p-t}, 1]$ *and* $s' \geq s \geq 1$, *and*

$$F(q,\ s) \geq F(t,\ s) \geq F(r,\ s) \geq F(r',\ s) \tag{ii}$$

holds for $q \in [t-1, t],\ s \geq 1$ *and* $r' \geq r \geq t$.

We would like to mention that Theorem 1 is surprising from the viewpoint of Tanahashi's work [20] on the best possibility of (GFI).

3. Operator inequalities of Ando-Hiai type. In this section, we propose operator inequalities of Ando-Hiai type. We will point out that Theorem 1 is reduced to our Ando-Hiai type theorem mentioned in the below.

For the sake of convenience, we cite a useful lemma which we will use frequently in the below.

Lemma 2. *For* $X,\ Y > 0$ *and* $a,\ b \in [0,1]$,
(i) *transposition:* $X \natural_a Y = Y \natural_{1-a} X$,
(ii) *multiplicativity:* $X \natural_{ab} Y = X \natural_a (X \natural_b Y)$.

Now we prepare the following lemma:

Lemma 3. *If* $A^{-r} \natural_{\frac{r}{p+r}} B^p \leq I$ *for some* $p,\ r \geq 0$, *then*

(i) $A^{-r} \natural_{\frac{\delta+r}{p+r}} B^p \leq B^\delta$ *for* $0 \leq \delta \leq p$,

(ii) $A^{-r} \natural_{\frac{\lambda+r}{p+r}} B^p \leq A^\lambda$ *for* $-r \leq \lambda \leq 0$.

Proof. We can prove them by the use of Lemma 2. For (i), we note that

$$A^{-r} \natural_{\frac{\delta+r}{p+r}} B^p = B^p \natural_{\frac{p-\delta}{p+r}} A^{-r} = B^p \natural_{\frac{p-\delta}{p}} (B^p \natural_{\frac{p}{p+r}} A^{-r}).$$

Since $B^p \natural_{\frac{p}{p+r}} A^{-r} = A^{-r} \natural_{\frac{r}{p+r}} B^p \leq I$ by the assumption, we have

$$A^{-r} \natural_{\frac{\delta+r}{p+r}} B^p \leq B^p \natural_{\frac{p-\delta}{p}} I = I \natural_{\frac{\delta}{p}} B^p = B^\delta.$$

Similarly, if $-r \leq \lambda \leq 0$, then we have

$$A^{-r} \natural_{\frac{\lambda+r}{p+r}} B^p = A^{-r} \natural_{\frac{\lambda+r}{r}} (A^{-r} \natural_{\frac{r}{p+r}} B^p) \leq A^{-r} \natural_{\frac{\lambda+r}{r}} I = I \natural_{\frac{-\lambda}{r}} A^{-r} = A^\lambda,$$

which shows (ii).

We here state our main theorem:

Theorem 4. *Suppose that* $A^{-r} \natural_{\frac{r}{p+r}} B^p \leq I$ *for some* $p,\ r \geq 0$. *Then* (i) *for each* $\delta \in [0,p]$

$$A^{-r} \natural_{\frac{\delta+r}{p+r}} B^p \leq A^{-r} \natural_{\frac{\delta+r}{\mu+r}} B^\mu \quad for\ \mu \in [\delta, p],$$

(ii) *for each* $t \in [0,r]$

$$A^{-r} \natural_{\frac{\delta+r}{p+r}} B^p \leq A^{-t} \natural_{\frac{\delta+t}{p+t}} B^p \quad for\ \delta \in [-t, p].$$

Proof. The former (i) follows from Lemma 2 (ii) and Lemma 3 (i) that

$$A^{-r} \natural_{\frac{\delta+r}{p+r}} B^p = A^{-r} \natural_{\frac{\delta+r}{\mu+r}} (A^{-r} \natural_{\frac{\mu+r}{p+r}} B^p) \leq A^{-r} \natural_{\frac{\delta+r}{\mu+r}} B^\mu.$$

The latter (ii) is obtained by Lemma 2 (ii) as follows:

$$\begin{aligned} A^{-r} \natural_{\frac{\delta+r}{p+r}} B^p &= B^p \natural_{\frac{p-\delta}{p+r}} A^{-r} = B^p \natural_{\frac{p-\delta}{p+t}} (B^p \natural_{\frac{p+t}{p+r}} A^{-r}) \\ &= B^p \natural_{\frac{p-\delta}{p+t}} (A^{-r} \natural_{\frac{-t+r}{p+r}} B^p) \leq B^p \natural_{\frac{p-\delta}{p+t}} A^{-t} = A^{-t} \natural_{\frac{\delta+t}{p+t}} B^p. \end{aligned}$$

4. Lemma related to a generalized Ando-Hiai inequality. In this section, we prepare two lemmas in order to apply Theorem 4 to Theorem 1. The first one is a generalized Ando-Hiai inequality itself, see [5] and [6].

Lemma 5. *If* $A \sharp_\alpha B \le I$ *for some* $A,\ B > 0$ *and* $\alpha \in [0,\ 1]$*, then*

(i) $$A^r \sharp_{\frac{\alpha r}{\alpha r+1-\alpha}} B \le I \quad \textit{for} \quad r \ge 1,$$

(ii) $$A \sharp_{\frac{\alpha}{\alpha+(1-\alpha)s}} B^s \le I \quad \textit{for} \quad s \ge 1,$$

(iii) $$A^r \sharp_{\frac{\alpha r}{\alpha r+(1-\alpha)s}} B^s \le I \quad \textit{for} \quad r, s \ge 1.$$

As prologue of a proof of Theorem 1, we put

(5) $$B_1 = (A^{-\frac{t}{2}} B^p A^{-\frac{t}{2}})^{\frac{1}{p-t}}$$

for fixed $A > 0,\ B \ge 0,\ t \in [0,\ 1]$ and $p \ge 1$. Then we remark that the operator function $F(\lambda,\ \mu)$ is expressed as

$$F(\lambda,\ \mu) = A^{-\lambda} \sharp_{\frac{1-t+\lambda}{(p-t)\mu+\lambda}} B_1^{(p-t)\mu},$$

and we have the following lemma connecting Theorem 1 with Theorem 4.

Lemma 6. *Notation as in above. If* $A \ge B \ge 0$*, then*

(i) $$A^{-r} \sharp_{\frac{r}{p-t+r}} B_1^{p-t} \le I \quad \textit{for} \quad r \ge t,$$

(ii) $$B_1^{(p-t)s} \sharp_{\frac{(p-t)s+q}{(p-t)s+t}} A^{-t} \le A^{-q} \quad \textit{for} \quad q \in [t-1, t].$$

Proof. Since $A \ge B \ge 0$ and $t \in [0, 1]$, Löwner-Heinz inequality ensures that $A^t \ge B^t$, so that

(†) $$A^{-t} \sharp_{\frac{t}{p}} B_1^{p-t} = A^{-\frac{t}{2}} B^t A^{-\frac{t}{2}} \le I.$$

Applying (i) in Lemma 5 for $r_1 = \frac{r}{t}$ and $\alpha = \frac{t}{p}$, we have

$$A^{-r} \sharp_{\frac{r}{p-t+r}} B_1^{p-t} = (A^{-t})^{r_1} \sharp_{\frac{\alpha r_1}{1-\alpha+\alpha r_1}} B_1^{p-t} \le I.$$

On the other hand, since it has been proved that $(A^t \natural_s B^p)^{\frac{1}{(p-t)s+t}} \le B \le A$ in [2; Theorem 2], we have

$$A^{-t} \sharp_{\frac{1}{(p-t)s+t}} B_1^{(p-t)s} = A^{-\frac{t}{2}} (A^t \natural_s B^p)^{\frac{1}{(p-t)s+t}} A^{-\frac{t}{2}} \le A^{1-t}.$$

Therefore it follows from Lemma 2 that

$$\begin{aligned} B_1^{(p-t)s} \sharp_{\frac{(p-t)s+q}{(p-t)s+t}} A^{-t} &= A^{-t} \sharp_{\frac{t-q}{(p-t)s+t}} B_1^{(p-t)s} \\ &= A^{-t} \sharp_{t-q} (A^{-t} \sharp_{\frac{1}{(p-t)s+t}} B_1^{(p-t)s}) \\ &\le A^{-t} \sharp_{t-q} A^{1-t} = A^{-q}, \end{aligned}$$

which proves (ii).

Now we give a proof of Theorem 1, in which Theorem 4 is the main tool and Lemma 6 is the starting point of a proof.

Proof of Theorem 1. (i) By Lemma 6 (i), we know that $A^{-r} \natural_{\frac{r}{p-t+r}} B_1^{p-t} \le I$. So we take $\delta = 1-t,\ p_1 = p-t$ and $\mu = (p-t)w$ in Theorem 4. Since $0 \le \delta \le \mu \le p_1$ by the assumption, it implies that

$$F(r,w) = A^{-r} \natural_{\frac{1-t+r}{(p-t)w+r}} B_1^{(p-t)w} \ge A^{-r} \natural_{\frac{1-t+r}{p-t+r}} B_1^{p-t} = F(r,1).$$

Successively we take $\delta = 1-t,\ p_1 = (p-t)s$ and $\mu = p-t$. Then we have $0 \le \delta \le \mu \le p_1$ by $p,\ s \ge 1$ and so the second inequality of (i) is obtained by Theorem 4 (i).

Finally we take $\delta = 1-t,\ \mu = (p-t)s$ and $p_1 = (p-t)s'$. Then $0 \le \delta \le \mu \le p_1$ by $s' \ge s \ge 1$ and so the final inequality of (i) holds.

(ii) The first inequality follows from Lemma 2 and Lemma 6 (ii). As a matter of fact, we have

$$\begin{aligned} F(t,s) &= A^{-t} \natural_{\frac{1}{(p-t)s+t}} B_1^{(p-t)s} \\ &= B_1^{(p-t)s} \natural_{\frac{(p-t)s-1+t}{(p-t)s+t}} A^{-t} \\ &= B_1^{(p-t)s} \natural_{\frac{(p-t)s-1+t}{(p-t)s+q}} (B_1^{(p-t)s} \natural_{\frac{(p-t)s+q}{(p-t)s+t}} A^{-t}) \\ &\le B_1^{(p-t)s} \natural_{\frac{(p-t)s-1+t}{(p-t)s+q}} A^{-q} \\ &= A^{-q} \natural_{\frac{1-t+q}{(p-t)s+q}} B_1^{(p-t)s} = F(q,s). \end{aligned}$$

Next we prove the second inequality by applying Theorem 4 (ii). For this, we have to obtain the inequality

$$A^{-r} \natural_{\frac{r}{(p-t)s+r}} B_1^{(p-t)s} \le I.$$

Fortunately it is implied by applying Lemma 5 (ii) to Lemma 6 (i) and $\frac{\alpha}{(1-\alpha)s+\alpha} = \frac{r}{(p-t)s+r}$ for $\alpha = \frac{r}{p-t+r}$. We here put $p_1 = (p-t)s$ for convenience. Then it is rephrased as

$$A^{-r} \natural_{\frac{r}{p_1+r}} B_1^{p_1} \le I.$$

Hence it follows from Theorem 4 (ii) that

$$A^{-r} \natural_{\frac{\delta+r}{p_1+r}} B_1^{p_1} \le A^{-t} \natural_{\frac{\delta+t}{p_1+t}} B_1^{p_1}$$

and putting $\delta = 1-t$,

$$F(r,s) = A^{-r} \natural_{\frac{1-t+r}{p_1+r}} B_1^{p_1} \le A^{-t} \natural_{\frac{1}{p_1+t}} B_1^{p_1} = F(t,s).$$

Finally we apply Lemma 5 (iii) for $\alpha = \frac{r}{p-t+r}$, $r_1 = \frac{r'}{r} \ge 1$ and $s \ge 1$ to Lemma 6 (i). Then

$$A^{-r'} \natural_{\frac{r'}{(p-t)s+r'}} B_1^{(p-t)s} = (A^{-r})^{r_1} \natural_{\frac{\alpha r_1}{(1-\alpha)s+\alpha r_1}} B_1^{(p-t)s} \le I.$$

Therefore it follows from Theorem 4 (ii) for $\delta = 0$ that

$$F(r',s) = A^{-r'} \natural_{\frac{r'}{(p-t)s+r'}} B_1^{(p-t)s} \le A^{-r} \natural_{\frac{r}{(p-t)s+r}} B_1^{(p-t)s} = F(r,s),$$

which completes the proof.

Remark 7. From our viewpoint, we review Theorem A by proving it. As a matter of fact, Lemma 6 (i) is extended by Lemma 5 (iii) as follows:

Notation as in Lemma 6. If $A \ge B \ge 0$, then

$$A^{-r} \natural_{\frac{r}{(p-t)s+r}} B_1^{(p-t)s} \le I \quad \text{for} \quad r \ge t \text{ and } s \ge 1.$$

Furthermore Theorem 4 (i) implies that

$$A^{-r} \natural_{\frac{\delta+r}{p_1+r}} B_1^{p_1} \le A^{-r} \natural_{\frac{\delta+r}{\mu+r}} B_1^{\mu}$$

for $0 \le \delta \le \mu \le p_1 = (p-t)s$.

Therefore we have Theorem A by taking $\delta = 1-t$ in above.

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