A NOTE ON COOK'S INEQUALITY FOR SIMPLE CLOSED CURVES

K.T. Hallenbeck

Received May 18, 2008

Abstract. The question of Howard Cook: "Do there exist, in the plane, two simple closed curves Y and X, such that X is in the bounded complementary domain of Y, and the span of X is greater than the span of Y?" is answered, in the negative, for several types of simple closed curve pairs.

1 Several definitions. In his 1964 paper "Disjoint mappings and the span of spaces" [4], Andrew Lelek introduced the concept of span. Let X be a connected nonempty metric space. The **span** $\sigma(X)$ of X is the least upper bound of the set of nonnegative numbers r that satisfy the following condition: there exists a connected space Y and a pair of continuous functions $f, g : Y \to X$ such that $f(Y) = g(Y)$ and dist $[f(y), g(y)] \geq r$ for every $y \in Y$. In the definition of the **semispan** $\sigma_0(X)$ of X, the equality $f(Y) = g(Y)$ is relaxed to the inclusion $f(Y) \supset g(Y)$. As usual, $\epsilon(X)$ denotes the infimum of the set of meshes of the chains that cover X, for any continuum X.

Let X be a simple closed curve in the Cartesian plane. Let L_α denote the line passing through the origin, such that the angle between the positive x-axis and L_α, measured counterclockwise, is α, $\alpha \in [0, \pi)$. The **directional diameter** $\mathrm{d}_\alpha(X)$ of X in the direction α is the length of the longest line segment (segments) with endpoints on X that is parallel to L_α[6]. The **breadth** $\mathrm{d}_{\inf}(X)$ of X is defined as $\inf[\mathrm{d}_\alpha(X) : \alpha \in [0, \pi)]$, [7]. For X that is a boundary of a convex region, it is known that $\sigma(X) = \mathrm{d}_{\inf}(X)$, [6].

The definition of the essential span $\sigma_{\mathrm{e}}(X)$ of X is obtained by restricting the functions in the definition of span to degree one maps exclusively, [1]. With that modification of the definition of the concept of span, the question of Cook has been answered in the negative, [1].

2 Theorems. Without changing the definition of span or extending the Cook's problem to topological objects other than simple closed curves, the following facts can be ascertained.

Theorem 1. *If R is a closed annulus in the plane, whose boundary consists of simple closed curves X and Y, where X is the boundary of the bounded component of the complement of R, then $\sigma(X) \leq \sigma(Y)$, provided R contains a curve that is a boundary of a convex region.*

Proof. Let X and Y be simple closed curves in the plane such that X is contained in the bounded component of the complement of Y, and let R be a closed annulus whose boundary consists of the union of X and Y. Let D be a convex region contained in the bounded component of the complement of Y and such that its boundary K is contained in R.

2000 *Mathematics Subject Classification.* 54F20.
Key words and phrases. span, breadth, simple closed curve, starlike curve.

By virtue of Lemma 1 in [6], $\sigma_o(X) \leq \mathrm{d_{inf}}(X)$. Since $\sigma(X) \leq \sigma_o(X)$ and $\mathrm{d_{inf}}(X) \leq \mathrm{d_{inf}}(K)$, it follows that

$$\sigma(X) \leq \mathrm{d_{inf}}(K). \tag{1}$$

Furthermore, since K is a boundary of a convex region, Thorem 3 in [6] implies that

$$\mathrm{d_{inf}}(K) = \sigma(K). \tag{2}$$

□

On the other hand, it follows from Themrem 3 in [7] that

$$\sigma(K) \leq \sigma(Y). \tag{3}$$

Combining (1),(2) and (3), we obtain the desired inequality.

Remark. In particular, Theorem 1 holds when $X = K$ or $Y = K$, though in general neither needs to be the case.

Our next theorem utilizes the concept of a span mate, introduced in [3]. We say that a starlike polygonal line is standard if it satisfies the conditions listed in Theorem 2.2 in [3].

Theorem 2. *If R is a closed annulus in the plane, whose boundary consisits of simple closed curves X and Y, where X is the boundary of the bounded component of the complement of R, then $\sigma(X) \leq \sigma(Y)$, provided R contains a standard starlike polygonal line.*

Proof. Let X and Y be simple closed curves in the plane such that X is contained in the bounded component of the complement of Y, and let R be a closed annulus whose boundary consists of the union of X and Y. Suppose S is a standard starlike polygonal line contained in R.

It is known that $\sigma(X) \leq \epsilon(X)$ (see [5] or [2]). Furthermore, by virtue of Theorem 2.2 in [3], $\epsilon(S) = \sigma(S)$. Since $\epsilon(X) \leq \epsilon(S)$, we have

$$\sigma(X) \leq \sigma(S). \tag{4}$$

The mappings used in the proof of Theorem 2.2 in [3] are of degree 1. Therefore, this proof implies that $\sigma(S) = \sigma_{\mathrm{e}}(S)$. In addition, it follows from Theorem 3.2 in [1] that $\sigma_{\mathrm{e}}(S) \leq \sigma_{\mathrm{e}}(Y)$. Since $\sigma_{\mathrm{e}}(Y) \leq \sigma(Y)$, we have

$$\sigma(S) \leq \sigma(Y). \tag{5}$$

Hence, by (4) and (5), $\sigma(X) \leq \sigma(Y)$.

□

Theorem 3. *If R is a closed annulus in the plane, whose boundary consists of simple closed curves X and Y, where X is the boundary of the bounded component of the complement of R, then $\sigma(X) \leq \sigma(Y)$, provided one of the following conditions holds*

a) $\sigma(X) = \sigma_{\mathrm{e}}(X)$

b) $\sigma(Y) = \mathrm{d_{inf}}(Y)$

c) $\sigma(Y) = \epsilon(Y)$

d) R contains a simple closed curve Z such that $\sigma(Z) = \sigma_{\mathrm{e}}(Z) = \epsilon(Z)$.

Proof. Let X and Y be simple closed curves in the plane such that X is contained in the bounded component of the complement of Y, and let R be a closed annulus whose boundary consists of the union of X and Y. If, in addition, $\sigma(X) = \sigma_{\mathrm{e}}(X)$ then $\sigma(X) \leq \sigma(Y)$ because $\sigma_{\mathrm{e}}(X) \leq \sigma_{\mathrm{e}}(Y)$ [1] and $\sigma_{\mathrm{e}}(Y) \leq \sigma(Y)$. Condition b) implies that $\sigma(X) \leq \sigma(Y)$ because $\sigma(X) \leq \mathrm{d}_{\mathrm{inf}}(X)$[6] and $\mathrm{d}_{\mathrm{inf}}(X) \leq \mathrm{d}_{\mathrm{inf}}(Y)$. Similarly, condition c) implies $\sigma(X) \leq \sigma(Y)$ because $\sigma(X) \leq \epsilon(X)$ [5], [2] and $\epsilon(X) \leq \epsilon(Y)$. Finally, if R contains a simple closed curve Z such that $\sigma(Z) = \sigma_{\mathrm{e}}(Z) = \epsilon(Z)$ then $\sigma(X) \leq \sigma(Z)$, by the application of c), and $\sigma(Z) \leq \sigma(Y)$, by the application of a). Hence, the assertion. □

References

[1] M. Barge, R. Swanson, The essential span of a simple closed curve, Continua (eds.Cook et al.), Lecture Notes in Pure and Applied Mathematics 170 (1995), Dekker, New York, 1983–1992.

[2] K.T. Hallenbeck, Estimates of span of a simple closed curve involving mesh, Houston J.Math. 26 (2000), 741–745.

[3] K.T. Hallenbeck, Span mates and mesh, Glasnik Mathematicki, vol. 37 (57) (2002), 175–186.

[4] A. Lelek, Disjoint mappings and the span of spaces, Fund. Math. 55 (1964), 199–214.

[5] A. Lelek, On the surjective span of connected metric spaces, Colloq. Math. 37 (1977), 35–45.

[6] K. Tkaczyńska (Hallenbeck), The span and semispan of some simple closed curves, Proc. Amer. Math. Soc. 111 (1991), 247–253.

[7] K. Tkaczyńska, On the span of simple closed curves, Houston J. Math., vol.20, no. 3 (1994), 507–528.

Department of mathematics Widener University Chester, Pa 19013
E-mail: hall@maths.widener.edu