INFORMATION LOSS OF EXTRACTED SERIES IN AR(1) MODEL

Etsuo Kumagai

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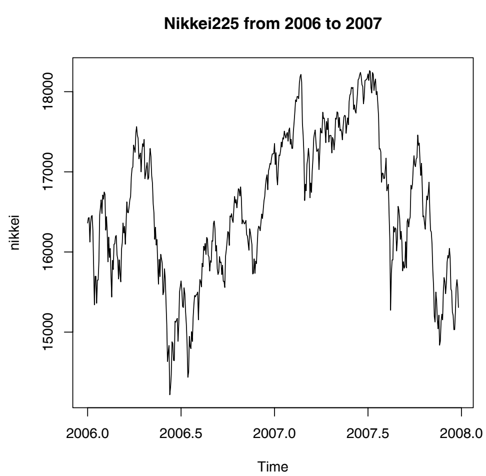
Abstract. Extracting data from the original series in AR(1) model makes the information loss and could not recover the information on the original series from one on the extracted data even if we extracted all data of the original series.

1 Introduction In time series, we are used to treat a daily data as a daily one, a weekly data as a weekly one, or a monthly data as a monthly one in stock prices, respectively, because we consider a daily data in order to examine the time series in detail and a monthly data in order to examine them in a long term. Besides, in a continuous model for the financial data, the discrete data is considered as the interval of data, Δ, which goes to zero as the total number n of data goes to infinity in the regularity condition (Bandi and Phillips[1], Kanaya[3], and Merton[4]). For a daily, weekly, and monthly data, the parameters are supposed to be set as $\Delta = 1/245, 1/52, 1/12$, respectively, based on the assumption that those three data have a homogeneous property which depends on the time interval only.

Indeed, the assumption may be reasonable and convenience for theoretical and computational aspects, but practically we have questions as follows: what is the differences between a daily data and a monthly data?, are they similar as time series?, and are their information simply proportional to the numbers of data?

We suppose that a weekly data and a monthly data in stock prices are regarded as the extracted data from the daily data. For example, we treat Nikkei225 daily stock prices, which is the most representative stock average in Japan, from January in 2006 to December in 2007 as follows:



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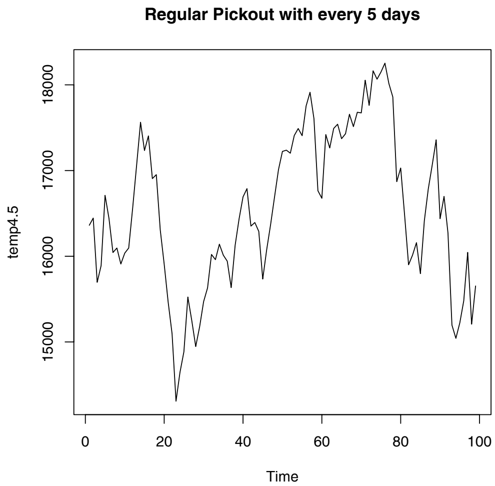
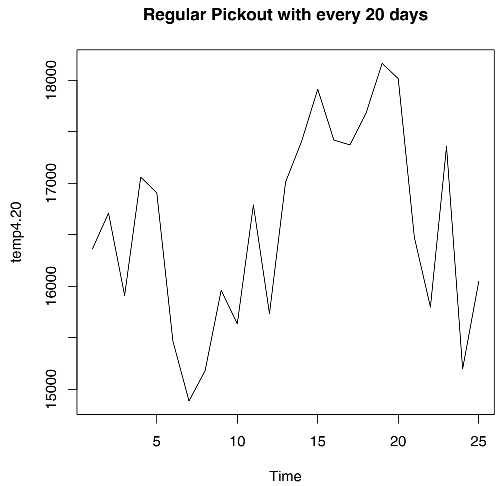
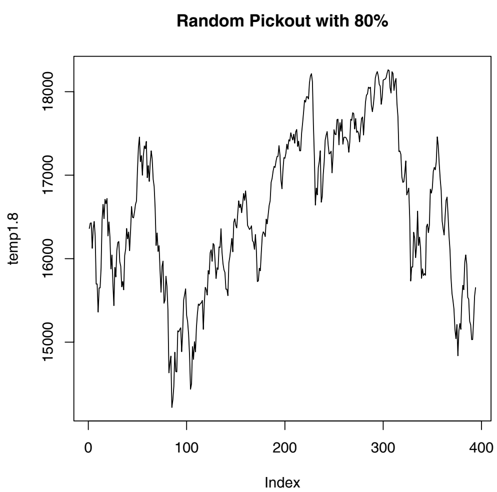
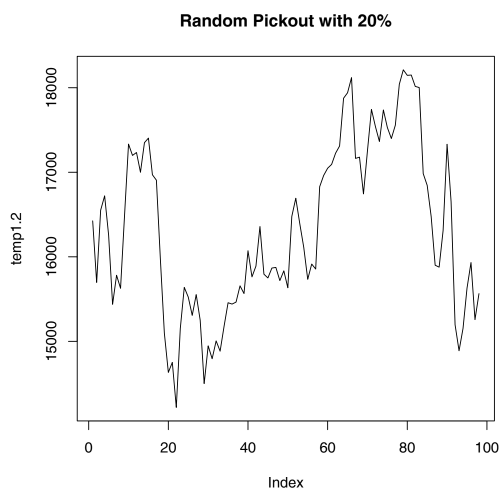
In this data, we simulate two cases, that is, the regular extracted case :





and the random extracted case :





From the above graphs, we could recognize that both cases indicate the loss of information compared to the original data and the random case is much better than the regular one. Here we consider the regular case only.

In this article, we consider the information loss between the original time series and the restricted ones in a simple AR(1) model (Brockwell and Davis[2] and Tsay[5]). At first, we considered that these information should be equivalent if $s = 1$ for $n = ms$, but the result we obtained was not true. That implies that extracting data from the original series makes the information loss and could not recover the information on the original series from one on the extracted data even if we extracted all data of the original series.

2 Information in AR(1) model We consider the following $AR(1)$ process $X = (X_t)_{t=0,1,\dots,n}$:

$$X_t = \phi X_{t-1} + Z_t, \quad Z_t \overset{i.i.d.}{\sim} N(0,\sigma^2), \quad X_0 = x_0, \quad |\phi| < 1.$$

Let $\boldsymbol{\theta} = (\phi, \sigma^2)$ be the parameter in this model. Since the conditional density given x_{t-1} is

$$f(x_t|x_{t-1}\,;\boldsymbol{\theta}) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x_t - \phi x_{t-1})^2}{2\sigma^2}\right\} \quad (t = 1, 2, \dots, n),$$

$$f(x_t|x_{t-1}, \dots, x_1, x_0\,;\boldsymbol{\theta}) = f(x_t|x_{t-1}\,;\boldsymbol{\theta}) \quad (t = 1, 2, \dots, n)$$

and the initial value $X_0 = x_0$, the joint density is represented by

$$f(x_n, x_{n-1}, \dots, x_1, x_0\,;\boldsymbol{\theta}) = \prod_{t=1}^{n} f(x_t|x_{t-1}\,;\boldsymbol{\theta}).$$

Then the log-likelihood function $\ell_n(\boldsymbol{\theta})$ is

$$\ell_n(\boldsymbol{\theta}) = -\frac{n}{2}\log(2\pi\sigma^2) - \sum_{t=1}^{n} \frac{(x_t - \phi x_{t-1})^2}{2\sigma^2}$$

and the derivatives are

$$\begin{aligned} \frac{\partial \ell_n(\boldsymbol{\theta})}{\partial \phi} &= \frac{1}{\sigma^2}\sum_{t=1}^{n} x_{t-1}(x_t - \phi x_{t-1}), \\ \frac{\partial \ell_n(\boldsymbol{\theta})}{\partial \sigma^2} &= -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{t=1}^{n}(x_t - \phi x_{t-1})^2, \end{aligned}$$

so that we have the maximum likelihood estimator (MLE) $\widehat{\boldsymbol{\theta}} = (\widehat{\phi}, \widehat{\sigma}^2)$ as follows:

$$\begin{aligned} \widehat{\phi} &= \frac{\sum_{t=1}^{n} x_{t-1}x_t}{\sum_{t=1}^{n} x_{t-1}^2}, \\ \widehat{\sigma}^2 &= \frac{1}{n}\sum_{t=1}^{n}(x_t - \widehat{\phi}\, x_{t-1})^2 = \frac{\sum_{t=1}^{n} x_t^2 \sum_{t=1}^{n} x_{t-1}^2 - (\sum_{t=1}^{n} x_t x_{t-1})^2}{n\sum_{t=1}^{n} x_{t-1}^2}. \end{aligned}$$

Also the second derivatives are

$$\begin{aligned} \frac{\partial^2 \ell_n(\boldsymbol{\theta})}{\partial \phi^2} &= -\frac{1}{\sigma^2}\sum_{t=1}^{n} x_{t-1}^2, \\ \frac{\partial^2 \ell_n(\boldsymbol{\theta})}{\partial \phi \partial \sigma^2} &= -\frac{1}{\sigma^4}\sum_{t=1}^{n} x_{t-1}(x_t - \phi x_{t-1}), \\ \frac{\partial^2 \ell_n(\boldsymbol{\theta})}{\partial \sigma^4} &= \frac{n}{2\sigma^4} - \frac{1}{\sigma^6}\sum_{t=1}^{n}(x_t - \phi x_{t-1})^2. \end{aligned}$$

THEOREM 1 For n time series which is distributed with AR(1) model, the exact Fisher information matrix $\boldsymbol{I}_n(\boldsymbol{\theta})$ is

$$\boldsymbol{I}_n(\boldsymbol{\theta}) = -E\begin{pmatrix} \frac{\partial^2 \ell_n(\boldsymbol{\theta})}{\partial \phi^2} & \frac{\partial^2 \ell_n(\boldsymbol{\theta})}{\partial \phi \partial \sigma^2} \\ \frac{\partial^2 \ell_n(\boldsymbol{\theta})}{\partial \phi \partial \sigma^2} & \frac{\partial^2 \ell_n(\boldsymbol{\theta})}{\partial \sigma^4} \end{pmatrix}$$
$$= \begin{pmatrix} \frac{n}{1-\phi^2} + \frac{1}{\sigma^2}\left(x_0^2 - \frac{\sigma^2}{1-\phi^2}\right)\frac{1-\phi^{2n}}{1-\phi^2} & 0 \\ 0 & \frac{n}{2\sigma^4} \end{pmatrix}$$

and the asymptotic formula of the information divided by n is

$$\lim_{n\to\infty} \frac{1}{n}\boldsymbol{I}_n(\boldsymbol{\theta}) = \begin{pmatrix} (1-\phi^2)^{-1} & 0 \\ 0 & (2\sigma^4)^{-1} \end{pmatrix}.$$

Proof: Since $X_t = \phi X_{t-1} + Z_t = \phi^t x_0 + \sum_{j=1}^{t} \phi^{t-j} Z_j$ $(t = 1, \dots, n)$, the expectation and variance are

$$E(X_t) = \phi^t x_0, \quad V(X_t) = \sigma^2 \sum_{j=1}^{t} \phi^{2(t-j)} = \sigma^2 \frac{1-\phi^{2t}}{1-\phi^2}$$

and the expectation of squared X_t is

$$E(X_t^2) = \sigma^2 \frac{1-\phi^{2t}}{1-\phi^2} + \phi^{2t} x_0^2.$$

And the covariance is

$$Cov(X_t, X_{t-h}) = \sigma^2 \sum_{j=1}^{t-h} \phi^{t-j}\phi^{t-h-j} = \sigma^2 \phi^h \frac{1-\phi^{2(t-h)}}{1-\phi^2}, \quad (h = 1, 2, \dots, t-1)$$

and

$$E(X_t X_{t-h}) = \sigma^2 \phi^h \frac{1-\phi^{2(t-h)}}{1-\phi^2} + \phi^{2t-h} x_0^2.$$

From these formulae, the expectations of the first derivatives of log-likelihood function are

$$E\left(\frac{\partial \ell_n(\boldsymbol{\theta})}{\partial \phi}\right) = \frac{1}{\sigma^2} E\left(\sum_{t=1}^{n} X_{t-1}(X_t - \phi X_{t-1})\right)$$
$$= \sum_{t=1}^{n} \phi \frac{1-\phi^{2(t-1)}}{1-\phi^2} + \frac{\sum_{t=1}^{n} \phi^{2t-1} x_0^2}{\sigma^2}$$
$$-\phi \sum_{t=1}^{n} \frac{1-\phi^{2(t-1)}}{1-\phi^2} - \phi \frac{\sum_{t=1}^{n} \phi^{2(t-1)} x_0^2}{\sigma^2} = 0,$$
$$E\left(\frac{\partial \ell_n(\boldsymbol{\theta})}{\partial \sigma^2}\right) = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4} E\left(\sum_{t=1}^{n} (X_t - \phi X_{t-1})^2\right)$$
$$= \frac{1}{2\sigma^2}\left(-n + \sum_{t=1}^{n} \frac{1-\phi^2}{1-\phi^2} - \sum_{t=1}^{n} \frac{\phi^{2t}}{1-\phi^2} + \phi^2 \sum_{t=1}^{n} \frac{\phi^{2(t-1)}}{1-\phi^2}\right) = 0.$$

With respect to the second derivatives, we have

$$E\left(-\frac{\partial^2 \ell_n(\boldsymbol{\theta})}{\partial \phi^2}\right) = \frac{1}{\sigma^2} E\left(\sum_{t=1}^{n} X_{t-1}^2\right) = \frac{n}{1-\phi^2} + \frac{1}{\sigma^2}\left(x_0^2 - \frac{\sigma^2}{1-\phi^2}\right)\frac{1-\phi^{2n}}{1-\phi^2},$$
$$E\left(-\frac{\partial^2 \ell_n(\boldsymbol{\theta})}{\partial \phi \partial \sigma^2}\right) = \frac{1}{\sigma^4} E\left(\sum_{t=1}^{n} X_{t-1}(X_t - \phi X_{t-1})\right) = 0,$$
$$E\left(-\frac{\partial^2 \ell_n(\boldsymbol{\theta})}{\partial \sigma^4}\right) = -\frac{n}{2\sigma^4} + \frac{1}{\sigma^6} E\left(\sum_{t=1}^{n} (X_t - \phi X_{t-1})^2\right) = \frac{n}{2\sigma^4},$$

so that we have the required result. □

3 Information in extracted series from AR(1) model We consider the extracted data $\{x_{t_1}, x_{t_2}, \ldots, x_{t_m}\}$ $(m < n)$ from the original data $\{x_1, \ldots, x_n\}$ with AR(1) model and calculate the exact information matrix with respect to the extracted data. In fact, there exists two ways to make the extracted data, that is, at random or preassigned, but in this article we consider the preassigned extracted data only.

Since $X_t = \phi X_{t-1} + Z_t = \phi^t x_0 + \sum_{j=1}^{t} \phi^{t-j} Z_j$ $(t = 1, \ldots, n)$, for $t_k > t_{k-1}$, we have

$$X_{t_k} = \phi^{t_k} x_0 + \sum_{j=1}^{t_k} \phi^{t_k - j} Z_j, \quad X_{t_{k-1}} = \phi^{t_{k-1}} x_0 + \sum_{j=1}^{t_{k-1}} \phi^{t_{k-1} - j} Z_j,$$

and

$$X_{t_k} = \phi^{t_k - t_{k-1}} X_{t_{k-1}} + \sum_{j=t_{k-1}+1}^{t_k} \phi^{t_k - j} Z_j.$$

Let $s_k = t_k - t_{k-1}$ for a convenient sake. Then, since the conditional expectation and variance are

$$E(X_{t_k} \mid X_{t_{k-1}}) = \phi^{s_k} X_{t_{k-1}}, \qquad V(X_{t_k} \mid X_{t_{k-1}}) = \sigma^2 \frac{1-\phi^{2s_k}}{1-\phi^2},$$

the conditional density is obtained by

$$f(x_{t_k} | x_{t_{k-1}}; \boldsymbol{\theta}) = \frac{\sqrt{1-\phi^2}}{\sqrt{2\pi\sigma^2 (1-\phi^{2s_k})}} \exp\left\{-\frac{(x_{t_k} - \phi^{s_k} x_{t_{k-1}})^2 (1-\phi^2)}{2\sigma^2 (1-\phi^{2s_k})}\right\},$$

the joint density is $f(x_{t_m}, x_{t_{m-1}}, \ldots, x_{t_1}, x_0; \boldsymbol{\theta}) = \prod_{k=1}^{m} f(x_{t_k} | x_{t_{k-1}}; \boldsymbol{\theta})$ and the log-likelihood function $\ell_{m|n}(\boldsymbol{\theta})$ is

$$\ell_{m|n}(\boldsymbol{\theta}) = -\frac{1}{2} \sum_{k=1}^{m} \log\left(2\pi\sigma^2 \frac{1-\phi^{2s_k}}{1-\phi^2}\right) - \sum_{k=1}^{m} \frac{(x_{t_k} - \phi^{s_k} x_{t_{k-1}})^2 (1-\phi^2)}{2\sigma^2 (1-\phi^{2s_k})}. \tag{1}$$

THEOREM 2 For the extracted time series, the exact Fisher information matrix is

$$\boldsymbol{I}_{m|n}(\boldsymbol{\theta}) = \begin{pmatrix} I_{\phi,\phi} & I_{\phi,\sigma^2} \\ I_{\sigma^2,\phi} & I_{\sigma^2,\sigma^2} \end{pmatrix},$$

where

$$\begin{aligned} I_{\phi,\phi} &= \frac{2m\phi^2}{(1-\phi^2)^2} + \sum_{k=1}^{m} \frac{s_k^2 \phi^{2s_k-2}(1-\phi^2)}{\sigma^2(1-\phi^{2s_k})} \left(\sigma^2 \frac{1-\phi^{2t_{k-1}}}{1-\phi^2} + \phi^{2t_{k-1}} x_0^2 \right) \\ &\quad + \sum_{k=1}^{m} \left(\frac{(3\phi^{2s_k}-1)s_k^2\phi^{2s_k-2}}{(1-\phi^{2s_k})^2} - \frac{4s_k\phi^{2s_k}}{(1-\phi^{2s_k})(1-\phi^2)} \right), \\ I_{\sigma^2,\phi} &= \frac{m\phi}{\sigma^2(1-\phi^2)} - \sum_{k=1}^{m} \frac{s_k\phi^{2s_k-1}}{\sigma^2(1-\phi^{2s_k})}, \\ I_{\sigma^2,\sigma^2} &= \frac{m}{2\sigma^4}. \end{aligned}$$

Proof: For the equation (1), since

$$\frac{\partial}{\partial\phi} \log\left(\frac{1-\phi^{2s_k}}{1-\phi^2}\right) = \frac{-2s_k\phi^{2s_k-1}}{1-\phi^{2s_k}} - \frac{-2\phi}{1-\phi^2},$$

the derivative of the first term in the right hand of the equation (1) is

$$-\frac{1}{2}\sum_{k=1}^{m} \frac{\partial}{\partial\phi} \log\left(\frac{1-\phi^{2s_k}}{1-\phi^2}\right) = \sum_{k=1}^{m} \frac{s_k\phi^{2s_k-1}}{1-\phi^{2s_k}} - \frac{m\,\phi}{1-\phi^2}.$$

Also, since

$$\begin{aligned} &\frac{\partial}{\partial\phi}\{(x_{t_k} - \phi^{s_k} x_{t_{k-1}})^2(1-\phi^2)\} \\ &= -2\,s_k\,\phi^{s_k-1}\,x_{t_{k-1}}(x_{t_k} - \phi^{s_k}\,x_{t_{k-1}})\,(1-\phi^2) - 2\,\phi\,(x_{t_k} - \phi^{s_k}\,x_{t_{k-1}})^2, \end{aligned}$$

the derivative of the second term is

$$\begin{aligned} &-\frac{\partial}{\partial\phi} \sum_{k=1}^{m} \frac{(x_{t_k} - \phi^{s_k}\,x_{t_{k-1}})^2\,(1-\phi^2)}{2\,\sigma^2\,(1-\phi^{2s_k})} \\ &= \sum_{k=1}^{m} \frac{(x_{t_k} - \phi^{s_k}\,x_{t_{k-1}})}{\sigma^2\,(1-\phi^{2s_k})^2} \\ &\quad \times \Big(s_k\,\phi^{s_k-1}\,(1-\phi^2)(x_{t_{k-1}} - \phi^{s_k}\,x_k) + \phi\,(x_{t_k} - \phi^{s_k}\,x_{t_{k-1}})\,(1-\phi^{2s_k}) \Big). \end{aligned}$$

Therefore the derivative of the equation (1) with respect to ϕ is

$$\begin{aligned} \frac{\partial}{\partial\phi}\,\ell_{m|n}(\boldsymbol{\theta}) &= \sum_{k=1}^{m} \frac{s_k\phi^{2s_k-1}}{1-\phi^{2s_k}} - \frac{m\,\phi}{1-\phi^2} \\ &\quad + \sum_{k=1}^{m} \frac{(x_{t_k} - \phi^{s_k}\,x_{t_{k-1}})}{\sigma^2\,(1-\phi^{2s_k})^2} \\ &\quad \times \Big(s_k\,\phi^{s_k-1}\,(1-\phi^2)(x_{t_{k-1}} - \phi^{s_k}\,x_{t_k}) + \phi\,(x_{t_k} - \phi^{s_k}\,x_{t_{k-1}})\,(1-\phi^{2s_k}) \Big). \end{aligned}$$

On the other hand, the derivative of the equation (1) with respect to σ^2 is

$$\frac{\partial}{\partial\sigma^2}\,\ell_{m|n}(\boldsymbol{\theta}) \;=\; -\frac{m}{2\sigma^2}+\sum_{k=1}^{m}\frac{(x_{t_k}-\phi^{s_k}x_{t_{k-1}})^2(1-\phi^2)}{2\sigma^4(1-\phi^{2s_k})}.$$

Since $V(X_{t_k}\,|\,X_{t_{k-1}})=E(X_{t_k}^2\,|\,X_{t_{k-1}})-\{E(X_{t_k}\,|\,X_{t_{k-1}})\}^2$, we have

$$E\left[(X_{t_k}-\phi^{s_k}X_{t_{k-1}})(X_{t_{k-1}}-\phi^{s_k}X_{t_k})\,\Big|\,X_{t_{k-1}}\right] \;=\; -\phi^{s_k}V(X_{t_k}\,|\,X_{t_{k-1}}),$$

so that the expectations of the first derivatives are

$$\begin{aligned}
&E\left(\frac{\partial}{\partial\phi}\,\ell_{m|n}(\boldsymbol{\theta})\right)\\
&= \sum_{k=1}^{m}\frac{s_k\phi^{2s_k-1}}{1-\phi^{2s_k}}-\frac{m\,\phi}{1-\phi^2}\\
&\quad+\sum_{k=1}^{m}\frac{E\left[(X_{t_k}-\phi^{s_k}X_{t_{k-1}})(X_{t_{k-1}}-\phi^{s_k}X_{t_k})\,\Big|\,X_{t_{k-1}}\right]}{\sigma^2\,(1-\phi^{2s_k})^2}s_k\,\phi^{s_k-1}\,(1-\phi^2)\\
&\quad+\sum_{k=1}^{m}\frac{V(X_{t_k}\,|\,X_{t_{k-1}})}{\sigma^2\,(1-\phi^{2s_k})^2}\phi\,(1-\phi^{2s_k})\\
&= \sum_{k=1}^{m}\frac{s_k\phi^{2s_k-1}}{1-\phi^{2s_k}}-\frac{m\,\phi}{1-\phi^2}-\sum_{k=1}^{m}\frac{s_k\,\phi^{2s_k-1}}{1-\phi^{2s_k}}+\sum_{k=1}^{m}\frac{\phi}{1-\phi^2} \;=\; 0,\\
&E\left(\frac{\partial}{\partial\sigma^2}\,\ell_{m|n}(\boldsymbol{\theta})\right)\\
&= -\frac{m}{2\sigma^2}+\sum_{k=1}^{m}\frac{V(X_{t_k}\,|\,X_{t_{k-1}})(1-\phi^2)}{2\sigma^4(1-\phi^{2s_k})} \;=\; -\frac{m}{2\sigma^2}+\sum_{k=1}^{m}\frac{1}{2\sigma^2} \;=\; 0.
\end{aligned}$$

Since the second derivative with respect to σ^2 is

$$\frac{\partial^2}{\partial\sigma^4}\,\ell_{m|n}(\boldsymbol{\theta}) \;=\; \frac{m}{2\sigma^4}-\sum_{k=1}^{m}\frac{(x_{t_k}-\phi^{s_k}x_{t_{k-1}})^2(1-\phi^2)}{\sigma^6(1-\phi^{2s_k})},$$

the $(2,2)$-element I_{σ^2,σ^2} is

$$I_{\sigma^2,\sigma^2} \;=\; -\frac{m}{2\sigma^4}+\sum_{k=1}^{m}\frac{1}{\sigma^4} \;=\; \frac{m}{2\sigma^4}.$$

Since the second derivative with respect to σ^2 and ϕ is

$$\begin{aligned}
&\frac{\partial^2}{\partial\sigma^2\partial\phi}\,\ell_{m|n}(\boldsymbol{\theta})\\
&= -\sum_{k=1}^{m}\frac{(x_{t_k}-\phi^{s_k}x_{t_{k-1}})s_kx_{t_{k-1}}\phi^{s_k-1}(1-\phi^2)}{\sigma^4(1-\phi^{2s_k})}-\sum_{k=1}^{m}\frac{\phi(x_{t_k}-\phi^{s_k}x_{t_{k-1}})^2}{\sigma^4(1-\phi^{2s_k})}\\
&\quad+\sum_{k=1}^{m}\frac{(x_{t_k}-\phi^{s_k}x_{t_{k-1}})^2(1-\phi^2)s_k\phi^{2s_k-1}}{\sigma^4(1-\phi^{2s_k})^2},
\end{aligned}$$

the $(2,1)$-element $I_{\sigma^2,\phi}$ is

$$I_{\sigma^2,\phi} = \frac{m\phi}{\sigma^2(1-\phi^2)} - \sum_{k=1}^{m} \frac{s_k\phi^{2s_k-1}}{\sigma^2(1-\phi^{2s_k})}.$$

We decompose the first derivative of the equation (1) with respect to ϕ as follows:

$$\frac{\partial}{\partial\phi}\ell_{m|n}(\boldsymbol{\theta}) = A - B + C \quad \text{(say)}.$$

Then the derivatives of the parts A, B are

$$\begin{aligned} \frac{\partial}{\partial\phi}A &= \sum_{k=1}^{m} \frac{s_k(2s_k-1)\phi^{2s_k-2} + s_k\phi^{4s_k-2}}{(1-\phi^{2s_k})^2}, \\ \frac{\partial}{\partial\phi}(-B) &= -\frac{m(1+\phi^2)}{(1-\phi^2)^2}. \end{aligned}$$

Here we decompose the derivative of the part C as follows:

$$\frac{\partial}{\partial\phi}C = \sum_{k=1}^{m}\frac{1}{\sigma^2}\left\{\frac{C_1}{(1-\phi^{2s_k})^2} + \frac{C_2}{(1-\phi^{2s_k})^4}\right\} \quad \text{(say)},$$

where

$$\begin{aligned} C_1 = \; & -s_k\phi^{s_k-1}x_{t_{k-1}}\{s_k\,\phi^{s_k-1}\,(1-\phi^2)(x_{t_{k-1}} - \phi^{s_k}\,x_{t_k}) + \phi\,(x_{t_k} - \phi^{s_k}\,x_{t_{k-1}})\,(1-\phi^{2s_k})\} \\ & +(x_{t_k} - \phi^{s_k}\,x_{t_{k-1}})\{s_k(s_k-1)\phi^{s_k-2}\,(1-\phi^2)(x_{t_{k-1}} - \phi^{s_k}\,x_{t_k}) \\ & \qquad -2s_k\phi^{s_k}(x_{t_{k-1}} - \phi^{s_k}\,x_{t_k}) - s_k^2\phi^{2s_k-2}\,(1-\phi^2)x_{t_k}\} \\ & +(x_{t_k} - \phi^{s_k}\,x_{t_{k-1}})\{(x_{t_k} - \phi^{s_k}\,x_{t_{k-1}})(1-\phi^{2s_k}) - s_k\phi^{s_k}x_{t_{k-1}}(1-\phi^{2s_k}) \\ & \qquad -2s_k\phi^{2s_k}(x_{t_k} - \phi^{s_k}\,x_{t_{k-1}})\}, \\ C_2 = \; & 4s_k(1-\phi^{2s_k})\phi^{2s_k-1} \\ & \times(x_{t_k} - \phi^{s_k}\,x_{t_{k-1}})\{s_k\,\phi^{s_k-1}\,(1-\phi^2)(x_{t_{k-1}} - \phi^{s_k}\,x_{t_k}) + \phi\,(x_{t_k} - \phi^{s_k}\,x_{t_{k-1}})\,(1-\phi^{2s_k})\}. \end{aligned}$$

Since

$$E(X_{t_{k-1}} - \phi^{s_k}X_{t_k}\,|\,X_{t_{k-1}}) = (1-\phi^{2s_k})X_{t_{k-1}}, \quad E(X_{t_k} - \phi^{s_k}X_{t_{k-1}}\,|\,X_{t_{k-1}}) = 0,$$

we have

$$\begin{aligned} E(C_1\,|\,X_{t_{k-1}}) &= -s_k^2\phi^{2s_k-2}(1-\phi^2)(1-\phi^{2s_k})X_{t_{k-1}}^2 \\ & \quad +\{(1-\phi^{2s_k}) - s_k(s_k-1)\phi^{2s_k-2}(1-\phi^2)\}\,V(X_{t_k}\,|\,X_{t_{k-1}}), \\ E(C_2\,|\,X_{t_{k-1}}) &= 4s_k(1-\phi^{2s_k})\phi^{2s_k-1}\{-s_k\phi^{2s_k-1}(1-\phi^2) + \phi(1-\phi^{2s_k})\}\,V(X_{t_k}\,|\,X_{t_{k-1}}), \end{aligned}$$

so that it holds that

$$\begin{aligned} & \left\{\frac{E(C_1\,|\,X_{t_{k-1}})}{(1-\phi^{2s_k})^2} + \frac{E(C_2\,|\,X_{t_{k-1}})}{(1-\phi^{2s_k})^4}\right\} \\ & = -\frac{s_k^2\phi^{2s_k-2}(1-\phi^2)\,X_{t_{k-1}}^2}{1-\phi^{2s_k}} \\ & \quad -\frac{\sigma^2}{(1-\phi^{2s_k})^2\,(1-\phi^2)}\Big(s_k(s_k-1)\phi^{2s_k-2}(1-\phi^2)(1-\phi^{2s_k}) \\ & \qquad -(1-\phi^{2s_k})^2 + 4s_k^2\phi^{4s_k-2}(1-\phi^2) - 4s_k\phi^{2s_k}(1-\phi^{2s_k})\Big). \end{aligned}$$

Therefore we have the $(1,1)$-element $I_{\phi,\phi}$ as follows:

$$
\begin{aligned}
I_{\phi,\phi} &= \frac{2m\phi^2}{(1-\phi^2)^2} + \sum_{k=1}^{m} \frac{s_k^2 \phi^{2s_k-2}(1-\phi^2)}{\sigma^2(1-\phi^{2s_k})}\left(\sigma^2 \frac{1-\phi^{2t_{k-1}}}{1-\phi^2} + \phi^{2t_{k-1}} x_0^2\right) \\
&\quad + \sum_{k=1}^{m}\left(\frac{(3\phi^{2s_k}-1)s_k^2\phi^{2s_k-2}}{(1-\phi^{2s_k})^2} - \frac{4s_k\phi^{2s_k}}{(1-\phi^{2s_k})(1-\phi^2)}\right).
\end{aligned}
$$

□

COROLLARY 1 Assume that $s_k = x_{t_k} - x_{t_{k-1}}$ is a constant s. Then $n = ms$ and the exact information is

$$
\boldsymbol{I}_{m|n}(\boldsymbol{\theta}) = \begin{pmatrix} I_{\phi,\phi} & I_{\phi,\sigma^2} \\ I_{\sigma^2,\phi} & I_{\sigma^2,\sigma^2} \end{pmatrix},
$$

where

$$
\begin{aligned}
I_{\phi,\phi} &= \frac{2m\phi^2}{(1-\phi^2)^2} + \frac{s^2\phi^{2s-2}(1-\phi^2)}{\sigma^2(1-\phi^{2s})}\left(\frac{m\sigma^2}{1-\phi^2} + \left(x_0^2 - \frac{\sigma^2}{1-\phi^2}\right)\frac{1-\phi^{2n}}{1-\phi^{2s}}\right) \\
&\quad + m\left(\frac{(3\phi^{2s}-1)s^2\phi^{2s-2}}{(1-\phi^{2s})^2} - \frac{4s\phi^{2s}}{(1-\phi^{2s})(1-\phi^2)}\right), \\
I_{\sigma^2,\phi} &= \frac{m}{\sigma^2}\left(\frac{\phi}{1-\phi^2} - \frac{s\,\phi^{2s-1}}{1-\phi^{2s}}\right), \\
I_{\sigma^2,\sigma^2} &= \frac{m}{2\sigma^4}.
\end{aligned}
$$

When $s = 1$, it holds that

$$
I_{\phi,\phi} = \frac{1}{\sigma^2}\left(x_0^2 - \frac{\sigma^2}{1-\phi^2}\right)\frac{1-\phi^{2n}}{1-\phi^2}, \quad I_{\sigma^2,\phi} = 0, \quad I_{\sigma^2,\sigma^2} = \frac{n}{2\sigma^4}.
$$

Proof: If s_k is a constant s, then $n = ms$. By Theorem 2, the $(2,1)$-element $I_{\sigma^2,\phi}$ is

$$
I_{\sigma^2,\phi} = \frac{m}{\sigma^2}\left(\frac{\phi}{1-\phi^2} - \frac{s\,\phi^{2s-1}}{1-\phi^{2s}}\right).
$$

It is obvious that this equals 0 if $s = 1$. Note that $I_{\sigma^2,\phi} > 0$ for $\phi > 0$ and $I_{\sigma^2,\phi} < 0$ for $\phi < 0$ unless $s = 1$.

Also the $(1,1)$-element $I_{\phi,\phi}$ is

$$
\begin{aligned}
I_{\phi,\phi} &= \frac{2m\phi^2}{(1-\phi^2)^2} + \frac{s^2\phi^{2s-2}(1-\phi^2)}{\sigma^2(1-\phi^{2s})}\left(\frac{m\sigma^2}{1-\phi^2} + \left(x_0^2 - \frac{\sigma^2}{1-\phi^2}\right)\frac{1-\phi^{2n}}{1-\phi^{2s}}\right) \\
&\quad + m\left(\frac{(3\phi^{2s}-1)s^2\phi^{2s-2}}{(1-\phi^{2s})^2} - \frac{4s\phi^{2s}}{(1-\phi^{2s})(1-\phi^2)}\right)
\end{aligned}
$$

and, when $s = 1$, we have

$$
I_{\phi,\phi} = \frac{1}{\sigma^2}\left(x_0^2 - \frac{\sigma^2}{1-\phi^2}\right)\frac{1-\phi^{2n}}{1-\phi^2}.
$$

□

The comparison between Theorem 1 and Corollary 1 would make it clear that the structure of information is changed by extracting data from the original time series and that we could not reduce the extracted information to the original one even if $s = 1$, that is, the information loss is not a zero matrix:

$$\boldsymbol{I}_n(\boldsymbol{\theta}) - \boldsymbol{I}_{m|n}(\boldsymbol{\theta}) = \begin{pmatrix} \dfrac{n}{1-\phi^2} & 0 \\ 0 & 0 \end{pmatrix},$$

so that the loss divided by n is not a zero matrix asymptotically:

$$\lim_{n\to\infty} \frac{1}{n}\Big(\boldsymbol{I}_n(\boldsymbol{\theta}) - \boldsymbol{I}_{m|n}(\boldsymbol{\theta})\Big) = \begin{pmatrix} \dfrac{1}{1-\phi^2} & 0 \\ 0 & 0 \end{pmatrix} \neq \mathbf{0}.$$

4 Conclusion Based on AR(1) model, we examined the information on a daily data and the one on the weekly or monthly data. At first, we considered that these information should be equivalent if $s = 1$ for $n = ms$, but the result we obtained was not true. That implies that extracting data from the original series makes the information loss and could not recover the information on the original series from one on the extracted data even if we extracted all data of the original series. This may be a counterexample for that the differences between a daily data and the weekly or monthly data depend only on the number of observations.

We omitted the case of the random extracted time series, but we will investigate this in future, because it relates to a kind of bootstrap or a random sampling with respect to time series.

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Division of Mathematical Sciences for Social Systems, Graduate School of Engineering Science, Osaka University, 1-3, Machikeneyama, Toyonaka, Osaka, 560-8531, JAPAN

E-mail address: kumagai@sigmath.es.osaka-u.ac.jp