WEYL TYPE THEOREMS FOR (p,k)-QUASIHYPONORMAL OPERATORS

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Received March 27, 2008; Revised February 25, 2009

Abstract. Let T be a bounded linear operator on a complex Hilbert space $\mathcal{H}$. T is called (p,k)-quasihyponormal if $T^*((T^*T)^p-(TT^*)^p)T \geq 0$ for $0 < p \leq 1$ and $k \in \mathbb{N}$. In this paper, we prove Weyl type theorems for (p,k)-hyponormal operators. Especially, we prove that generalized a-Weyl's theorem holds for T if T^* is (p,k)-quasihyponormal.

1 Introduction.

Let $B(\mathcal{H})$ denote the algebra of all bounded linear operators acting on an infinite dimensional separable Hilbert space $\mathcal{H}$. An operator $T \in B(\mathcal{H})$ is called Fredholm if the range $R(T)$ is closed, the null space $N(T)$ has finite dimension and $\dim \mathcal{H}/R(T) < \infty$. Moreover, if ind $(T) = \dim N(T) - \dim \mathcal{H}/R(T) = 0$, then T is called Weyl. The Weyl spectrum $\sigma_W(T)$ is defined by

$$\sigma_W(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl}\}.$$

We say that Weyl's theorem holds for T if

$$\sigma(T) \setminus \sigma_W(T) = \pi_{00}(T)$$

where $\pi_{00}(T)$ is the set of all isolated points $\lambda \in \sigma(T)$ with $0 < \dim N(T-\lambda) < \infty$.

T is called normal if $T^*T = TT^*$, hyponormal if $T^*T - TT^* \geq 0$ and p-hyponormal $(0 < p \leq 1)$ if $(T^*T)^p - (TT^*)^p \geq 0$. In this paper, we invesigate (p,k)-quasihyponormal operators, i.e., $T^{*k}((T^*T)^p - (TT^*)^p)T^k \geq 0$ $(0 < p \leq 1, k \in \mathbb{N})$. T is called k-quasihyponormal and p-quasihyponormal if $p = 1$ and $k = 1$, respectively. Hence the notion of (p,k)-quasihyponormal operator is an extension the notions of hyponormal, p-hyponormal, p-quasihyponormal and k-quasihyponormal operator([1], [2], [9]).

H. Weyl [23] proved that Weyl's theorem holds for hermitian operators. Weyl's theorem has been extended for hyponormal, p-hyponormal and algebraically p-hyponormal operators ([11], [10], [14].) More generally, M. Berkani proved that generalized Weyl's theorem holds for hyponormal operators ([5, 6, 7]). Recently, X. Cao, M. Guo and B. Meng [8] proved Weyl type theorems for p-hyponormal operators and one of the author [19] proved that generalized Weyl's theorem holds for (p,k)-quasihyponormal operators. In this paper, we prove Weyl type theorems for (p,k)-hyponormal operators. Especially, we prove that generalized a-Weyl's theorem holds for T if T^* is (p,k)-quasihyponormal.

2 Weyl's Theorem.

I.H. Kim proved many interesting properties of (p,k)-qusihyponormal operators ([17], [18]). The following (1) is due to [17], (2) and (3) are due to [20].

2000 *Mathematics Subject Classification.* Primary 47A10, 47A12, 47B20.

Key words and phrases. (p,k)-quasihyponormal operator, Weyl's theorem, Generalized a-Weyl's theorem.

This research was supported by Grant-in-Aid Research No. 20540184 .

Lemma 2.1. *Let $T \in B(\mathcal{H})$ be (p,k)-quasihyponormal. Then the following assertions hold.*

(1) *Let the range $R(T^k)$ be not dense. Decompose*

$$T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \quad on \quad \mathcal{H} = [R(T^k)] \oplus N(T^{*k})$$

where $[R(T^k)]$ is the closure of $R(T^k)$. Then T_1 is p-hyponormal, $T_3^k = 0$ and $\sigma(T) = \sigma(T_1) \cup \{0\}$.

(2) *The restriction $T|_{\mathcal{M}}$ to an invariant subspace $\mathcal{M}$ of T is also (p,k)-quasihyponormal.*

(3) *Let λ be an isolated point of $\sigma(T)$ and E_λ be the Riesz idempotent for λ of T. If $\lambda \neq 0$, then E_λ is self-adjoint and $E_\lambda\mathcal{H} = N(T-\lambda_0) = N((T-\lambda_0)^*)$. If $\lambda = 0$, then $E_\lambda\mathcal{H} = N(T^k)$.*

Lemma 2.2. *Let T be (p,k)-quasihyponormal. Then T has the single valued extension property, i.e., if $f(z)$ is analytic and $(T-z)f(z) = 0$ on a some open set $D \subset \mathbb{C}$, then $f(z) = 0$ on D.*

Proof. If $R(T^k)$ is dense, then T is p-hyponormal and T has the the single valued extension property by [13, Theorem 1]. Hence we may assume $R(T^k)$ is not dense. Hence we can write

$$\begin{pmatrix} T_1 - z & T_2 \\ 0 & T_3 - z \end{pmatrix} \begin{pmatrix} f_1(z) \\ f_2(z) \end{pmatrix} = \begin{pmatrix} (T_1 - z)f_1(z) + T_2 f_2(z) \\ (T_3 - z)f_2(z) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

by Lemma 2.1. Since $\sigma(T_3) = \{0\}$ and $f_2(z)$ is analytic on D, we have $f_2(z) = 0$ on D. Hence $(T_1 - z)f_1(z) = 0$ and so $f_1(z) = 0$ on D by [13, Theorem 1] because T_1 is p-hyponormal by Lemma 2.1. □

Remark 2.3. *We can prove that (p,k)-quasihyponormal operator has Bishop's property (β), similarly.*

The following result is due to I.H. Kim [17]. We show another proof.

Proposition 2.4. *Weyl's theorem holds for (p,k)-quasihyponormal operators.*

Proof. Let T be (p,k)-quasihyponormal and $\lambda \in \sigma(T) \setminus \sigma_W(T)$. Then $T - \lambda$ is Weyl and not invertible. If λ is an interior point of $\sigma(T)$, there exists an open set G such that $\lambda \in G \subset \sigma(T) \setminus \sigma_W(T)$. Hence $\dim N(T-\mu) > 0$ for all $\mu \in G$ and T does not have the single valued extension property by [15, Theorem 9]. This is a contradiction. Hence λ is a boundary point of $\sigma(T)$, and hence an isolated point of $\sigma(T)$ by [12, Theorem XI 6.8]. Thus $\lambda \in \pi_{00}(T)$.

Let $\lambda \in \pi_{00}(T)$ and E_λ be the Riesz idempotent for λ of T. Then $0 < \dim N(T-\lambda) < \infty$,

$$T = T|E_\lambda\mathcal{H} \oplus T|(I - E_\lambda)\mathcal{H}$$

and

$$\sigma(T|E_\lambda\mathcal{H}) = \{\lambda\}, \quad \sigma(T|(I-E_\lambda)\mathcal{H}) = \sigma(T) \setminus \{\lambda\}.$$

We remark that $T|E_\lambda\mathcal{H}$ is (p,k)-quasihyponormal by Lemma 2.1.

If $\lambda \neq 0$, then $T|E_\lambda\mathcal{H} = \lambda$ by Lemma 2.1. Hence $E_\lambda\mathcal{H} \subset N(T-\lambda)$ and E_λ is of finite rank. Since $(T-\lambda)|(I-E_\lambda)\mathcal{H}$ is invertible, $T - \lambda = 0|E_\lambda\mathcal{H} \oplus (T-\lambda)|(I-E_\lambda)\mathcal{H}$ is Weyl. Hence $\lambda \in \sigma(T) \setminus \sigma_W(T)$.

If $\lambda = 0$, then $(T|E_0\mathcal{H})^k = 0$ by Lemma 2.1. Hence $E_0\mathcal{H} \subset N(T^k)$ and

$$\dim E_0\mathcal{H} \leq \dim N(T^k) \leq k \dim N(T) < \infty$$

by [21, Lemma 3.3]. Then $T|E_\lambda\mathcal{H}$ is compact. Since $T|(I - E_0)$ is invertible, $\lambda \in \sigma(T) \setminus \sigma_W(T)$ by [12, Proposition XI 6.9]. □

3 Generalized a-Weyl's theorem.

More generally, M. Berkani investigated B-Fredholm theory as follows (see [3, 5, 6, 7]). An operator T is called B-Fredholm if there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed and the induced operator

$$T_{[n]} : R(T^n) \ni x \to Tx \in R(T^n)$$

is Fredholm, i.e., $R(T_{[n]}) = R(T^{n+1})$ is closed, $\dim N(T_{[n]}) < \infty$ and $\dim R(T^n)/R(T_{[n]}) < \infty$. Similarly, a B-Fredholm operator T is called B-Weyl if ind $T_{[n]} = \dim N(T_{[n]}) - \dim R(T^n)/R(T_{[n]}) = 0$. The following results are due to M. Berkani and M. Sarih [7].

Proposition 3.1. *Let $T \in B(\mathcal{H})$.*

(1) *If $R(T^n)$ is closed and $T_{[n]}$ is Fredholm, then $R(T^m)$ is closed and $T_{[m]}$ is Fredholm for every $m \geq n$. Moreover,* ind $T_{[m]} =$ ind $T_{[n]}(=$ ind $T)$.

(2) *T is B-Fredholm (B-Weyl) if and only if there exist T-invariant subspaces $\mathcal{M}$ and $\mathcal{N}$ such that $T = T|\mathcal{M} \oplus T|\mathcal{N}$ where $T|\mathcal{M}$ is Fredholm (Weyl) and $T|\mathcal{N}$ is nilpotent.*

The B-Weyl spectrum $\sigma_{BW}(T)$ is defined by

$$\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda \text{ is not } B\text{-Weyl}\} \subset \sigma_W(T).$$

We say that generalized Weyl's theorem holds for T if

$$\sigma(T) \setminus \sigma_{BW}(T) = E(T)$$

where $E(T)$ denotes the set of all isolated points of $\sigma(T)$ which are eigenvalues (no restriction on multiplicity). Berkani and Koliha ([6]) proved that if generalized Weyl's theorem holds for T, then Weyl's theorem for T. Recently, M. Berkani and A. Arroud [5] prove that generalized Weyl's theorem holds for hyponormal operators and one of the authors [19] proved the same result holds for (p, k)-quasihyponormal operators.

Next result is due to B.P. Duggal and S.V. Djordjević [14].

Proposition 3.2. *If T^* is p-hyponormal, then Weyl's theorem holds for T.*

We extend above result as follows.

Theorem 3.3. *If T^* is (p, k)-quasihyponormal, then Weyl's theorem holds for T.*

Proof. [19, Theorem 2.6] implies that

$$\sigma(T^*) \setminus \sigma_{BW}(T^*) = E(T^*).$$

It is obvious that

$$(\sigma(T^*) \setminus \sigma_{BW}(T^*))^* = \sigma(T) \setminus \sigma_{BW}(T),$$

hence we have to prove

$$(E(T^*))^* = E(T).$$

Let $\lambda^* \in E(T^*)$. Then λ is an isolated point of $\sigma(T)$. Let F_{λ^*} be the Riesz idempotent for λ^* of T^*. If $\lambda^* \neq 0$, then F_{λ^*} is self-adjoint,

$$\{0\} \neq F_{\lambda^*}\mathcal{H} = N((T-\lambda)^*) = N(T-\lambda)$$

by Lemma 2.1. Hence $\lambda \in E(T)$. If $\lambda^* = 0$, then $T^*|F_0$ is (p,k)-quasihyponormal by Lemma 2.1 and $(T^*|F_0\mathcal{H})^k = 0$ by Lemma 2.1. Hence $T^{*k}F_0 = 0$. Let $E_0 = F_0^*$ be the Riesz idempotent for 0 of T. Then $T^kE_0 = (T^{*k}F_0)^* = 0$. Hence $T|E_0\mathcal{H}$ is nilpotent. Thus $\lambda = 0 \in E(T)$.

Conversely, let $\lambda \in E(T)$. Then λ^* is an isolated point of $\sigma(T^*)$. Let F_{λ^*} be the Riesz idempotent for λ^* of T^*. If $\lambda \neq 0$, then F_{λ^*} is self-adjoint and

$$\{0\} \neq F_{\lambda^*}\mathcal{H} = N((T-\lambda)^*) = N(T-\lambda)$$

by Lemma 2.1. Hence $\lambda^* \in E(T^*)$. Let $\lambda = 0$. Since $T^*|F_0\mathcal{H}$ is (p,k)-quasihyponormal and $\sigma(T^*|F_0\mathcal{H}) = \{0\}$, we have $(T^*|F_0\mathcal{H})^k = 0$ by Lemma 2.1. This implies that $T^*|F_0\mathcal{H}$ is nilpotent. Thus $\lambda^* = 0 \in E(T^*)$. □

Next we investigate a-Weyl's theorem (cf. [3]).

We define $T \in SF_+^-$ if $R(T)$ is closed, $\dim N(T) < \infty$ and $\operatorname{ind} T \leq 0$. Let $\sigma_{SF_+^-}(T) = \{\lambda \in \mathbb{C} \mid T-\lambda \notin SF_+^-\} \subset \sigma_W(T)$. Let $\sigma_a(T)$ be the set of all approximate eigen values of T and let $\pi_{00}^a(T)$ be the set of all isolated points $\lambda \in \sigma_a(T)$ with $0 < \dim N(T-\lambda) < \infty$.

We say that a-Weyl's theorem holds for T if

$$\sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \pi_{00}^a(T).$$

V. Rakočević [22, Corollary 2.5] proved that if a-Weyl's theorem holds for T, then Weyl's theorem holds for T.

Theorem 3.4. *If T^* is (p,k)-quasihyponormal, then a-Weyl's theorem holds for T.*

Proof. Since T^* has the single valued extension property by Lemma 2.2, we have $\sigma(T) = \sigma_a(T)$ and $\pi_{00}(T) = \pi_{00}^a(T)$ ([3, Corollary 2.45]).

Let $\lambda \in \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$. If λ is an interior point of $\sigma_a(T)$, then there exists an open set G such that $\lambda \in G \subset \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$. Since T^* has the single valued extension property, $\operatorname{ind}\,(T-\mu)^* \leq 0$ for all $\mu \in \mathbb{C}$ by [3, Corollary 3.19]. Let $\mu \in G$. Then $T-\mu \in SF_+^-$ and $\operatorname{ind}\,(T-\mu) = 0$. On the otherhand, $R(T-\mu)$ is closed, $T-\mu$ is not invertible and $0 < \dim N(T-\mu) < \infty$. Hence $0 < \dim N((T-\mu)^*) < \infty$ and T^* does not have a single valued extension property by [15, Theorem 9]. This is a contradiction. Hence we may assume that λ is a boundary point of $\sigma(T)$. Since $T-\lambda \in SF_+^-$, λ is an isolated point of $\sigma(T)$ by [12, Theorem XI 6.8]. Thus $\lambda \in \pi_{00}(T) = \pi_{00}^a(T)$.

Conversely, let $\lambda \in \pi_{00}^a(T) = \pi_{00}(T)$. Then $\dim N(T) < \infty$ and the conjugate number λ^* of λ is an isolated point of $\sigma(T^*)$. Let F_{λ^*} be the Riesz idempotent for λ^* of T^*.

If $\lambda^* \neq 0$, then F_{λ^*} is self-adjoint and @

$$F_{\lambda^*}\mathcal{H} = N((T-\lambda)^*) = N(T-\lambda)$$

by Lemma 2.1. Since $\dim N(T-\lambda) < \infty$, F_{λ^*} is compact. We decompose

$$(T-\lambda)^* = 0|F_{\lambda^*}\mathcal{H} \oplus (T-\lambda)^*|(I-F_{\lambda^*})\mathcal{H}.$$

Then $(T-\lambda)^*|(I-F_{\lambda^*})\mathcal{H}$ is invertible and

$$T-\lambda = 0|F_{\lambda^*}\mathcal{H} \oplus (T-\lambda)|(I-F_{\lambda^*})\mathcal{H}.$$

Hence $R(T-\lambda) = (I - F_{\lambda^*})\mathcal{H}$ is closed and ind $(T-\lambda) = 0$. Thus $\lambda \in \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$.

If $\lambda^* = 0$, then

$$T^{*k}|F_0\mathcal{H} = (T^*|F_0\mathcal{H})^k = 0$$

by Lemma 2.1. Since $E_0 = F_0^*$ is the Riesz idempotent for 0 of T and $T^k E_0 = (T^{*k}F_0)^* = 0$, we have $E_0\mathcal{H} \subset N(T^k)$. Then

$$\dim E_0\mathcal{H} \leq \dim N(T^k) \leq k \dim N(T) < \infty$$

by [21, Lemma 3.3]. This implies E_0 is compact. We decompose

$$T = T|E_0\mathcal{H} \oplus T|(I-E_0)\mathcal{H}.$$

Since $T|(I-E_0)\mathcal{H}$ is invertible, $R(T) = R(T|E_0\mathcal{H}) \oplus (I-E_0)\mathcal{H}$ is closed, $N(T) \subset E_0\mathcal{H}$ and ind $T = 0$. Thus $0 \in \sigma_a(T) \setminus \sigma_{SF_+^-}(T)$. □

Next we investigate generalized a-Weyl's theorem (cf. [3]).

We define $T \in SBF_+^-$ if there exists $n \in \mathbb{N}$ such that $R(T^n)$ is closed, $T_{[n]} : R(T^n) \ni x \to Tx \in R(T^n)$ is upper semi-Fredholm (i.e., $R(T_{[n]}) = R(T^{n+1})$ is closed, $\dim N(T_{[n]}) = \dim N(T) \cap R(T^n) < \infty$) and $0 \geq$ ind $T_{[n]}(=$ ind $T)$ ([7]). We define $\sigma_{SBF_+^-}(T) = \{\lambda \in \mathbb{C} \mid T - \lambda \notin SBF_+^-\} \subset \sigma_{SF_+^-}(T)$. Let $E^a(T)$ denote the set of all isolated points $\lambda \in \sigma_a(T)$ with $0 < \dim N(T-\lambda)$. We say that generalized a-Weyl's theorem holds for T if

$$\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E^a(T).$$

M. Berkani and J.J. Koliha [6] proved that if generalized a-Weyl's theorem holds for T, then a-Weyl's theorem holds for T.

Theorem 3.5. *If T^* is (p,k)-quasihyponormal, then generalized a-Weyl's theorem holds for T.*

Proof. Since T^* has the single valued extension property by Lemma 2.2, we have $\sigma(T) = \sigma_a(T)$, $\pi_{00}(T) = \pi_{00}^a(T)$ and $E(T) = E^a(T)$.

Let $\lambda_0 \in \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$. If λ_0 is an interior point of $\sigma_a(T)$, then there exists an open set G such that $\lambda_0 \in G \subset \sigma_a(T) \setminus \sigma_{SBF_+^-}(T)$. Let $\lambda \in G$. Then $T - \lambda \in SBF_+^-$, i.e., there exists $n \in \mathbb{N}$ such that $R((T-\lambda)^n)$ is closed, $\dim N(T_{[n]} - \lambda) < \infty$ and ind $(T-\lambda) =$ ind $(T_{[n]} - \lambda) \leq 0$. Then there exists a positive number ε such that if $0 < |\lambda - \mu| < \varepsilon$ then $T - \mu$ is upper semi-Fredholm, ind $(T-\mu) =$ ind $(T-\lambda) \leq 0$ and $\mu \in G$ by [7, Theorem 3.1]. Since T^* has a single valued extension property, ind $(T-\mu)^* \leq 0$ by [3, Corollary 3.19]. Hence ind $(T-\mu) = 0$. If $0 = \dim N(T-\mu)$, then $T-\mu$ is invertible. This is a contradiction. Hence $0 < \dim N(T-\mu) < \infty$, and $0 < \dim N((T-\mu)^*) < \infty$. Then T^* does not have the single valued extension property by [15]. This is a contradiction.

Hence we may assume that λ_0 is a boundary point of $\sigma(T)$. Since $T - \lambda_0 \in SBF_+^-$, $T - \lambda_0$ is topologically uniform descent by [7, Proposition 2.5], and λ_0 is an isolated point of $\sigma(T)$ by [16, Corollary 4.9]. We decompose

$$T - \lambda_0 = (T-\lambda_0)|\mathcal{M} \oplus (T-\lambda_0)|\mathcal{N}$$

where $(T-\lambda_0)|\mathcal{N}$ is nilpotent and $(T-\lambda_0)|\mathcal{M}$ is semi-Fredholm by [7, Theorem 2.6]. If $\mathcal{N} = \{0\}$, then

$$\lambda_0 \in \sigma_a(T) \setminus \sigma_{SF_+^-}(T) = \pi_{00}^a(T) = \pi_{00}(T) \subset E(T) = E^a(T)$$

by Theorem 3.4. If $\mathcal{N} \neq \{0\}$, then λ_0 is an eigen-value of $T|\mathcal{N}$ as $T|\mathcal{N}$ is nilpotent. Hence $\lambda_0 \in E(T) = E^a(T)$. Thus $\sigma_a(T) \setminus \sigma_{SBF_+^-}(T) \subset E^a(T)$.

The converse inclusion is clear because

$$\begin{aligned} E^a(T) = E(T) \subset \pi_{00}(T) = \pi_{00}^a(T) \\ = \sigma_a(T) \setminus \sigma_{SF_+^-}(T) \subset \sigma_a(T) \setminus \sigma_{SBF_+^-}(T) \end{aligned}$$

by Theorem 3.4.

□

Remark 3.6.

(1) *If $f(z)$ is an analytic function on $\sigma(T)$, then generalized a-Weyl's theorem holds for $f(T)$. (The proof is similar to [8, Theorem 3.3]).*

(2) *Generalized a-Weyl's theorem does not hold for (p, k)-quasihyponormal operators as seen in [4, Example 2.13].*

Acknowledgment. The authors would like to express their sincere thanks to the referee for his kind comments.

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