MULTIPLICATIVE QUADRATIC FORMS ON THE QUATERNIONS AND RELATED FUNCTIONAL EQUATIONS

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Abstract. Let $\mathcal{R}$ be a skew field. We consider the functional equation

$$F(x) + m(x)G(x^{-1}) = 0, \quad (\forall x \in \mathcal{R}, x \neq 0),$$

where $F, G : \mathcal{R} \to \mathcal{R}$ are additive and $m : \mathcal{R} \to \mathcal{R}$ is multiplicative. We obtain its general solution in the case $\mathcal{R}$ is the quaternions $\mathcal{H}$ over a subfield of the reals $\mathbb{R}$.

In due course we determine the general form of a quadratic multiplicative m on $\mathcal{H}$, i.e. solutions of $m(xy) = m(x)m(y), m(x + y) + m(x - y) = 2m(x) + 2m(y)$ $(\forall x, y \in \mathcal{H})$. The Euclidean norm $m(x_0+x_1i+x_2j+x_3k) = x_0^2+x_1^2+x_2^2+x_3^2$ is a particular solution.

For $G = -F$ $(F(1) = 1)$ and m quadratic multiplicative on $\mathcal{R}$ we show that F is not multiplicative and that $F(t^2)$ is not equal to $F(t)^2$ for some $t \in \mathcal{R}$.

1. Introduction

Let k be a commutative field and let U, V be k-vector spaces. A functional (form) $T : U \to k$ is functionally homogeneous if, for some scalar function $M : k \to k$, $T(\lambda u) = M(\lambda)T(u)$ for all $\lambda \in k$ and $u \in U$. In earlier works, biadditive functions $T : V \times V \to k$ which are functionally homogeneous have being completely determined [4]. The functional equation

$$F(x) + m(x)G(x^{-1}) = 0, \quad (\forall x \in k^* := k\backslash\{0\}),$$

plays a key role. In the current article we consider some similar functional equations on a skew field (i.e., a non-commutative division ring).

Notations:

$\mathcal{R}$ – a skew field, $\mathrm{Char}(\mathcal{R}) \neq 2$.

$\mathcal{R}^* := \mathcal{R}\backslash\{0\}$.

$S : \mathcal{R} \times \mathcal{R} \to \mathcal{R}$ – a symmetric biadditive form, $S \not\equiv 0$.

$m : \mathcal{R} \to \mathcal{R}$ – a multiplicative form, $m(0) = 0, m(1) = 1$.

$\mathcal{F}$ – a subfield of the reals $\mathbb{R}$.

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$\mathcal{H}(\mathcal{F})$ – the quaternions over $\mathcal{F}$.

$\psi : \mathcal{F} \to \mathcal{H}(\mathcal{F})$ – a field embedding.

2. Two functional equations

Let $B : \mathcal{R} \to \mathcal{R}$ be additive and $m : \mathcal{R} \to \mathcal{R}$ be multiplicative. We first consider the functional equation

$$B(x) - m(x)B(x^{-1}) = 0, \quad (\forall x \in \mathcal{R}^*). \tag{2.1}$$

Later, in parallel, we consider a similar equation

$$C(x) - C(x^{-1})m(x) = 0, \quad (\forall x \in \mathcal{R}^*), \tag{2.2}$$

where $C : \mathcal{R} \to \mathcal{R}$ is additive and $m : \mathcal{R} \to \mathcal{R}$ is multiplicative.

The value of m at 0 has no impact on the two equations. For convenience we shall assume

$$m(0) = 0. \tag{2.3}$$

Clearly $B \equiv 0$ is a trivial solution which goes with any multiplicative m. We need only treat the case $B \not\equiv 0$.

Proposition 2.1. *Let $B : \mathcal{R} \to \mathcal{R}$ be additive, $B \not\equiv 0$, and multiplicative $m : \mathcal{R} \to \mathcal{R}$ be functions satisfying the functional equation* (2.1) *with* $m(0) = 0$.

Then, for all $x, y \in R$,

$$m(x)B(x^{-1}y) - m(y)B(y^{-1}x) = 0, \quad (x \neq 0, y \neq 0), \tag{2.4}$$

$$m(1) = 1 \quad \text{and} \quad m(-x) = m(x), \tag{2.5}$$

$$B(x) = 2^{-1}[1 + m(x) - m(1 - x)]B(1), \tag{2.6}$$

$$B(1) \neq 0, \tag{2.7}$$

$$S(x, y) := m(x)B(x^{-1}y)B(1)^{-1} \quad (x \neq 0) \quad \textit{and} \quad S(0, y) := 0 \tag{2.8}$$

$$\textit{is symmetric, biadditive and normalized:} \quad S(1, 1) = 1, \tag{2.9}$$

$$m(x) = S(x, x),\ m \textit{ is quadratic}, \tag{2.10}$$

$$S(x, y) = 2^{-1}[m(x) + m(y) - m(x - y)], \tag{2.11}$$

$$f(x) := 2^{-1}[1 + m(x) - m(1 - x)] = S(x, 1), \tag{2.12}$$

$$B(x) = f(x)B(1), \tag{2.13}$$

$$f \textit{ and } m \textit{ again satisfy } (2.1) \textit{ with } f \textit{ normalized: } f(1) = 1. \tag{2.14}$$

Proof. Replacing x by $x^{-1}y$ in (2.1) we get

$$B(x^{-1}y) - m(x^{-1}y)B(y^{-1}x) = 0.$$

Multiplying by $m(x)$ on the left and using the multiplicativity of m we get (2.4). Because $B \not\equiv 0$, it is clear from (2.1) that $m \not\equiv 0$. Being multiplicative, we get $m(1) = 1$. Because B is odd, (2.1) implies $m(-x)B(x^{-1}) = m(x)B(x^{-1}) = B(x)$. Fixing an x_0 with $B(x_0^{-1}) \neq 0$ we get $m(x_0) = m(-x_0)$. Being multiplicative, it extends to $m(x) = m(-x)$ for all x, proving (2.5). Consider the simple algebraic identity

$$(1 - x)^{-1} - 1 = (1 - x)^{-1}x. \tag{2.15}$$

Applying B to the equation side by side using its additivity and the multiplicativity of m liberally we step through the following:

$$B[(1 - x)^{-1}] - B(1) = B[(1 - x)^{-1}x],$$

$$m(1-x)B[(1-x)^{-1}] - m(1-x)B(1) = m(1-x)B[(1-x)^{-1}x],$$

$$B(1-x) - m(1-x)B(1) = m(x)B[x^{-1}(1-x)],$$

$$B(1) - B(x) - m(1-x)B(1) = m(x)[B(x^{-1}) - B(1)],$$

$$B(1) - B(x) - m(1-x)B(1) = m(x)B(x^{-1}) - m(x)B(1),$$

$$B(1) - B(x) - m(1-x)B(1) = B(x) - m(x)B(1),$$

holding for all $x \neq 0, 1$. Extracting $B(x)$ we get (2.6) for all $x \neq 0, 1$. The equation also holds at $x = 0, 1$ by (2.3). In view of (2.6), the assumption $B \not\equiv 0$ implies (2.7). The symmetry of S is seen from its definition, (2.8), and (2.4), noting that $B(0) = 0$. For each fixed x, the additivity of $S(x, y)$ in the variable y is seen from its definition, observing that B is additive. Being symmetric, S is biadditive. The normalization of $S(1, 1)$ is clear as $m(1) = 1$. This proves (2.9). Immediate from the definition of S we get $S(x, x) = m(x)$ for all x. S being symmetric and biadditive, this proves (2.10) and leads to (2.11). Putting $y = 1$ in (2.11) we get (2.12). (2.13) is a repeat of (2.6). Because B and m satisfy (2.1), (2.14) obviously follows from (2.13). □

Proposition 2.2. *Let $C : \mathcal{R} \to \mathcal{R}$ be additive, $C \not\equiv 0$, and multiplicative $m : \mathcal{R} \to \mathcal{R}$ be functions satisfying the functional equation* (2.2) *with* $m(0) = 0$.

Then, for all $x, y \in \mathcal{R}$,

(2.16) $C(yx^{-1})m(x) - C(xy^{-1})m(y) = 0, \quad (x \neq 0, y \neq 0),$

(2.17) $m(1) = 1 \quad \text{and} \quad m(-x) = m(x),$

(2.18) $C(x) = 2^{-1}C(1)[1 + m(x) - m(1-x)],$

(2.19) $C(1) \neq 0,$

(2.20) $S(x, y) := C(1)^{-1}C(yx^{-1})m(x) \quad (x \neq 0) \quad$ *and* $\quad S(0, y) := 0$

(2.21) *is symmetric, biadditive and normalized:* $S(1, 1) = 1,$

(2.22) $m(x) = S(x, x)$, *m is quadratic,*

(2.23) $S(x, y) = 2^{-1}[m(x) + m(y) - m(x - y)],$

(2.24) $f(x) := 2^{-1}[1 + m(x) - m(1-x)] = S(x, 1),$

(2.25) $C(x) = C(1)f(x),$

(2.26) *f and m again satisfy* (2.2) *with f normalized:* $f(1) = 1$.

Proof. Replacing x by yx^{-1} in (2.2) we get

$$C(yx^{-1}) - C(xy^{-1})m(yx^{-1}) = 0.$$

Multiplying by $m(x)$ on the right and using the multiplicativity of m we get (2.16). Because $C \not\equiv 0$, it is clear from (2.1) that $m \not\equiv 0$. Being multiplicative, we get $m(1) = 1$. Because C is odd, (2.2) implies $C(x^{-1})m(-x) = C(x^{-1})m(x)$. Fixing an x_0 with $C(x_0^{-1}) \neq 0$ we get $m(x_0) = m(-x_0)$. Being multiplicative, it extends to $m(x) = m(-x)$ for all x, proving (2.17). Consider the simple algebraic identity

(2.27) $$(1-x)^{-1} - 1 = x(1-x)^{-1}.$$

Applying C to the equation side by side using its additivity and the multiplicativity of m liberally we step through the following:

$$C[(1-x)^{-1}] - C(1) = C[x(1-x)^{-1}],$$

$$C[(1-x)^{-1}]m(1-x) - C(1)m(1-x) = C[x(1-x)^{-1}]m(1-x),$$

$$C(1-x) - C(1)m(1-x) = C[(1-x)x^{-1}]m(x),$$

$$C(1) - C(x) - C(1)m(1-x) = [C(x^{-1}) - C(1)]m(x),$$

$$C(1) - C(x) - C(1)m(1-x) = C(x^{-1})m(x) - C(1)m(x),$$

$$C(1) - C(x) - C(1)m(1-x) = C(x) - C(1)m(x),$$

holding for all $x \neq 0, 1$. Extracting $C(x)$ we get (2.18) for all $x \neq 0, 1$. The equation also holds at $x = 0, 1$ by (2.3). In view of (2.18), the assumption $C \not\equiv 0$ implies (2.19). The symmetry of S is seen from its definition, (2.20), and (2.16), noting that $C(0) = 0$. For each fixed x, the additivity of $S(x, y)$ in the variable y is seen from its definition, observing that C is additive. Being symmetric, S is biadditive. All remaining claims follow in a similar way as Proposition 2.1. □

In light of (2.10) and (2.6), and of (2.22) and (2.18) respectively, the study of the functional equation (2.1), and of (2.2), is tied to that of quadratic and multiplicative m. The next section starts with such m on a general $\mathcal{R}$. We obtain some general relations and make some intermediate observations.

3. Quadratic and multiplicative m

Let $m \not\equiv 0$ be a multiplicative quadratic form on $\mathcal{R}$, i.e. it satisfies $m(xy) = m(x)m(y)$ and $m(x+y) + m(x-y) = 2m(x) + 2m(y)$ for all $x, y \in \mathcal{R}$ and $m(1) = 1$. We consider

$$(3.1)\qquad \begin{aligned} S(x,y) :&= \frac{1}{2}(m(x+y) - m(x) - m(y)) \\ &= \frac{1}{2}(m(x) + m(y) - m(x-y)) \\ &= \frac{1}{4}(m(x+y) - m(x-y)). \\ f(x) :&= S(x,1). \end{aligned}$$

Proposition 3.1. *Let $m \not\equiv 0$ be a multiplicative quadratic form on $\mathcal{R}$ and let S and f be defined by* (3.1)*. Then, for general $x, y, u, v, r, s \in \mathcal{R}$,*

(i) *S is a symmetric biadditive form and*

$$S(px, qy) = pqS(x,y), \quad \textit{and} \quad S(x,x) = m(x), \quad (\forall p, q \in \mathbb{Z}),$$

(ii) *f is additive, and $f(1) = 1$,*

(iii) $S(x,y) = m(y)f(y^{-1}x) = f(xy^{-1})m(y), \quad (y \neq 0)$,

(iv) $f(x) = f(x^{-1})m(x) = m(x)f(x^{-1}), \quad (x \neq 0)$,

(v)

$$(3.2)\qquad S(rxs, rys) = m(r)S(x,y)m(s),$$

(vi) $m(2) = 4$,

(vii)

$$(3.3)\qquad S(rx, sy) + S(sx, ry) = 2S(r,s)S(x,y),$$

(viii)

$$S(u,v) = 2f(u)f(v) - f(uv), \tag{3.4}$$

(ix)

$$m(x) = 2f(x)^2 - f(x^2), \tag{3.5}$$

(x)

$$2f(u)f(v) - f(uv) = 2f(v)f(u) - f(vu). \tag{3.6}$$

Proof. (i) See [2]. (ii) The additivity of f follows from the biadditivity of S, and $f(1) = S(1,1) = m(1) = 1$. (iii)

$$\begin{aligned} S(x,y) &= \frac{1}{2}(m(x+y) - m(x) - m(y)) \\ &= \frac{1}{2}(m(xy^{-1}+1) - m(xy^{-1}) - m(1))m(y) \\ &= f(xy^{-1})m(y). \end{aligned}$$

$$\begin{aligned} S(x,y) &= \frac{1}{2}(m(x+y) - m(x) - m(y)) \\ &= \frac{1}{2}m(y)(m(y^{-1}x+1) - m(y^{-1}x) - m(1)) \\ &= m(y)f(y^{-1}x). \end{aligned}$$

(iv)

$$f(x) = S(x,1) = S(1,x) = f(x^{-1})m(x).$$

$$f(x) = S(x,1) = S(1,x) = m(x)f(x^{-1}).$$

(v)

$$\begin{aligned} S(rxs, rys) &= \frac{1}{2}(m(rxs+rys) - m(rxs) - m(rys)) \\ &= \frac{1}{2}m(r)(m(x+y) - m(x) - m(y))m(s) \\ &= m(r)S(x,y)m(s). \end{aligned}$$

(vi)

$$m(2) = S(2,2) = 4S(1,1) = 4m(1) = 4.$$

(vii) By (3.2) we have $S(rx,ry) = m(r)S(x,y)$ in particular. Polarize each sides with respect to the variable r we get (3.3). (viii) Letting $s = y = 1$ in (3.3) we get $f(rx) + S(x,r) = 2f(r)f(x)$. As S is symmetric, it follows that $f(rx) + S(r,x) = 2f(r)f(x)$. Renaming the variables we get (3.4). (ix)-(x) The diagonal of (3.4) gives (3.5) and the symmetry of S gives (3.6). ☐

Lemma 3.2. *f is not multiplicative.*

Proof. Suppose to the contrary that f is multiplicative.

As f is additive, and $f(1) = 1$, it is then an injective ring homomorphism of $\mathcal{R}$ into $\mathcal{R}$. Hence (3.6) reduces to $f(uv) = f(vu)$, leading further to $uv = vu$ for all $u, v \in \mathcal{R}$. It contradicts the non-commutativity assumption of $\mathcal{R}$. ☐

Lemma 3.3. *Suppose that*

$$f(t^2) = f(t)^2 \quad \text{for all } t \in \mathcal{R}. \tag{3.7}$$

Then the following equations hold for all $t, u, v \in \mathcal{R}$:

$$m(t) = f(t)^2, \tag{3.8}$$

$$S(u,v) = \frac{1}{2}f(u)f(v) + \frac{1}{2}f(v)f(u), \tag{3.9}$$

$$2f(uv) = 3f(u)f(v) - f(v)f(u). \tag{3.10}$$

Furthermore, for any given $u, v \in \mathcal{R}$,

$$f(uv) = f(vu) \quad \text{iff} \quad f(u)f(v) = f(v)f(u) \quad \text{iff} \quad f(uv) = f(u)f(v). \tag{3.11}$$

Proof. (i) Putting (3.7) in (3.5) we get (3.8). Polarizing (3.8) we get (3.9).

(ii) Polarizing both sides of (3.7) we obtain

$$f(uv) + f(vu) = f(u)f(v) + f(v)f(u) \quad \text{for all } u, v \in \mathcal{R}. \tag{3.12}$$

Eliminating the term $f(vu)$ from (3.6) using (3.12) we obtain

$$2f(u)f(v) - f(uv) = 2f(v)f(u) - [f(u)f(v) + f(v)f(u) - f(uv)].$$

Collecting like terms we arrive at (3.10).

(iii) For given u, v, the equivalences follow simply from (3.10). Details: (a) Suppose that $f(uv) = f(vu)$. Then (3.10) yields $3f(u)f(v) - f(v)f(u) = 3f(v)f(u) - f(u)f(v)$. So $4f(u)f(v) = 4f(v)f(u)$, proving $f(u)f(v) = f(v)f(u)$. (b) Suppose that $f(u)f(v) = f(v)f(u)$. Then the right hand side of (3.10) equals $2f(u)f(v)$ and gives $2f(uv) = 2f(u)f(v)$. This proves $f(uv) = f(u)f(v)$. (c) Suppose that $f(uv) = f(u)f(v)$. By (3.10), $2f(uv) = 3f(u)f(v) - f(v)f(u)$. Substitution gives $2f(u)f(v) = 3f(u)f(v) - f(v)f(u)$. Simplifying we get $f(u)f(v) = f(v)f(u)$. Therefore $3f(u)f(v) - f(v)f(u) = 3f(v)f(u) - f(u)f(v)$. By (3.10), it translates into to $2f(uv) = 2f(vu)$. This proves $f(uv) = f(vu)$. □

Proposition 3.4. *$f(t^2) \neq f(t)^2$ for some t.*

Proof. Suppose to the contrary that (3.7) holds. Using (3.10) and computing $f(uv^2)$ in two ways we get

$$\begin{aligned} f(uv^2) &= \frac{1}{2}[3f(u)f(v^2) - f(v^2)f(u)] \\ &= \frac{1}{2}[3f(u)f(v)^2 - f(v)^2 f(u)] \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} f(uv^2) = f((uv)v) &= \frac{1}{2}[3f(uv)f(v) - f(v)f(uv)] \\ &= \frac{3}{4}[3f(u)f(v) - f(v)f(u)]f(v) - \frac{1}{4}f(v)[3f(u)f(v) - f(v)f(u)] \\ &= \frac{9}{4}f(u)f(v)^2 - \frac{3}{2}f(v)f(u)f(v) + \frac{1}{4}f(v)^2 f(u). \end{aligned} \tag{3.14}$$

Comparing the right hand sides we get

$$3f(u)f(v)^2 - 6f(v)f(u)f(v) + 3f(v)^2 f(u) = 0. \tag{3.15}$$

$$\begin{aligned}
f(vuv) &= \frac{1}{2}[3f(v)f(uv) - f(uv)f(v)] \\
&= \frac{1}{4}[3f(v)[3f(u)f(v) - f(v)f(u)] - [3f(u)f(v) - f(v)f(u)]f(v)] \\
&= \frac{10}{4}f(v)f(u)f(v) - \frac{3}{4}f(v)^2 f(u) - \frac{3}{4}f(u)f(v)^2 \\
&= \frac{10}{4}f(v)f(u)f(v) - \frac{3}{4}f(v)^2 f(u) - \frac{1}{4}[6f(v)f(u)f(v) - 3f(v)^2 f(u)] \\
&\quad \text{(by (3.15))} \\
&= f(v)f(u)f(v). \qquad (3.16)
\end{aligned}$$

Hence

$$\begin{aligned}
f(uvuv) &= \frac{1}{2}[3f(u)f(vuv) - f(vuv)f(u)] \\
&= \frac{1}{2}[3f(u)f(v)f(u)f(v) - f(v)f(u)f(v)f(u)]. \qquad (3.17)
\end{aligned}$$

Comparing that with

$$\begin{aligned}
f(uvuv) &= f((uv)^2) = f(uv)^2 \\
&= \frac{1}{4}[3f(u)f(v) - f(v)f(u)]^2 \\
&= \frac{1}{4}[9f(u)f(v)f(u)f(v) - 3f(u)f(v)f(v)f(u) \\
&\quad - 3f(v)f(u)f(u)f(v) + f(v)f(u)f(v)f(u)] \qquad (3.18)
\end{aligned}$$

we get

$$\begin{aligned}
&2[3f(u)f(v)f(u)f(v) - f(v)f(u)f(v)f(u)] \\
&= 9f(u)f(v)f(u)f(v) - 3f(u)f(v)f(v)f(u) \\
&\quad - 3f(v)f(u)f(u)f(v) + f(v)f(u)f(v)f(u).
\end{aligned}$$

Simplifying and factoring we get

$$3[f(u)f(v) - f(v)f(u)]^2 = 0. \qquad (3.19)$$

There are two cases to consider.

Case 1. Suppose that $char(\mathcal{R}) \neq 3$.

Then (3.19) yields $f(u)f(v) = f(v)f(u)$. By (3.11) $f(uv) = f(u)f(v)$ follows. That contradicts Lemma 3.2.

Case 2. Suppose that $char(\mathcal{R}) = 3$.

Then (3.10) reduces to $2f(uv) = -f(v)f(u) = 2f(v)f(u)$. Hence f is a skew morphism:

$$f(uv) = f(v)f(u), \quad (\forall u, v \in \mathcal{R}). \qquad (3.20)$$

Because f is not identically zero, (3.20) implies that it has a trivial kernel. So f is injective.

Since $m(u) = f(u)^2 = f(u^2)$, the multiplicativity of m translates into

$$f((uv)^2) = f(u^2)f(v^2), \quad (\forall u, v \in \mathcal{R}). \qquad (3.21)$$

By (3.20) we have $f(u^2)f(v^2) = f(v^2u^2)$ and so (3.21) gives $f((uv)^2) = f(v^2u^2)$. Injectivity of f then yields the relation

$$(uv)^2 = v^2u^2, \quad (\forall u, v \in \mathcal{R}). \tag{3.22}$$

Using (3.22) in a sequence of successive calculates we obtain a collection of identities, each holding for all $u, v \in \mathcal{R}$:

$$(uv)^2u = (v^2u^2)u = v^2u^3, \tag{3.23}$$

$$(uv)^2u = u(vu)^2 = u(u^2v^2) = u^3v^2, \tag{3.24}$$

$$v^2u^3 = u^3v^2, \quad \text{i.e. } u^3 \text{ and } v^2 \text{ commute}, \tag{3.25}$$

$$(uv)^3 = ((uv)^2u)v = (u^3v^2)v = u^3v^3, \tag{3.26}$$

$$uv^2uv^2 = (uv^2)^2 = (v^2)^2u^2 = v^4u^2 = v(v^3u^2) = vu^2v^3,$$

$$uv^2u = vu^2v, \quad \text{i.e. } uv \text{ and } vu \text{ commute}, \tag{3.27}$$

$$(uv)^4 = ((uv)^2)^2 = (v^2u^2)^2 = (u^2)^2(v^2)^2 = u^4v^4,$$

$$(uv)^4 = (uv)(uv)^3 = (uv)(u^3v^3) = uvu^3v^3,$$

$$u^4v^4 = uvu^3v^3,$$

$$u^3v = vu^3, \quad \text{i.e. } u^3 \text{ and } v \text{ commute}. \tag{3.28}$$

Because uv and vu commute, (3.27), and since $char(\mathcal{R}) = 3$, expansion gives

$$(uv - vu)^3 = (uv)^3 - (vu)^3. \tag{3.29}$$

By (3.26), $(uv)^3 - (vu)^3 = u^3v^3 - v^3u^3$; and by (3.28), $u^3v^3 - v^3u^3 = 0$. Therefore (3.29) yields $(uv - vu)^3 = 0$, implying that $uv = vu$ for all u, v. It contradicts that $\mathcal{R}$ is non-commutative. □

4. QUADRATIC AND MULTIPLICATIVE m ON THE QUATERNIONS

Let $\mathcal{F}$ be a subfield of the reals and let $\mathcal{H}(\mathcal{F})$ be the quaternions over $\mathcal{F}$. We shall solve for quadratic and multiplicative m on $\mathcal{H}(\mathcal{F})$. Let

$$\tilde{u} = u_1i + u_2j + u_3k, \quad (u_1, u_2, u_3 \in \mathcal{F}),$$

denote a unit vector, i.e. it satisfies $\tilde{u}^2 = -1$. From $m(\tilde{u})^2 = m(\tilde{u}^2) = m(-1) = 1$, we get $m(\tilde{u}) = \pm 1$. Using $m(\tilde{u}) = 2f(\tilde{u})^2 - f(\tilde{u}^2) = 2f(\tilde{u}^2) - f(-1) = 2f(\tilde{u})^2 + 1$, we get that either

$$m(\tilde{u}) = 1 \quad \text{and} \quad f(\tilde{u}) = 0,$$

or

$$m(\tilde{u}) = -1 \quad \text{and} \quad f(\tilde{u}) \text{ is a unit vector}.$$

There are four cases to consider at this time:

Case 1. $m(i) = 1$, $m(j) = 1$, $m(k) = 1$, $f(i) = 0$, $f(j) = 0$ and $f(k) = 0$.
Case 2. $m(i) = -1$, $m(j) = -1$, $m(k) = 1$, $f(i) = \tilde{u}$, $f(j) = \tilde{v}$ and $f(k) = 0$.
Case 3. $m(i) = -1$, $m(j) = 1$, $m(k) = -1$, $f(i) = \tilde{u}$, $f(j) = 0$ and $f(k) = \tilde{v}$.
Case 4. $m(i) = 1$, $m(j) = -1$, $m(k) = -1$, $f(i) = 0$, $f(j) = \tilde{u}$ and $f(k) = \tilde{v}$.
Here $\tilde{v}$ also refers to a unit vector.

First we consider Case 2. Let $x = 1+i+k$ and $\bar{x} = 1-i-k$. Then $f(x) = 1+f(i)+f(k) = 1 + \tilde{u}$ and similarly $f(\bar{x}) = 1 - \tilde{u}$. As $x^2 = -1 + 2i + 2k$ and $\bar{x}^2 = -1 - 2i - 2k$, we get $f(x^2) = -1 + 2\tilde{u}$ and $f(\bar{x}^2) = -1 - 2\tilde{u}$.

Hence $m(x) = 2f(x)^2 - f(x^2) = 2(1+\tilde{u})^2 - (-1+2\tilde{u}) = 2+4\tilde{u}+2\tilde{u}^2+1-2\tilde{u} = 1+2\tilde{u}$, and $m(\bar{x}) = 2f(\bar{x})^2 - f(\bar{x}^2) = 2(1-\tilde{u})^2 - (-1-2\tilde{u}) = 2-4\tilde{u}+2\tilde{u}^2+1+2\tilde{u} = 1-2\tilde{u}$. So, $m(x)m(\bar{x}) = (1+2\tilde{u})(1-2\tilde{u}) = 5$. Comparing with $m(x\bar{x}) = m(1+1+0+1) = m(3) = 9$, we see that $m(x)m(\bar{x}) \neq m(x\bar{x})$ and is a contradiction to the multiplicity of m.

This proves that Case 2 is inadmissible.

The 3-cycle $i \mapsto j$, $j \mapsto k$, $k \mapsto i$ generates an automorphism ϕ on $\mathcal{H}$ leaving elements of the center $\mathcal{F}$ fixed. An m falls under Case 2 iff $m \circ \phi$ falls under Case 3 iff $m \circ \phi \circ \phi$ falls under Case 4. Since Case 2 is inadmissible, Case 3 and Case 4 are also inadmissible.

Next, we consider Case 1 which is the only case left:

$$m(i) = m(j) = m(k) = 1 \quad \text{and} \quad f(i) = f(j) = f(k) = 0. \tag{4.1}$$

Consider (3.6) at $u = i$ and $v = x_3 j$:

$$2f(i)f(x_3j) - f(ix_3j) = 2f(x_3j)f(i) - f(x_3ji).$$

Because $f(i) = 0$, we get $f(x_3k) = f(-x_3k) = -f(x_3k)$. This proves

$$f(x_3k) = 0, \quad (\forall x_3 \in \mathcal{F}).$$

The 3-cycle $i \mapsto j$, $j \mapsto k$, $k \mapsto i$ generates an automorphism ϕ on $\mathcal{H}(\mathcal{F})$ leaving the center $\mathcal{F}$ fixed. Consideration of $m \circ \phi$ and $m \circ \phi \circ \phi$ immediately extends the above to

$$f(x_1i) = f(x_2j) = f(x_3k) = 0, \quad (\forall x_1, x_2, x_3 \in \mathcal{F}).$$

Therefore

$$f(x) = f(x_0 + x_1i + x_2j + x_3k) = f(x_0), \quad (\forall x \in \mathcal{H}(\mathcal{F})). \tag{4.2}$$

It leads to

$$\begin{aligned} m(x) &= 2f(x)^2 - f(x^2) = 2f(x_0)^2 - f(x_0^2 - x_1^2 - x_2^2 - x_3^2) \\ &= [2f(x_0)^2 - f(x_0^2)] + f(x_1^2) + f(x_2^2) + f(x_3^2). \end{aligned} \tag{4.3}$$

The equality $m(xi) = m(x)m(i) = m(x)$ then implies

$$[2f(x_0)^2 - f(x_0^2)] + f(x_1^2) + f(x_2^2) + f(x_3^2) = [2f(x_1)^2 - f(x_1^2)] + f(x_0^2) + f(x_3^2) + f(x_2^2).$$

Setting $x_0 = t$ and $x_1 = 0$, it reduces to

$$f(t)^2 = f(t^2), \quad (\forall t \in \mathcal{F}). \tag{4.4}$$

With that, (4.3) becomes

$$m(x) = f(x_0^2) + f(x_1^2) + f(x_2^2) + f(x_3^2). \tag{4.5}$$

Polarizing (4.4) we see that

$$f(st) = f(s)f(t), \quad (\forall s, t \in \mathcal{F}). \tag{4.6}$$

Let $\psi : \mathcal{F} \to \mathcal{H}(\mathcal{F})$ be the restriction of f to $\mathcal{F}$. Then ψ is both additive and multiplicative, with $\psi(1) = 1$. So ψ is a field embedding of $\mathcal{F}$ into $H(\mathcal{F})$. We rewrite (4.5) as

$$m(x) = \psi(x_0^2 + x_1^2 + x_2^2 + x_3^2) = \psi(x\bar{x}). \tag{4.7}$$

Polarizing we get

$$S(x, y) = \psi(x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3) = \psi(x_0y_0 + \tilde{x} \cdot \tilde{y}). \tag{4.8}$$

Conversely, it is straight forward to check that a function m having the representation (4.7) is indeed quadratic and multiplicative.

We sum the section up:

Proposition 4.1. *$m : H(\mathcal{F}) \to H(\mathcal{F})$, $m \not\equiv 0$, is quadratic and multiplicative iff it has the representation*

$$m(x) = \psi(x_0^2 + x_1^2 + x_2^2 + x_3^2) = \psi(x\bar{x})$$

for some field embedding $\psi : \mathcal{F} \to \mathcal{H}(\mathcal{F})$.

Note*: The only $\psi : \mathbb{Q} \to \mathcal{H}(\mathbb{Q})$ is the natural inclusion $\psi(x_0) = x_0$. However, non-trivial embeddings $\psi : \mathbb{R} \to \mathcal{H}(\mathbb{R})$ exist* [1, 3].

5. The equation $F(x) + m(x)G(x^{-1}) = 0$

We only treat the functional equation over $H(\mathcal{F})$:

$$F(x) + m(x)G(x^{-1}) = 0, \quad (\forall x \in \mathcal{H}(\mathcal{F}), x \neq 0), \tag{5.1}$$

where m is multiplicative, $m(0) = 0$, $m(1) = 1$ and F and G are additive.

Replacing x by x^{-1} in (5.1) and multiply the resulting equation by $m(x)$ we get

$$G(x) + m(x)F(x^{-1}) = 0, \quad (\forall x \in \mathcal{H}(\mathcal{F}), x \neq 0). \tag{5.2}$$

Adding and subtracting the equations we get, respectively,

$$A(x) + m(x)A(x^{-1}) = 0, \quad (\forall x \in \mathcal{H}(\mathcal{F}), x \neq 0), \tag{5.3}$$

and

$$B(x) - m(x)B(x^{-1}) = 0, \quad (\forall x \in \mathcal{H}(\mathcal{F}), x \neq 0), \tag{5.4}$$

where $A := F + G$, $B := F - G$ are additive. The latter is (2.1) and, if B is not identically zero, its solution is seen via Proposition 4.1 and (4.2):

Proposition 5.1. *The solution of*

$$B(x) - m(x)B(x^{-1}) = 0, \quad (\forall x \in \mathcal{H}(\mathcal{F}), x \neq 0), \tag{5.5}$$

where $B \not\equiv 0$ is additive, m is multiplicative, $m(0) = 0$ and $m(1) = 1$, is given by

$$B(x) = \psi(x_0)B(1), \tag{5.6}$$

$$m(x) = \psi(x\bar{x}). \tag{5.7}$$

Here, $B(1)$ is a non-zero constant.

We now attend to the former equation (5.3).

Proposition 5.2. *Let $A \not\equiv 0$ be additive and m be multiplicative on $\mathcal{H}(\mathcal{F})$, $m(0) = 0$ and $m(1) = 1$. The solution of*

$$A(x) + m(x)A(x^{-1}) = 0, \quad (\forall x \in \mathcal{H}(\mathcal{F}), x \neq 0), \tag{5.8}$$

is given by

$$A(x) = \psi(x_1)A(i) + \psi(x_2)A(j) + \psi(x_3)A(k), \tag{5.9}$$

$$m(x) = \psi(x\bar{x}) \tag{5.10}$$

for all $x \in \mathcal{H}(\mathcal{F})$. Here $\psi : \mathcal{F} \to \mathcal{H}(\mathcal{F})$ is a field embedding. The constants $A(i), A(j)$ and $A(k)$ are not all zero.

Proof. In the proof for the sake of brevity we write $\mathcal{H}$ in place of $\mathcal{H}(\mathcal{F})$.

Suppose that (5.8) holds and that A and M have the prescribed properties. Replacing x by $x^{-1}y$ in (5.8) we get

$$A(x^{-1}y) + m(x^{-1}y)A(y^{-1}x) = 0.$$

Multiplying by $m(x)$ on the left and using the multiplicativity of m we get

$$m(x)A(x^{-1}y) + m(y)A(y^{-1}x) = 0 \tag{5.11}$$

for all $x \neq 0, y \neq 0 \in \mathcal{H}$.

We define T by

$$T(x, y) = m(x)A(x^{-1}y),$$
$$T(0, y) = T(x, 0) = T(0, 0) = 0$$

for all $x \neq 0, y \neq 0 \in \mathcal{H}$. Using (5.11) we deduce further that T is skew-symmetric and biadditive. Furthermore

$$T(sx, sy) = m(s)T(x, y), \quad (\forall s, x, y \in \mathcal{H}). \tag{5.12}$$

Symmetric polarization of the variable s gives

$$0 = [m(s + t) + m(s - t) - 2m(s) - 2m(t)]T(x, y), \quad (\forall s, t, x, y \in \mathcal{H}). \tag{5.13}$$

The assumption $A \not\equiv 0$ implies that T is not identically zero. Hence, the above equation implies that m is quadratic. Therefore, by Proposition 4.1, it admits the representation (5.10).

We replace x by xi in (5.8), using $m(xi) = m(x)$ and observing that A is odd, to get

$$A(xi) - m(x)A(ix^{-1}) = 0, \quad (\forall x \in \mathcal{H}, x \neq 0). \tag{5.14}$$

Similarly, replacing x by ix in (5.8) we get

$$A(ix) - m(x)A(x^{-1}i) = 0, \quad (\forall x \in \mathcal{H}, x \neq 0). \tag{5.15}$$

Adding (5.14) and (5.15) side by side and letting

$$b(x) := A(xi) + A(ix), \quad (\forall x \in \mathcal{H}), \tag{5.16}$$

we get

$$b(x) - m(x)b(x^{-1}) = 0, \quad (\forall x \in \mathcal{H}, x \neq 0). \tag{5.17}$$

It is clear that b is additive. According to Proposition 5.1, b is given by

$$b(x) = \psi(x_0)b(1), \quad (\forall x \in \mathcal{H}). \tag{5.18}$$

Here, $b(1) = 0$ is allowed so as to carry the trivial solution $b \equiv 0$.

For $x = x_0 + x_1 i + x_2 j + x_3 k$,

$$\begin{aligned} b(x) &= A(xi) + A(ix) \\ &= A(x_0 i - x_1 - x_2 k + x_3 j) + A(x_0 i - x_1 + x_2 k - x_3 j) \\ &= -2A(x_1) + 2A(x_0 i). \end{aligned}$$

Hence (5.18) translates into

$$-2A(x_1) + 2A(x_0 i) = \psi(x_0)2A(i).$$

Dropping the factor 2 and renaming the variables we obtain

$$A(x_0) = 0 \quad \text{and} \quad A(x_1 i) = \psi(x_1)A(i), \quad (\forall x_0, x_1 \in \mathcal{F}). \tag{5.19}$$

In view of the symmetric roles played by i, j, and k in the quaternions, parallel arguments give

$$A(x_2 j) = \psi(x_2)A(j), \quad (\forall x_2 \in \mathcal{F}), \tag{5.20}$$

$$A(x_3 k) = \psi(x_3)A(k), \quad (\forall x_3 \in \mathcal{F}). \tag{5.21}$$

A being additive, (5.19), (5.20) and (5.21) imply that A has the representation (5.9).

Conversely, suppose that A and m are given by (5.9) and (5.10). The following computation justifies that (5.3) is indeed satisfied.

$$x^{-1} = \frac{\bar{x}}{x\bar{x}} = \frac{x_0 - x_1 i - x_2 j - x_3 k}{x\bar{x}},$$

$$A(\frac{\bar{x}}{x\bar{x}}) = \psi(\frac{-x_1}{x\bar{x}})A(i) + \psi(\frac{-x_2}{x\bar{x}})A(j) + \psi(\frac{-x_3}{x\bar{x}})A(k),$$

$$\begin{aligned}\psi(x\bar{x})A(\frac{\bar{x}}{x\bar{x}}) &= \psi(x\bar{x})\psi(\frac{-x_1}{x\bar{x}})A(i) \\ &\quad + \psi(x\bar{x})\psi(\frac{-x_2}{x\bar{x}})A(j) + \psi(x\bar{x})\psi(\frac{-x_3}{x\bar{x}})A(k) \\ &= -\psi(x_1)A(i) - \psi(x_2)A(j) - \psi(x_3)A(k) \\ &= -A(x).\end{aligned}$$

□

Combining the above two Propositions where $A : F + G$, $B := F - G$ we arrive at the following solution for (5.1).

Theorem 5.3. *The general solution of the equation*

$$F(x) + m(x)G(x^{-1}) = 0, \quad (\forall x \in \mathcal{H}(\mathcal{F}), x \neq 0), \tag{5.22}$$

where m is multiplicative, $m(0) = 0$, $m(1) = 1$, $F, G \not\equiv 0$ are additive, is given by:

$$m(x) = \psi(x\bar{x}), \tag{5.23}$$

$$F(x) = \frac{1}{2}[\psi(x_0)B(1) + \psi(x_1)A(i) + \psi(x_2)A(j) + \psi(x_3)A(k)], \tag{5.24}$$

$$G(x) = \frac{1}{2}[-\psi(x_0)B(1) + \psi(x_1)A(i) + \psi(x_2)A(j) + \psi(x_3)A(k)]. \tag{5.25}$$

Here, at least one of the constants $B(1), A(i), A(j), A(k)$ is non-zero.

6. The equation $F(x) + G(x^{-1})m(x) = 0$

With parallel deductions, using Proposition 2.2 instead of Proposition 2.1, we get

Theorem 6.1. *The general solution of the equation*

$$F(x) + G(x^{-1})m(x) = 0, \quad (\forall x \in \mathcal{H}(\mathcal{F}), x \neq 0), \tag{6.1}$$

where m is multiplicative, $m(0) = 0$, $m(1) = 1$, $F, G \not\equiv 0$ are additive, is given by:

$$m(x) = \psi(x\bar{x}), \tag{6.2}$$

$$F(x) = \frac{1}{2}[B(1)\psi(x_0) + A(i)\psi(x_1) + A(j)\psi(x_2) + A(k)\psi(x_3)], \tag{6.3}$$

$$G(x) = \frac{1}{2}[-B(1)\psi(x_0) + A(i)\psi(x_1) + A(j)\psi(x_2) + A(k)\psi(x_3)]. \tag{6.4}$$

Here, at least one of the constants $B(1), A(i), A(j), A(k)$ is non-zero.

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