ON TOPOLOGICAL STRUCTURE OF SPACES THAT ARE MILDLY CONNECTED-CLEAVABLE OVER THE REALS

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Abstract. We prove the following theorem (8): If X is a nowhere hereditarily disconnected homogeneous space metrizable by a complete metric, and X is cleavable over R along every punctured closed connected subset, then X is locally connected. Using this result, we establish the next theorem (Theorem 15): Suppose that X is an infinite homogeneous connected locally compact metrizable space. Suppose also that X is cleavable over R along every punctured closed connected subset. Then X is homeomorphic to the space R of real numbers.

All spaces considered in this article are assumed to be topological spaces in which every finite subset is closed. However, some of our main results concern separable metrizable spaces. In terminology we follow [6].

One of the first articles in which the general concept of cleavability of one space over another along a given subset was considered is [2]. We refer the reader to this survey for a general discussion of the idea of cleavability and its origins, and for some basic results. Here it is enough to say that cleavability is a certain way to compare topological spaces by means of continuous mappings.

A space X is said to be *cleavable* or *splittable* over a space Y along a subset A of X if there exists a continuous mapping f of X to Y such that the sets $f(A)$ and $f(X \setminus A)$ are disjoint. The last condition is equivalent to the following: $f^{-1}(f(A)) = A$. Further, we say that a space X is *cleavable* over a space Y, if X is cleavable over Y along every subset A of X. Notice that the cleaving mapping f depends on the set A. Of course, if f is a one-to-one continuous mapping of X to Y, then f cleaves X over Y along every subset A of X, and hence, X is cleavable over Y. Having this in mind, we may say that continuous one-to-one mappings present an absolute case of cleavability of one space over another.

It was proved in [1] that if a compact space X is cleavable over the space R of reals, then X topologically embeds into R. Deep results on cleavability of compacta over linearly ordered spaces were obtained by R.Z. Buzyakova [4], [5]. Recently, new interesting results on cleavability and embeddings were obtained by Derrick Stover [10].

The above theorem on cleavability of compacta over R shows that this type of cleavability is a rather strong property. However, we do not know the answer to the following question:

Question 1. *Suppose that X is a connected Tychonoff space cleavable over R. Does it follow that there exists a one-to-one continuous function from X to R?*

Much weaker and more flexible than cleavability of a space over another space along every subset is the condition that X is cleavable over Y along a fixed subset A of X. Observe that every perfectly normal space $X$is obviously cleavable over R along every closed subset

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of X. Therefore, every such space is cleavable over R along every open subset. However, even a metric space needn't be cleavable over R along every locally closed subset (recall that a *locally closed* subset is the intersection of an open set with a closed set).

When we consider cleavability along concrete subsets, we run into many curious questions, some of which are not easy to answer.

For example, the plane R^2 is easily seen not to be cleavable over R along the set Q^2 of points with two rational coordinates. However, the answer to the next question is nor obvious:

Question 2. *Is the space R^3 cleavable over R^2 along the set Q^3 of points with three rational coordinates?*

When we cleave a space X along a given subset A of X, it may be natural to impose additional restrictions on the image of A. For example, we might require $f(A)$ to consist of more than one point, or f restricted to A to be a homeomorphism. We may also impose various restrictions on the subsets along which we want to cleave.

The main goal of this article is to establish some structure theorems on separable spaces metrizable by a complete metric and cleavable over R along a rather narrow family of locally closed connected sets.

We first establish several elementary results on cleavability of spaces over R that will be used to prove the structure theorems.

A nonempty subset A of a space X will be called a *punctured connected* subset of X if $A \cup \{b\}$ is connected, for some $b \in X \setminus A$. If b can be selected in $X \setminus A$ so that $A \cup \{b\}$ is closed in X and connected, then we call A a *punctured closed connected* subset of X.

It is good to keep in mind the next obvious statement.

Proposition 1. *A punctured connected subset of a space X is never closed in X, and every non-closed connected subset of X is punctured connected.*

Lemma 2. *Suppose that A is a punctured connected subset of a space X, and that X is cleavable over R along A. Then $A = U_A \cup F_A$ where U_A is a nonempty open subset of X, and F_A is closed and nowhere dense in X (we fix this notation for the future use).*

Proof. Fix some $b \in X \setminus A$ such that the set $B = A \cup \{b\}$ is connected. Take a continuous mapping f of X to R that cleaves X along A. Notice that A is not closed, since B is connected. Now from the formula $f^{-1}(f(A)) = A$ and the continuity of f it follows that $f(A)$ is not closed in R. Hence, the sets $f(B)$ and $f(A)$ are infinite. The set $f(B)$ is connected, as a continuous image of a connected set B. It follows, since $f(B)$ is infinite, that $f(B)$ is the union of a nonempty open subset V of R and a finite subset P of R. However, either $f(A) = f(B)$ or $f(A) = f(B) \setminus \{f(b)\}$. Therefore, the same is true for $f(A)$:

$$f(A) = V_1 \cup P_1,$$

where V_1 is nonempty and open in R, and P_1 is a finite subset of R. By continuity of f, it follows that A is the union of a nonempty open set W and a closed set F.

Let F_A be the set obtained by subtracting from F the interior of F, and let U_A be the union of W with the interior of F. These sets are exactly what we need. □

Lemma 3. *Let X be a space cleavable over R along every punctured closed connected subset. Then every infinite closed connected subset C of X has a nonempty interior.*

Proof. Take any $b \in C$ and put $A = \{x \in C : x \neq b\}$. Obviously, A is a punctured closed connected set. Therefore, X is cleavable over R along A. Now it follows from Lemma 2 that A contains a nonempty open set. Hence, the interior of C is nonempty. □

Proposition 4. *Suppose that A is a connected subset of a space X which is cleavable over R along every connected subset. Then the set $B = \overline{A} \setminus A$ is closed and discrete in X.*

Proof. Assume the contrary. Then some $c \in X$ is an accumulation point for B. Clearly, $c \in \overline{A}$. Put $C = A \cup \{c\}$. Then C is connected, and there exists a continuous mapping f of X to R cleaving X along A. Put $B_0 = B \setminus \{c\}$. Then $f(B_0)$ doesn't intersect $f(C)$, $f(B_0)$ is contained in the closure of $f(C)$, and $f(B_0)$ accumulates to the point $f(c)$ of $f(C)$. Since $f(C)$ is a connected subset of R, this is clearly impossible. □

A space X will be called *neighbourhood-connected at a point* $a \in X$ if for each open neighbourhood $O(a)$ of a in X there exist an open neighbourhood V of a and a connected subset C such that $V \subset C \subset O(a)$. Such spaces are called in [8] *weakly locally connected at* a. If a space X is neighbourhood-connected at every point, then it is said to be *neighbourhood-connected.*

A space X will be called *network-connected at a point* $a \in X$ if for each open neighbourhood $O(a)$ of a there exists an infinite connected subset C of X such that $a \in C \subset O(a)$. If a space X is network-connected at every point, then X is said to be *network-connected.*

We also need a condition that is much weaker than network-connectedness. A space X is *nowhere hereditarily disconnected* if every nonempty open subset of X contains an infinite connected subset.

Clearly, if a space X is locally connected at some point $a \in X$, then X is also neighbourhood-connected at this point. It is also clear that if a space X is neighbourhood-connected at some non-isolated $a \in X$, then X is also network-connected at a. But it is the converse statements that we are really interested in.

The following statement (see [8]) is easy to prove:

Proposition 5. *If a space X is neighbourhood connected, then X is locally connected.*

The $sin(1/x)$ curve, with the limit closed segment (the topologists sine curve), is network-connected but not locally connected. The existence of such an example is indeed hardly astonishing, since network-connectedness sounds as a much weaker condition than local connectedness.

We will show below that cleavability over R strongly influences the structure of network-connected spaces.

Proposition 6. *Let X be a regular nowhere hereditarily disconnected space, γ be an open covering of X, and M be the set of $x \in X$ such that $x \in O(x) \subset C \subset U \in \gamma$, for some open neighbourhood $O(x)$ of x, some $U \in \gamma$, and some connected subset C. Suppose further that X is cleavable over R along every punctured closed connected subset. Then M is an open dense subset of X.*

Proof. It is clear from the definition of M that M is open: for every $x \in M$, its neighbourhood $O(x)$ mentioned in the definition is also contained in M. Assume that M is not dense in X. Then $G = X \setminus \overline{M}$ is a nonempty open subset of X. Fix $z \in G$, and take $U \in \gamma$ such that $z \in U$. Since X is regular, we can find a nonempty open subset W such that $z \in W \subset \overline{W} \subset G \cap U$. Since X is nowhere hereditarily disconnected, there exists an infinite closed connected subset C such that $C \subset \overline{W} \subset U \cap G$.

It follows from Lemma 3 that there exists a nonempty open set H such that $H \subset C$. Then, by the definition of M, H is a subset of M. On the other hand, by the construction H is clearly disjoint from M. This contradiction completes the proof. □

Now we present the most general of our main results.

Theorem 7. *Suppose that X is a nowhere hereditarily disconnected Moore space with the Baire property. Suppose further that X is cleavable over R along every punctured closed connected subset. Then X is neighbourhood-connected at a dense set of points.*

Proof. Fix a development $\{\gamma_n : n \in \omega\}$ for X. Fix also $n \in \omega$. By Proposition 6, there exists a dense open subset M_n of X such that for every $x \in M_n$ we can find an open neighbourhood $O_n(x)$, a connected subset C_n of X and $U_n \in \gamma_n$ satisfying the following conditions:

$$x \in O_n(x) \subset C_n \subset U_n.$$

Put $Y = \cap\{M_n : n \in \omega\}$. The set Y is dense in X, since X has the Baire property.

Let us show that X is neighbourhood-connected at every point of Y.

Fix $x \in Y$ and $n \in \omega$. Since $x \in M_n$, we can also fix $U_n \in \gamma_n$, a connected subset C_n of X, and an open subset V_n of X such that

$$x \in V_n \subset C_n \subset U_n.$$

According to the definition of neighbourhood-connectedness at a point, it is enough to show that the family $\{C_n : n \in \omega\}$ is a network for X at x.

Let $O(x)$ be any open neighbourhood of x in X. Since $\{\gamma_n : n \in \omega\}$ is a development of X, there exists $n \in \omega$ such that every element of γ_n containing the point x is a subset of $O(x)$. Then $x \in V_n \subset C_n \subset U_n \subset O(x)$. Thus, X is neighbourhood-connected at x. □

Theorem 7 shows that cleavability over R along every punctured closed connected subset is a property that may considerably influence the structure of a space. This provides a motivation for introducing the following concept.

A space X will be called *mildly connected-cleavable* over R if X is cleavable over R along every punctured closed connected subset. Recall that a topological space X is *homogeneous* if every point of X can be brought to any other point of X by a homeomorphism of X onto itself.

Theorem 8. *Suppose that X is a nowhere hereditarily disconnected homogeneous space metrizable by a complete metric and mildly connected-cleavable over R. Then X is locally connected.*

Proof. Notice that the space X has the Baire property, since X is metrizable by a complete metric. Being metrizable, it is also a Moore space. It follows now from Theorem 7 that X is neighbourhood-connected at a dense set of points. Since X is homogeneous, we conclude that X is neighbourhood-connected at every point. It remains to apply Proposition 5. □

We are now going to show that the class of nowhere hereditarily disconnected spaces is quite wide. This will allow to derive several strong corollaries from the above theorem.

First, we introduce a short name for the spaces in this class. We will call a space X *π-connected* if every nonempty open subset U of X contains an infinite connected subset. Clearly, a space X is π-connected if and only if it is nowhere hereditarily disconnected.

Theorem 9. *If X is an infinite open dense subspace of a connected compact Hausdorff space K, then X is π-connected.*

Proof. Assume the contrary. Then, by regularity of K, the space X contains a nonempty open subspace V the closure of which in X is hereditarily disconnected and compact (and therefore, coincides with the closure of V in K). It follows that the space $\overline{V}$ is zero-dimensional, that is, $\overline{V}$ has a base consisting of nonempty open and closed subsets of $\overline{V}$. Some of these open and closed subsets W is contained in V and is open and closed in K as well. Notice that $V \neq K$, since K is connected. It follows that $W \neq K$. Hence, K is disconnected, a contradiction. □

Corollary 10. *Every infinite connected locally compact Hausdorff space X is π-connected.*

Proof. Take any Hausdorff compactification bX of X. Then X is open in bX (see [6]), and bX is connected, since X is dense in bX and connected. It remains to apply Theorem 9. □

Proposition 11. *Suppose that X is the union of the family of its locally connected subspaces none of which has an isolated point (in itself). Then X is π-connected.*

Proof. This statement easily follows from the fact that the union of any family of network-connected spaces is network-connected. □

Theorem 12. *Suppose that f is a continuous mapping of a separable locally connected space X onto a regular connected space Y. Then Y is π-connected.*

Proof. Let M be the set of all points $y \in Y$ such that the interior of $f^{-1}(y)$ in X is nonempty. Then M is countable, since X is separable. Notice that any nonempty subspace of Y is uncountable, since Y is regular and connected [6]. It follows that the set $A = Y \setminus M$ is dense in Y. Clearly, Y is network-connected at each $y \in A$, since X is locally connected and f is continuous. Since A is dense in Y, it follows that the space Y is π-connected. □

By a very similar argument, we can prove the following statement:

Theorem 13. *Suppose that f is a continuous mapping of a separable locally connected space X onto a space Y with the Baire property. Then Y is π-connected.*

For the terminology related to topological groups and their continuous actions on topological spaces, the reader can refer to [3].

Theorem 14. *Suppose that a locally connected separable topological group G acts continuously and transitively on a regular Moore space Y with the Baire property, and that Y is mildly connected-cleavable over R. Then Y is locally connected.*

Proof. The space Y is homogeneous, since G acts transitively on Y. For the same reason, Y is a continuous image of G. Therefore, by Theorem 13, Y is π-connected. Now it follows from Theorem 7 that the space X is neighbourhood-connected at a dense set of points. Since X is homogeneous, we conclude that X is locally connected. □

Several results we present next show how, in fact, strong is the requirement that a space be mildly connected-cleavable over R. The following theorem is one of our main results on the structure of mildly connected-cleavable spaces.

We say that a space is *trivial* if it is either empty or consists of one point.

Theorem 15. *Suppose that X is a homogeneous connected locally compact metrizable space. Suppose also that X is mildly connected-cleavable over R. Then either X is trivial, or X is homeomorphic to the space R of real numbers.*

Proof. Assume that X is non-trivial. Then X is infinite, since it is connected. It follows from Corollary 10 that X is π-connected. Observe that the space X is metrizable by a complete metric, since it is locally compact and metrizable [6]. Therefore, by Theorem 8, X is locally connected. Hence, by a theorem of Hahn-Mazurkiewicz (see [7] p. 256), X is locally arcwise connected. It follows that X contains a subspace P homeomorphic to the closed unit interval $[0, 1]$. By Lemma 3, the interior of P is nonempty. Hence X contains an open subspace U homeomorphic to R. Taking into account that X is homogeneous, we conclude that the space X is locally homeomorphic to the space R at every point, that is, X is a 1-manifold. However, upto a homeomorphism, there are only two non-trivial connected metrizable 1-manifolds: the real line R and the circle S^1 (see a discussion of topological manifolds in [9]). But the circle is not mildly connected-cleavable over R (it is not cleavable over R along any half-open arc, see an argument in [1]). Thus, X is homeomorphic to R. □

Theorem 16. *Suppose that X is an infinite homogeneous connected and π-connected complete metric space. Then X is mildly connected-cleavable over R if and only if X is homeomorphic to R.*

Proof. The sufficiency is obvious. The proof of the necessity is similar to the proof of the preceding theorem. □

We will call a space X *mildly locally connected at a point* $a \in X$ if there exists an infinite connected locally connected subspace A of X such that $a \in \overline{A}$.

Corollary 17. *Suppose that X is a nonempty homogeneous connected complete metric space, and that X is mildly locally connected at some point. Then X is mildly connected-cleavable over R if and only if X is homeomorphic to R.*

Proof. Since X is homogeneous, X is mildly locally connected at each point. It is easy to derive from this that X is π-connected. It remains to observe that X is infinite, and to apply Theorem 16. □

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