UNIQUENESS OF VON NEUMANN BORNOLOGY IN LOCALLY C^*-ALGEBRAS. A BORNOLOGICAL ANALOGUE OF JOHNSON'S THEOREM

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Abstract. All locally C^*- structures on a commutative complex algebra have the same bound structure. It is also shown that a Mackey complete C^*-convex algebra is semisimple.

By the well-known Johnson's theorem [4], there is on a given complex semi-simple algebra a unique (up to an isomorphism) Banach algebra norm. R. C. Carpenter extended this result to commutative Fréchet locally m-convex algebras [3]. Without metrizability, it is not any more valid even in the rich context of locally C^*-convex algebras. Below there are given telling examples where even a C^*-algebra structure is involved.

We follow the terminology of [5], pp. 101-102. Let E be an involutive algebra and p a vector space seminorm on E. We say that p is a C^*-seminorm if $p(x^*x) = [\ p(x)]^2$, for every x. An involutive topological algebra whose topology is defined by a (saturated) family of C^*-seminorms is called a C^*-convex algebra. A complete C^*-convex algebra is called a locally C^*-algebra (by Inoue). A Fréchet C^*-*convex algebra* is a metrizable C^*-convex algebra, that is equivalently a metrizable locally C^*-algebra, or also a Fréchet locally C^*-algebra.

All the bornological notions can be found in [6]. The references for m-convexity are [5], [8] and [9]. Let us recall for convenience that the bounded structure (bornology) of a locally convex algebra (*l.c.a.*) (E, τ) is the collection $\mathbb{B}\tau$ of all the subsets B of E which are bounded in the sense of Kolmogorov and von Neumann, that is B is absorbed by every neighborhood of the origin (see e.g. [7], p. 108, Definition 2 and the following comments therein).

In [2], G. R. Allan proved the uniqueness of the pseudo-Banach structure in GB^*-algebras. Here we consider the entire (von Neumann) bound structure in the frame of locally C^*-algebras. In this context, we do have an analogue of Johnson's theorem. It is shown that on a commutative complex algebra E there is a unique bound structure relatively to all locally C^*-structures on E (cf. the Theorem below). Locally C^*-algebras are complete, by definition. It is then worth to notice that the Theorem is actually obtained under the weaker Mackey completeness condition.

Example 1. Let Ω be the first non countable ordinal and endow the set $[0, \Omega)$ with the order topology. Consider $C([0, \Omega))$ the complex algebra of continuous functions, on $[0, \Omega)$, endowed with the topology of uniform convergence on compacta. It is a commutative locally C^*-convex algebra. Endowed with the norm $\|.\|_\infty$, it is a C^*-algebra. The two structures are not equivalent.

Example 2. Let $C([0, 1])$ be the algebra of complex continuous functions on the interval $[0, 1]$. Endowed with the topology of uniform convergence on denumerable compact subsets

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Remark 1. In the previous (commutative) examples the two topologies have the same bounded sets. This is actually a general fact as we will see, even under a weaker completeness condition. Thus in the proof of the following proposition, we are led to specify Theorem 2.1 of [10].

Recall that an *l.c.s.* (E,τ) is *Mackey complete* (M-complete) if its bounded structure $\mathbb{B}\tau$ admits a fundamental system $\mathcal{B}$ of Banach discs (disc "*complétant*"); that is, for every B in $\mathcal{B}$, the vector space generated by B is a Banach space when endowed with the gauge $\|.\|_B$ of B.

Theorem. On a complex commutative algebra E all M-complete C^*-convex structures have the same bornology.

Proof. Suppose that (E,τ) and $(E,\tau^{'})$ are M-complete C^*-convex algebras. Let $(|.|_\lambda)_\lambda$ and $(|.|_\alpha)_\alpha$ be families of C^*-seminorms defining the topologies τ and $\tau^{'}$ respectively. By a result of Sebestyén ([11]), every such seminorm is submultiplicative. Now for $(B_i)_i$ a basis of the bornology of (E,τ) put, for every B_i and every λ, $|B_i|_\lambda = \sup\{|x|_\lambda : x \in B_i\}$ and $\Lambda_n^i = \{\lambda \in \Lambda : |B_i|_\lambda < n\}$. One has $\Lambda = \cup\{\Lambda_n^i : n = 1,2,...\}$. Now, for every n, put $q_n^i(x) = \sup\{|x|_\lambda : \lambda \in \Lambda_n^i\}$ and $E_i = \{x : q_n^i(x) < \infty; n = 1,2,...\}$. Then $(E_i,(q_n^i)_n)$ is a commutative Fréchet locally C^*-algebra. Moreover, $(E,\mathbb{B}\tau)$ is the bornological inductive limit of these algebras $(E_i,(q_n^i)_n)$. Now the restriction of any $|.|_\alpha$ to every E_i is, of course, a C^*-seminorm. Thus it is continuous (cf. [5], Corollary 28.14, p. 360) and so bounded. Hence $\mathbb{B}\tau \subset \mathbb{B}\tau^{'}$. The inverse inclusion is proved in the same way.

Remark 2. In Johnson's theorem semisimplicity is essential. Though the latter is not apparent in our theorem, it is actually inherent therein: It is known that a locally C^*-algebra is necessarily semisimple. We remark that already M-completeness is sufficient. Indeed such an algebra is pseudo-complete with a continuous inverse map $x \longmapsto x^{-1}$. So we can use the expression of the spectral radius via the seminorms as in [1]. If x is a hermitian element, then

$$\left|x^2\right|_\lambda = |x^*x|_\lambda = |x|_\lambda^2, \quad for \;\; every \;\; \lambda.$$

Whence,

$$\rho(x) = \sup_\lambda \lim_n \sup[|x^n|_\lambda]^{\frac{1}{n}} = \sup_\lambda |x|_\lambda .$$

Now, if x is in the (Jacobson) radical, then $\rho(x) = 0$, whence $x = 0$. If x is not hermitian, consider as usual x^*x which is also in the radical.

Next we present a consequence of the previous theorem without any completeness.

Corollary. Let (E,τ) be a complex commutative algebra. Then all Hausdorff C^*-convex structures on it, having the same bounded Cauchy nets, have the same bounded structure.

Remark 3. Let E be a commutative complex C^*-algebra. Then its bounded structure is the only one for any other locally C^*-algebra structure on it. This is the case for the examples $C([0,1])$ and $C([0,\Omega))$ above.

Remark 4. As a matter of application, the locally C^*-algebra ΠE_i, the standard product of C^*-algebras, admits a unique bounded structure associated to each one of its locally C^*-algebra structures . It is the product bornology.

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