# A MARKOV CHAIN-BASED MODEL FOR ACTOMYOSIN DYNAMICS

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Abstract. We propose a stochastic model based on a continuous-time Markov chain to describe certain features of the displacements performed by single myosin heads along actin filaments during the chemical cycle of an ATP molecule. We obtain an expression of the transient probabilities of the model and determine a continuous approximation of the chain by a Wiener-type diffusion process subject to jumps and restricted between two reflecting boundaries.

**1 Introduction** By the present paper we intend to pay a tribute to the cherished memory of Professor Tadashige Ishihara. We believe that the choice of this topic is particularly appropriate, as it falls well within the area of his wide interests and scientific expertise. These include applications of mathematics to biology and to operations research (see, for instance, Ishihara [12], [13], [14], [15], Ishihara and Sato [16], [20]) in which he has left an indelibly positive mark. We would like to mention that Prof. Ishihara's interest in biomathematics and related computational problems was also witnessed by his enthusiastic agreement to edit special volumes of Scientiae Mathematicae Japonicae dedicated to collections of peer reviewed articles presented in some international conferences dealing with current topics of biomathematics and its applications ([22], [23], [24]). We are very thankful to Prof. Ishihara for his being over the years a mentor and an invaluable advisor for us as well as for a group of our colleagues at the Naples Federico II University and at the University of Salerno. This paper is meant to be an expression of our deep gratitude for his scientific advice, encouragement and patronage.

As is well-known, $M/M/1$ queueing systems have long been the object of thorough systematic investigations. Among these, a particular attention appears to have been directed to the mathematical analysis of such systems in which, in addition, the possibility of the so-called "catastrophe" is included. Essentially, such a possibility consists of adding to the customary hypotheses the existence of a non-zero probability that the system enters a special condition: either a sudden vanishing of the queue or its transition to an intermediate particular state from which the zero state can successively be reached at some randomly distributed instants. In a previous paper [8] an $M/M/1$ model in the presence of catastrophes has been proposed, and transition probabilities, busy period density and waiting times for the occurrence of catastrophes have been explicitly obtained. An heavy-traffic approximation has also been determined, which leads to a Wiener process subject to randomly distributed jumps. The possibility of exploiting mathematical models of such a kind in a variety of areas within the realm of science and technology has also been realized in the literature, as well as its appropriateness for the description of certain problems within the context of population dynamics (see, for instance, [1], [5] and [25]).

Making us of the theory of $M/M/1$ queueing systems with catastrophes, in the present paper we propose and discuss a mathematical model suggested by a problem of high current

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interest: the interaction between a Myosin II (simply “myosin” in the sequel) head and an actin filament, which are part of the biological motor responsible for the generation of forces during muscle contraction processes. Here, we are in the presence of a protein motor whose dynamics appears to be based on an intriguing, apparently unrealistic phenomenon: the harnessing of a part of the heat bath energy randomly generated within the natural medium (mostly an aqueous solution) in which these motors are immersed, such as to make it synergetically contribute to an efficient generation of the output of the motor. In recent years, thanks to highly sophisticated measurements and monitoring systems [19], it has been possible to accurately monitor the sliding of a myosin head along an actin filament. It has thus been proved that such a sliding occurs by sequences of small “steps”, each a few nanometers long and time randomly distributed, followed by sudden resets time exponentially distributed. Such conclusions, thanks also to a well-known article [6] have successively been tested in numerous other laboratories, where somewhat similar behaviors have been observed even for other types of biological motors. It should be recalled that in the dynamics of motor proteins an essential role is played by ATP (Adenosine triphosphates) molecules that are present in the environment in which the motor is requested to operate. Indeed, these molecules are the source of the fuel used by the motor in order to work. In particular, with reference to myosin, a specific site has been localized which is apt to host a single ATP molecule whose chemical cycle is synchronized with the various phases through which the motor operates. A synthetic outline of the occurring events can be summarized as follows:

1. The myosin head is rigidly attached to the actin filament (“Rigor” state);
2. One single ATP molecule binds on myosin, whose head then detaches from the filament;
3. The ATP molecule hydrolizes into ADP (Adenosine diphosphates) + Pi with the consequent store of a certain amount of energy; the myosin head undergoes a number of attachment-detachment processes from the actin filament in correspondence to existing actin-binding sites (hereafter referred to as the “rising phase”);
4. While the myosin head is attached at an actin-binding site, a drastic conformational change may occur, known as “power stroke”, consisting of a sudden reorientation of part of myosin head;
5. The loss of the Pi anion then occurs, which completes the sliding;
6. The ADP molecule is released, the myosin ATP site thus becoming empty, and the system is reset.

With reference to the above synthesized phenomena, the sliding of the myosin head on the actin filament modeled as a Brownian particle in a periodic, elastic-type potential subject to tilting has been discussed for instance in [4] and [2], whereas a model consisting of a Wiener process perturbed by Gaussian-distributed jumps has been considered in [3].

By a totally different approach, a Markov chain-based model will be discussed in the sequel, which is based on the $M/M/1/K$-catastrophe paradigm. In such a model the states $0, 1, \ldots, k$ will represent the possible integer-valued positions of the myosin head during the rising phase (see 3. above). These are complemented by two additional states $R$ (*Rigor*) and $C$ (*Conformational Change*) representing the conditions specified in 1. and 5. above, respectively.

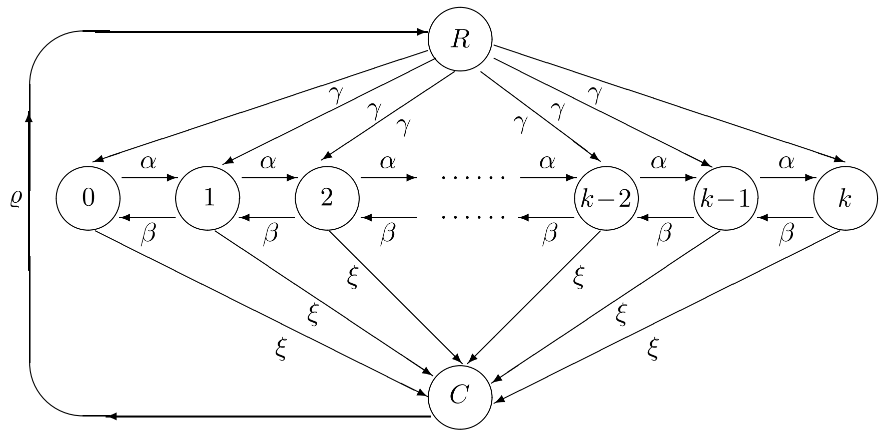


Figure 1: State diagram of the Markov chain modeling actomyosin dynamics.

The probabilities of states $R$, $C$ and of positions $0, 1, \dots, k$ of the myosin head will be calculated in Section 2, as well as the time probability density of the first attainment of state $C$, namely of the ending process originated by the current ATP molecule.

A continuous approximation to the discrete model will finally be worked out and thoroughly discussed in Section 3.

**2 A discrete model** We shall view the process responsible for the sliding of the myosin head along the actin filament as the continuous-time Markov chain sketched in Fig. 1. Integers $0, 1, \dots, k$ represent the different states occupied by the myosin head during the rising phase (see, 3. above), state $C$ is occupied when the power stroke takes place, and state $R$ corresponds to the phase when the myosin head is rigidly attached to the actin filament.

Parameters $\alpha$ and $\beta$ denote forward and backward transition rates of the myosin head during the rising phase, respectively; $\xi$ is the rate at which the rising phase ends and the chain enters state $C$, whereas $\varrho$ is the rate at which the chain leaves state $C$ to enter state $R$. Finally, $\gamma$ is the rate at which from state $R$ a transition to one of states $0, 1, \dots, k$ occurs, modeling the beginning of the rising phase.

This leads us to define a stochastic process $\{Z(t),\ t \geq 0\}$ with space-state $\{-1, 0, 1\}$ where:
$Z(t) = -1$, if at time $t$ the chain is in state $R$;
$Z(t) = 0$, if at time $t$ the chain is in any of states $0, 1, \dots, k$;
$Z(t) = 1$, if at time $t$ the chain is in state $C$.

The assumed initial condition is $Z(0) = -1$, so that at initial time the chain is in state $R$. The following probabilities

$$u_i(t) = P\{Z(t) = i \mid Z(0) = -1\} \qquad (i = -1, 0, 1)$$

are then associated to $Z(t)$. For $t > 0$ they satisfy the following system, where we have set

$\nu := (k+1)\,\gamma$:

$$\begin{cases} \dfrac{d}{dt}u_{-1}(t) = -\nu\, u_{-1}(t) + \varrho\, u_1(t), \\ \dfrac{d}{dt}u_0(t) = -\xi\, u_0(t) + \nu\, u_{-1}(t), \\ \dfrac{d}{dt}u_1(t) = -\varrho\, u_1(t) + \xi\, u_0(t), \end{cases} \tag{2.1}$$

with initial condition

$$u_i(0) = \begin{cases} 1, & \text{if } i=-1 \\ 0, & \text{otherwise.} \end{cases} \tag{2.2}$$

The above scheme can also be viewed as a simplified version of a three-state model studied in Shimokawa *et al.* [21] that describes the chemical reactions between a myosin head and a neighboring actin filament.

Let now

$$r_i = \lim_{t\to+\infty} u_i(t) \qquad (i=-1,0,1)$$

be the steady-state probabilities. Making use of (2.1) one obtains:

$$r_{-1} = \frac{\varrho\,\xi}{\nu\,\varrho+\nu\,\xi+\varrho\,\xi}, \quad r_0 = \frac{\nu\,\varrho}{\nu\,\varrho+\nu\,\xi+\varrho\,\xi}, \quad r_1 = \frac{\nu\,\xi}{\nu\,\varrho+\nu\,\xi+\varrho\,\xi}\,. \tag{2.3}$$

To determine the transient solution of system (2.1) with condition (2.2), we denote by $x_1$ and $x_2$ (with $x_2 < x_1 < 0$) the solutions of the following equation:

$$x^2 + (\nu+\varrho+\xi)\,x + \nu\,\varrho + \nu\,\xi + \varrho\,\xi = 0$$

and set

$$\Delta := (\nu-\varrho-\xi)^2 - 4\,\varrho\,\xi. \tag{2.4}$$

**Proposition 2.1** *For $t \geq 0$ the following holds:*
*(i) If $\Delta > 0$,*

$$\begin{aligned} u_{-1}(t) &= r_{-1} + \frac{\nu}{x_1-x_2}\left\{-\frac{\varrho+\xi+x_1}{x_1}\,e^{x_1 t} + \frac{\varrho+\xi+x_2}{x_2}\,e^{x_2 t}\right\}, \\ u_0(t) &= r_0 + \frac{\nu}{x_1-x_2}\left\{\frac{\varrho+x_1}{x_1}\,e^{x_1 t} - \frac{\varrho+x_2}{x_2}\,e^{x_2 t}\right\}, \\ u_1(t) &= r_1 + \frac{\nu\,\xi}{x_1-x_2}\left\{\frac{1}{x_1}\,e^{x_1 t} - \frac{1}{x_2}\,e^{x_2 t}\right\}; \end{aligned}$$

*(ii) If $\Delta = 0$,*

$$\begin{aligned} u_{-1}(t) &= r_{-1} + \frac{\nu}{x_1}\,e^{x_1 t}\left\{\frac{\varrho+\xi}{x_1} - (\varrho+\xi+x_1)t\right\}, \\ u_0(t) &= r_0 + \frac{\nu}{x_1}\,e^{x_1 t}\left\{-\frac{\varrho}{x_1} + (\varrho+x_1)t\right\}, \\ u_1(t) &= r_1 + \frac{\nu\,\xi}{x_1}\,e^{x_1 t}\left\{-\frac{1}{x_1} + t\right\}; \end{aligned}$$

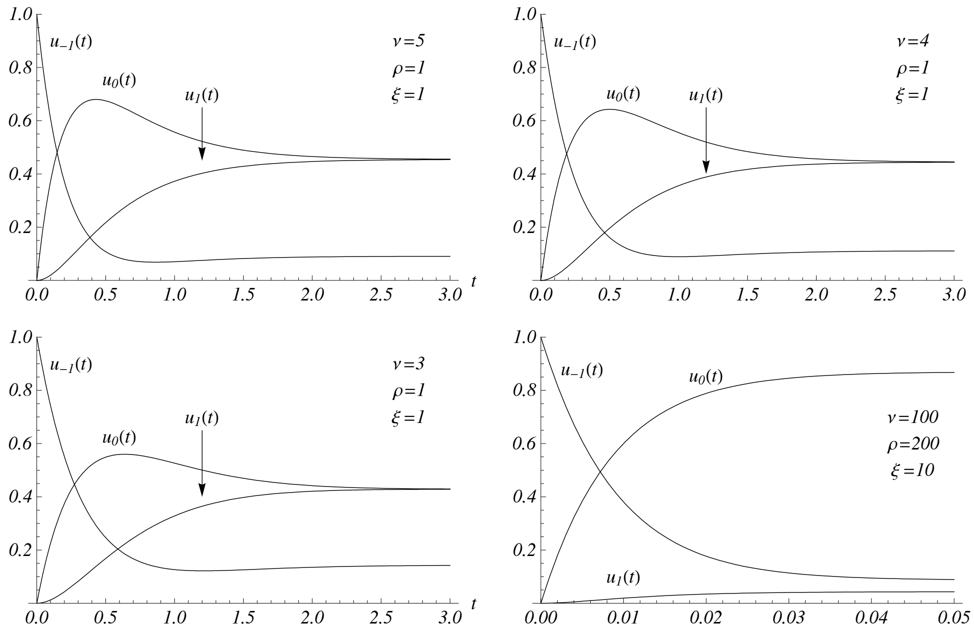


Figure 2: Plots of functions $u_i(t)$ for the indicated parameters choices.

*(iii) If* $\Delta < 0$,

$$u_{-1}(t) = r_{-1} + \frac{\nu\, e^{at}}{\nu\,\varrho + \nu\,\xi + \varrho\,\xi}\left\{(\varrho+\xi)\cos(b\,t) - \frac{1}{b}\Big[(\varrho+\xi)(\nu+a) + \varrho\,\xi\Big]\sin(b\,t)\right\},$$

$$u_0(t) = r_0 + \frac{\nu\, e^{at}}{\nu\,\varrho + \nu\,\xi + \varrho\,\xi}\left\{-\varrho\cos(b\,t) + \Big[\varrho(\nu+\xi+a) + \xi\,\nu\Big]\sin(b\,t)\right\},$$

$$u_1(t) = r_1 + \frac{\nu\,\xi\, e^{at}}{\nu\,\varrho + \nu\,\xi + \varrho\,\xi}\left\{-\cos(b\,t) + \frac{a}{b}\sin(b\,t)\right\},$$

*where*

$$a = -\frac{\nu+\varrho+\xi}{2}, \qquad b = \frac{\sqrt{|\Delta|}}{2}. \tag{2.5}$$

The proof of Proposition 2.1 is omitted, as it follows by standard techniques and from condition $u_{-1}(t) + u_0(t) + u_1(t) = 1$, $t \geq 0$. Plots of functions $u_i(t)$ are shown in Fig. 2. The parameterization of the fourth plot is adherent to biologically acceptable values.

Note that probabilities $u_i(t)$ in the case $\Delta < 0$ exhibit time-oscillating terms. However, quite similarly to a phenomenon disclosed in [7], the parameters choices leading to $\Delta < 0$ make predominant the exponential damping effects, so that no oscillation is disclosed by the numerical computations.

We denote by $\mathcal{T}$ the continuous random variable

$$\mathcal{T} = \inf\{t > 0 : Z(t) = 1\},$$

that describes the time in which the rising phase ends and the chain enters state $C$ for the first time, namely when the power stroke occurs. In order to obtain the probability density of $\mathcal{T}$ we set an absorbing boundary in state $C$. (See Fig. 1.) In other terms, we assume that $\varrho \downarrow 0$ in the system (2.1) with condition (2.2). We thus have:

$$P(\mathcal{T} \le t) = \lim_{\varrho \downarrow 0} u_1(t).$$

From Proposition 2.1 we note that if $\varrho \downarrow 0$ only cases (i) and (ii) can occur. Hence, due to Proposition 2.1 the probability density of $\mathcal{T}$ is given by:

$$g(t) = \frac{d}{dt} P(\mathcal{T} \le t) = \begin{cases} \nu\,\xi\, \dfrac{e^{-\nu\,t} - e^{-\xi\,t}}{\xi - \nu}, & \text{if } \xi \ne \nu, \\ \nu^2\, t\, e^{-\nu\,t}, & \text{if } \xi = \nu. \end{cases} \tag{2.6}$$

As is expected, $\mathcal{T}$ is the sum of two independent exponentially distributed random variables having means $\nu^{-1}$ and $\xi^{-1}$, so that the mean and the coefficient of variation of $\mathcal{T}$ are given by:

$$E(\mathcal{T}) = \frac{1}{\nu} + \frac{1}{\xi}, \qquad CV(\mathcal{T}) = \frac{\sqrt{\nu^2 + \xi^2}}{\nu + \xi}.$$

We shall now describe the state of the process when the myosin head is in the rising phase. To this purpose, we analyze the properties of the state $Z(t) = 0$ by introducing another stochastic process $\{N(t),\ t \ge 0\}$ with state-space $\{0, 1, \ldots, k\}$. The integers $0, 1, \ldots, k$ represent the possible positions of the myosin head during the discretely approximated rising phase. By setting

$$p_n(t) = P\{N(t) = n \mid Z(0) = -1\} \qquad (n = 0, 1, \ldots, k)$$

we have:

$$u_0(t) = \sum_{n=0}^{k} p_n(t).$$

For $t > 0$, the probabilities $p_n(t)$ satisfy the forward equations:

$$\begin{cases} \dfrac{d}{dt} p_0(t) = -(\alpha + \xi)\, p_0(t) + \beta\, p_1(t) + \gamma\, u_{-1}(t), \\ \dfrac{d}{dt} p_n(t) = -(\alpha + \beta + \xi)\, p_n(t) + \alpha\, p_{n-1}(t) + \beta\, p_{n+1}(t) + \gamma\, u_{-1}(t), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (n = 1, 2, \ldots, k-1)\,, \\ \dfrac{d}{dt} p_k(t) = -(\beta + \xi)\, p_k(t) + \alpha\, p_{k-1}(t) + \gamma\, u_{-1}(t), \end{cases} \tag{2.7}$$

with the initial conditions:

$$p_n(0) = 0 \qquad (n = 0, 1, \ldots, k). \tag{2.8}$$

Let us now define the steady-state probabilities:

$$q_n := \lim_{t \to +\infty} p_n(t) \qquad (n = 0, 1, \ldots, k).$$

Making use of system (2.7) we have:

$$q_n = \frac{\gamma\, r_{-1}}{\xi} \left\{ 1 - \frac{\alpha - \beta}{\xi}\, \frac{(z_1 - 1)\,(1 - z_2^{k+1})\, z_1^{k-n} - (1 - z_2)\,(z_1^{k+1} - 1)\, z_2^{k-n}}{z_1^{k+1} - z_2^{k+1}} \right\} \qquad (n = 0, 1, \ldots, k), \tag{2.9}$$

where $z_1$ and $z_2$ $(0 < z_2 < 1 < z_1)$ are the solutions of the equation

$$\alpha\, z^2 - (\alpha + \beta + \xi)\, z + \beta = 0.$$

In order to determine certain useful relations involving $p_n(t)$, let us define the Laplace transforms of $u_i(t)$ and $p_n(t)$, with $\lambda > 0$:

$$U_i(\lambda) = \int_0^{+\infty} e^{-\lambda\, t}\, u_i(t)\, dt, \qquad i = -1, 0, 1,$$

$$\Pi_n(\lambda) = \int_0^{+\infty} e^{-\lambda\, t}\, p_n(t)\, dt, \qquad n = 0, 1, \dots, k.$$

First of all, from (2.1) and (2.2) one obtains:

$$U_{-1}(\lambda) = \frac{(\lambda+\xi)\,(\lambda+\varrho)}{\lambda\, D(\lambda)}, \qquad U_0(\lambda) = \frac{\nu\,(\lambda+\varrho)}{\lambda\, D(\lambda)}, \qquad U_1(\lambda) = \frac{\nu\, \xi}{\lambda\, D(\lambda)}, \tag{2.10}$$

where

$$D(\lambda) = (\lambda+\nu)(\lambda+\varrho+\xi) + \varrho\, \xi. \tag{2.11}$$

**Proposition 2.2** *One has*

$$\Pi_n(\lambda) = \frac{U_0(\lambda)}{k+1} + (\beta - \alpha)\, \frac{U_0(\lambda)}{k+1}\, H_n(\lambda), \qquad n = 0, 1, \dots, k, \tag{2.12}$$

*where*

(2.13)
$$H_n(\lambda) = \frac{1}{\lambda+\xi}\, \frac{[\Psi_1(\lambda)-1]\left[1-\Psi_2^{k+1}(\lambda)\right] \Psi_1^{k-n}(\lambda) - [1-\Psi_2(\lambda)]\left[\Psi_1^{k+1}(\lambda)-1\right] \Psi_2^{k-n}(\lambda)}{\Psi_1^{k+1}(\lambda) - \Psi_2^{k+1}(\lambda)},$$

*and where $\Psi_1(\lambda)$ and $\Psi_2(\lambda)$ are solutions of*

$$\alpha\, x^2 - (\lambda + \alpha + \beta + \xi)\, x + \beta = 0,$$

*with $0 < \Psi_2(\lambda) < 1 < \Psi_1(\lambda)$.*

**Proof.** Applying the Laplace transform to (2.7), with initial condition (2.8), we obtain the following system:

$$\begin{cases} (\lambda+\alpha+\xi)\,\Pi_0(\lambda) = \beta\,\Pi_1(\lambda) + \gamma\, U_{-1}(\lambda), \\ (\lambda+\alpha+\beta+\xi)\,\Pi_n(\lambda) = \alpha\,\Pi_{n-1}(\lambda) + \beta\,\Pi_{n+1}(\lambda) + \gamma\, U_{-1}(\lambda) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (n = 1, 2, \dots, k-1), \\ (\lambda+\beta+\xi)\Pi_k(\lambda) = \alpha\,\Pi_{k-1}(\lambda) + \gamma\, U_{-1}(\lambda). \end{cases} \tag{2.14}$$

The general solution of the second of (2.14) is

(2.15)
$$\Pi_n(\lambda) = \frac{\gamma}{\lambda+\xi}\, U_{-1}(\lambda) + A(\lambda)\,[\Psi_1(\lambda)]^{-n} + B(\lambda)\,[\Psi_2(\lambda)]^{-n} \qquad (n = 1, 2, \dots, k-1).$$

Using (2.15) in the first and the last equation of (2.14) we obtain:

$$A(\lambda) = \frac{\gamma\,(\beta-\alpha)}{(\lambda+\xi)^2}\,U_{-1}(\lambda)\,\frac{[\Psi_1(\lambda)-1]\left[1-\Psi_2^{k+1}(\lambda)\right]\Psi_1^k(\lambda)}{\Psi_1^{k+1}(\lambda)-\Psi_2^{k+1}(\lambda)},$$

$$B(\lambda) = \frac{\gamma\,(\alpha-\beta)}{(\lambda+\xi)^2}\,U_{-1}(\lambda)\,\frac{[1-\Psi_2(\lambda)]\left[\Psi_1^{k+1}(\lambda)-1\right]\Psi_2^k(\lambda)}{\Psi_1^{k+1}(\lambda)-\Psi_2^{k+1}(\lambda)}.$$

Substituting $A(\lambda)$ and $B(\lambda)$ in (2.15) we have:

$$\Pi_n(\lambda) = \frac{\gamma}{\lambda+\xi}\,U_{-1}(\lambda) + \frac{\gamma\,(\beta-\alpha)}{\lambda+\xi}\,U_{-1}(\lambda)\,H_n(\lambda), \qquad n=0,1,\dots,k, \tag{2.16}$$

where $H_n(\lambda)$ is given in (2.13). Recalling that $\gamma=\nu/(k+1)$, by virtue of (2.10) and (2.11) one has:

$$\frac{\gamma}{\lambda+\xi}\,U_{-1}(\lambda) = \frac{1}{k+1}\,\frac{\nu\,(\lambda+\varrho)}{\lambda\,D(\lambda)} = \frac{1}{k+1}\,U_0(\lambda),$$

so that (2.12) immediately follows from (2.16). ♢

**Proposition 2.3** *For all $t>0$ we have:*

$$p_n(t) = \frac{u_0(t)}{k+1} + (\beta-\alpha)\int_0^t \frac{u_0(\tau)}{k+1}\,h_n(t-\tau)\,d\tau, \qquad n=0,1,\dots,k, \tag{2.17}$$

*where*

$$h_n(t) = e^{-(\alpha+\beta+\xi)t}\sum_{i=0}^{k}\left(\frac{\alpha}{\beta}\right)^{(n-i)/2}\sum_{r=-\infty}^{\infty}\left[I_{2(k+1)r+n+i}(2t\sqrt{\alpha\beta}) - I_{2(k+1)r+n+i+2}(2t\sqrt{\alpha\beta})\right] \tag{2.18}$$
$$(n=0,1,\dots,k)$$

*is the inverse Laplace transform of (2.13) and where $I_m(\cdot)$ denotes the modified Bessel function of the first kind.*

**Proof.** Recalling Proposition 2.2, we point out that

$$[\Psi_1(\lambda)-1][1-\Psi_2(\lambda)] = \frac{\lambda+\xi}{\alpha}, \qquad \Psi_1(\lambda)\,\Psi_2(\lambda) = \frac{\beta}{\alpha}.$$

Using such identities in (2.13), we obtain:

$$\begin{aligned} H_n(\lambda) &= \frac{1}{\alpha}\left[\frac{\Psi_1^{k-n}(\lambda)}{\Psi_1^{k+1}(\lambda)-\Psi_2^{k+1}(\lambda)}\sum_{i=0}^{k}\Psi_2^i(\lambda) - \frac{\Psi_2^{k-n}(\lambda)}{\Psi_1^{k+1}(\lambda)-\Psi_2^{k+1}(\lambda)}\sum_{i=0}^{k}\Psi_1^i(\lambda)\right] \\ &= \frac{1}{\alpha}\left\{\sum_{r=0}^{\infty}\sum_{i=0}^{k}\left(\frac{\beta}{\alpha}\right)^{(k+1)r+i}[\Psi_1(\lambda)]^{-[2(k+1)r+n+i+1]}\right. \\ &\qquad \left.+\sum_{r=0}^{\infty}\sum_{i=0}^{k}\left(\frac{\beta}{\alpha}\right)^{(k+1)(r+1)-n-1}[\Psi_1(\lambda)]^{-[2(k+1)(r+1)-n-i-1]}\right\}. \end{aligned} \tag{2.19}$$

Since (see, for instance, Eq. (49) of [10], p. 237)

$$\int_0^{+\infty} e^{-\lambda t}\left(\frac{\beta}{\alpha}\right)^{-j/2}\frac{j}{t}\,e^{-(\alpha+\beta+\xi)t}\,I_j(2t\sqrt{\alpha\beta})\,dt = [\Psi_1(\lambda)]^{-j},$$

via the inverse Laplace transform of (2.19) and recalling that $I_{-m}(z) = I_m(z)$, after some calculations we obtain

$$h_n(t) = \frac{e^{-(\alpha+\beta+\xi)t}}{\alpha\, t} \sum_{i=0}^{k} \left(\frac{\alpha}{\beta}\right)^{(n+1-i)/2} \sum_{r=-\infty}^{\infty} [2(k+1)r + n + i + 1]\, I_{2(k+1)r+n+i+1}(2t\sqrt{\alpha\beta}).$$

Finally, making use of identity (cf. Eq. (23) of [9], p. 79)

$$2m\, I_m(z) = z\, [I_{m-1}(z) - I_{m+1}(z)]$$

we are led to (2.18). ♢

We remark that in the symmetric case $\alpha = \beta$ Eq. (2.17) becomes

$$p_n(t) = \frac{u_0(t)}{k+1}, \qquad n = 0, 1, \ldots, k.$$

**3 A continuous approximation** We now consider a continuous approximation to the discrete stochastic process that describes dynamics of the myosin head. To this end, in the discrete model we perform the following substitutions, with $\varepsilon > 0$:

$$k = \frac{\widehat{k}}{\varepsilon}, \qquad \alpha = \frac{\widehat{\alpha}}{\varepsilon} + \frac{\sigma^2}{2\varepsilon^2}, \qquad \beta = \frac{\widehat{\beta}}{\varepsilon} + \frac{\sigma^2}{2\varepsilon^2}, \qquad \gamma = \widehat{\gamma}\, \varepsilon. \tag{3.1}$$

The parameter $\varepsilon$ appearing above can be considered as a measure of the size of $\widehat{k}$. It plays a crucial role in the approximating procedure indicated below. We set:

$$\widehat{u}_i(t) = \lim_{\varepsilon \to 0^+} u_i(t)\Big|_{k=\frac{\widehat{k}}{\varepsilon},\, \gamma=\widehat{\gamma}\,\varepsilon,\, \alpha=\frac{\widehat{\alpha}}{\varepsilon}+\frac{\sigma^2}{2\varepsilon^2},\, \beta=\frac{\widehat{\beta}}{\varepsilon}+\frac{\sigma^2}{2\varepsilon^2}} \qquad (i = -1, 0, 1), \tag{3.2}$$

with $\widehat{u}_{-1}(t) + \widehat{u}_0(t) + \widehat{u}_1(t) = 1$, for $t \geq 0$, and

$$\widehat{r}_i = \lim_{t \to +\infty} \widehat{u}_i(t) \qquad (i = -1, 0, 1)$$

denoting the steady-state probabilities. Making use of (2.3) and (3.1) one obtains:

$$\widehat{r}_{-1} = \frac{\varrho\, \xi}{\widehat{\nu}\, \varrho + \widehat{\nu}\, \xi + \varrho\, \xi}, \quad \widehat{r}_0 = \frac{\widehat{\nu}\, \varrho}{\widehat{\nu}\, \varrho + \widehat{\nu}\, \xi + \varrho\, \xi}, \quad \widehat{r}_1 = \frac{\widehat{\nu}\, \xi}{\widehat{\nu}\, \varrho + \widehat{\nu}\, \xi + \varrho\, \xi}, \tag{3.3}$$

where we have set $\widehat{\nu} = \widehat{k}\, \widehat{\gamma}$. To determine $\widehat{u}_i(t)$ $(i = -1, 0, 1)$ we denote by $\widehat{x}_1$ and $\widehat{x}_2$ (with $\widehat{x}_2 < \widehat{x}_1 < 0$) the solutions of the following equation:

$$x^2 + (\widehat{\nu} + \varrho + \xi)\, x + \widehat{\nu}\, \varrho + \widehat{\nu}\, \xi + \varrho\, \xi = 0$$

and set

$$\widehat{\Delta} := (\widehat{\nu} - \varrho - \xi)^2 - 4\, \varrho\, \xi. \tag{3.4}$$

Making use of (3.1) from Proposition 2.1 one obtains the following

**Proposition 3.1** *For $t \geq 0$ one has:*
*(i) If $\widehat{\Delta} > 0$,*

$$\widehat{u}_{-1}(t) = \widehat{r}_{-1} + \frac{\widehat{\nu}}{\widehat{x}_1 - \widehat{x}_2} \left\{ -\frac{\varrho + \xi + \widehat{x}_1}{\widehat{x}_1}\, e^{\widehat{x}_1 t} + \frac{\varrho + \xi + \widehat{x}_2}{\widehat{x}_2}\, e^{\widehat{x}_2 t} \right\},$$

$$\widehat{u}_0(t) = \widehat{r}_0 + \frac{\widehat{\nu}}{\widehat{x}_1 - \widehat{x}_2} \left\{ \frac{\varrho + \widehat{x}_1}{\widehat{x}_1}\, e^{\widehat{x}_1 t} - \frac{\varrho + \widehat{x}_2}{\widehat{x}_2}\, e^{\widehat{x}_2 t} \right\},$$

$$\widehat{u}_1(t) = \widehat{r}_1 + \frac{\widehat{\nu}\, \xi}{\widehat{x}_1 - \widehat{x}_2} \left\{ \frac{1}{\widehat{x}_1}\, e^{\widehat{x}_1 t} - \frac{1}{\widehat{x}_2}\, e^{\widehat{x}_2 t} \right\};$$

*(ii) If* $\widehat{\Delta} = 0$,

$$\widehat{u}_{-1}(t) = \widehat{r}_{-1} + \frac{\widehat{\nu}}{\widehat{x}_1}\, e^{\widehat{x}_1 t}\Big\{\frac{\varrho+\xi}{\widehat{x}_1} - (\varrho+\xi+\widehat{x}_1)t\Big\},$$
$$\widehat{u}_0(t) = \widehat{r}_0 + \frac{\widehat{\nu}}{\widehat{x}_1}\, e^{\widehat{x}_1 t}\Big\{-\frac{\varrho}{\widehat{x}_1} + (\varrho+\widehat{x}_1)t\Big\},$$
$$\widehat{u}_1(t) = \widehat{r}_1 + \frac{\widehat{\nu}\,\xi}{\widehat{x}_1}\, e^{\widehat{x}_1 t}\Big\{-\frac{1}{\widehat{x}_1} + t\Big\};$$

*(iii) If* $\widehat{\Delta} < 0$,

$$\widehat{u}_{-1}(t) = \widehat{r}_{-1} + \frac{\widehat{\nu}\, e^{\widehat{a}t}}{\widehat{\nu}\,\varrho + \widehat{\nu}\,\xi + \varrho\,\xi}\Big\{(\varrho+\xi)\cos(\widehat{b}\,t) - \frac{1}{\widehat{b}}\Big[(\varrho+\xi)(\widehat{\nu}+\widehat{a}) + \varrho\,\xi\Big]\sin(\widehat{b}\,t)\Big\},$$
$$\widehat{u}_0(t) = \widehat{r}_0 + \frac{\widehat{\nu}\, e^{\widehat{a}t}}{\widehat{\nu}\,\varrho + \widehat{\nu}\,\xi + \varrho\,\xi}\Big\{-\varrho\cos(\widehat{b}\,t) + \Big[\varrho(\widehat{\nu}+\xi+\widehat{a}) + \xi\,\widehat{\nu}\Big]\sin(\widehat{b}\,t)\Big\},$$
$$\widehat{u}_1(t) = \widehat{r}_1 + \frac{\widehat{\nu}\,\xi\, e^{\widehat{a}t}}{\widehat{\nu}\,\varrho + \widehat{\nu}\,\xi + \varrho\,\xi}\Big\{-\cos(\widehat{b}\,t) + \frac{\widehat{a}}{\widehat{b}}\sin(\widehat{b}\,t)\Big\},$$

*where*

$$\widehat{a} = -\frac{\widehat{\nu}+\varrho+\xi}{2}, \qquad \widehat{b} = \frac{\sqrt{|\widehat{\Delta}|}}{2}. \tag{3.5}$$

Furthermore, in the continuous approximation let

$$\widehat{g}(t) = \lim_{\varepsilon\downarrow 0} f_{\mathcal{T}}(t)\Big|_{k=\frac{\widehat{k}}{\varepsilon},\ \gamma=\widehat{\gamma}\,\varepsilon,\ \alpha=\frac{\widehat{\alpha}}{\varepsilon}+\frac{\sigma^2}{2\varepsilon^2},\ \beta=\frac{\widehat{\beta}}{\varepsilon}+\frac{\sigma^2}{2\varepsilon^2}}$$

be the probability density of the time in which the rising phase ends and the chain enters state $C$ for the first time, namely when the power stroke occurs. By virtue of (2.6) and (3.1) one has:

$$\widehat{g}(t) = \begin{cases} \widehat{\nu}\,\xi\, \dfrac{e^{-\widehat{\nu}\,t} - e^{-\xi\,t}}{\xi - \widehat{\nu}}, & \text{if } \xi \neq \widehat{\nu}, \\[2ex] \widehat{\nu}^2\, t\, e^{-\widehat{\nu}\,t}, & \text{if } \xi = \widehat{\nu}. \end{cases}$$

Now we describe the state of the process when the myosin head is in the rising phase in the continuous approximation. To this purpose, first of all we consider a scaled process $N^*(t)$ with space-state $\{0, \varepsilon, \ldots, k\,\varepsilon\}$ (where $\varepsilon$ is a positive constant) and characterized by transition probabilities

$$p_n^*(t) = P\{N^*(t) = n\,\varepsilon \mid Z(0) = -1\} \equiv p_n(t).$$

Then, we introduce a continuous Markov process $\{X(t),\ t \geq 0\}$ with transition pdf

$$f(x,t) = \frac{\partial}{\partial x}\, P\{X(t) < x\},$$

such that for all $t \geq 0$ there holds:

$$\int_0^{\widehat{k}} f(x,t)\, dx = \widehat{u}_0(t).$$

For $\varepsilon \downarrow 0$ one has:

$$\varepsilon\, f(n\,\varepsilon, t) \simeq P\{n\,\varepsilon \le N^*(t) < (n+1)\,\varepsilon\}.$$

Then, along similar lines as in [8], in order to come to a continuous approximation of the $M/M/1$ queue with catastrophes, the forward equations (2.7) for $p_n(t)$ become:

$$\begin{cases} \dfrac{\partial f(x,t)}{\partial t} = -\xi\, f(x,t) - (\widehat{\alpha} - \widehat{\beta})\, \dfrac{\partial f(x,t)}{\partial x} + \dfrac{\sigma^2}{2}\, \dfrac{\partial^2 f(x,t)}{\partial x^2} + \widehat{\gamma}\, \widehat{u}_{-1}(t), \\ \lim\limits_{x\downarrow 0} \Big[ (\widehat{\alpha} - \widehat{\beta})\, f(x,t) - \dfrac{\sigma^2}{2}\, \dfrac{\partial f(x,t)}{\partial x} \Big] = 0, \\ \lim\limits_{x\uparrow \widehat{k}} \Big[ (\widehat{\alpha} - \widehat{\beta})\, f(x,t) - \dfrac{\sigma^2}{2}\, \dfrac{\partial f(x,t)}{\partial x} \Big] = 0. \end{cases} \tag{3.6}$$

with the initial condition

$$\lim_{t\downarrow 0} f(x,t) = 0 \qquad (0 \le x \le \widehat{k}). \tag{3.7}$$

Let us now define the steady-state density:

$$W(x) := \lim_{t\to +\infty} f(x,t) \qquad (0 \le x \le \widehat{k}).$$

From system (3.6) we have:

$$W(x) = \frac{\widehat{\gamma}\, \widehat{r}_{-1}}{\xi} \left\{ 1 - \frac{\widehat{\alpha} - \widehat{\beta}}{\xi}\, \frac{y_1 \left(e^{y_2\, \widehat{k}} - 1\right) e^{-y_1\,(\widehat{k}-x)} + y_2 \left(1 - e^{-y_1\, \widehat{k}}\right) e^{-y_2\,(\widehat{k}-x)}}{e^{-y_1\, \widehat{k}} - e^{-y_2\, \widehat{k}}} \right\}, \tag{3.8}$$

where $y_1$ and $y_2$ ($y_2 < 0 < y_1$) are solutions of the equation:

$$\sigma^2\, y^2 - 2\,(\widehat{\alpha} - \widehat{\beta})\, y - 2\,\xi = 0.$$

In order to determine certain useful relations involving $f(x,t)$, let us define the Laplace transforms of $\widehat{u}_i(t)$ and $f(x,t)$, with $\lambda > 0$:

$$\widehat{U}_i(\lambda) = \int_0^{+\infty} e^{-\lambda\, t}\, \widehat{u}_i(t)\, dt, \qquad i = -1, 0, 1,$$

$$\varphi(x,\lambda) = \int_0^{+\infty} e^{-\lambda\, t}\, f(x,t)\, dt \qquad (0 \le x \le \widehat{k}).$$

Recalling (3.2), from (2.10) and (2.11) one immediately obtains:

$$\widehat{U}_{-1}(\lambda) = \frac{(\lambda+\xi)\,(\lambda+\varrho)}{\lambda\, \widehat{D}(\lambda)}, \qquad \widehat{U}_0(\lambda) = \frac{\widehat{\nu}\,(\lambda+\varrho)}{\lambda\, \widehat{D}(\lambda)}, \qquad \widehat{U}_1(\lambda) = \frac{\widehat{\nu}\,\xi}{\lambda\, \widehat{D}(\lambda)}, \tag{3.9}$$

where

$$\widehat{D}(\lambda) = (\lambda + \widehat{\nu})(\lambda + \varrho + \xi) + \varrho\,\xi. \tag{3.10}$$

**Proposition 3.2** *One has*

$$\varphi(x,\lambda) = \frac{\widehat{U}_0(\lambda)}{\widehat{k}} + (\beta - \alpha)\, \frac{\widehat{U}_0(\lambda)}{\widehat{k}}\, R(x,\lambda), \qquad (0 \le x \le \widehat{k}) \tag{3.11}$$

*where*

(3.12)
$$R(x,\lambda) = \frac{1}{\lambda+\xi}\,\frac{\chi_1(\lambda)\Big(e^{-\chi_2(\lambda)\widehat{k}}-1\Big)\,e^{-\chi_1(\lambda)(\widehat{k}-x)}+\chi_2(\lambda)\Big(1-e^{-\chi_1(\lambda)\widehat{k}}\Big)\,e^{-\chi_2(\lambda)(\widehat{k}-x)}}{e^{-\chi_1(\lambda)\widehat{k}}-e^{-\chi_2(\lambda)\widehat{k}}}$$

*and where* $\chi_1(\lambda)$ *and* $\chi_2(\lambda)$ *are solutions of*

$$\sigma^2\,s^2-2(\widehat{\alpha}-\widehat{\beta})\,s-2\,(\lambda+\xi)=0,$$

*with* $\chi_2(\lambda)<0<\chi_1(\lambda)$.

**Proof.** Applying the Laplace transform to (3.6) and recalling the initial condition (3.7), we obtain the following system:

$$\begin{cases} (\lambda+\xi)\varphi(x,\lambda) = -(\widehat{\alpha}-\widehat{\beta})\dfrac{d\varphi(x,\lambda)}{dx}+\dfrac{\sigma^2}{2}\dfrac{d^2\varphi(x,\lambda)}{dx^2}+\widehat{\gamma}\,\widehat{U}_{-1}(\lambda), \\ \lim\limits_{x\downarrow 0}\Big[(\widehat{\alpha}-\widehat{\beta})\,\varphi(x,\lambda)-\dfrac{\sigma^2}{2}\,\dfrac{d\varphi(x,\lambda)}{\partial x}\Big]=0, \\ \lim\limits_{x\uparrow \widehat{k}}\Big[(\widehat{\alpha}-\widehat{\beta})\,\varphi(x,\lambda)-\dfrac{\sigma^2}{2}\,\dfrac{\partial\varphi(x,\lambda)}{\partial x}\Big]=0. \end{cases} \tag{3.13}$$

The general solution of the first of (3.13) is:

(3.14)
$$\varphi(x,\lambda)=\frac{\widehat{\gamma}}{\lambda+\xi}\,\widehat{U}_{-1}(\lambda)+\widehat{A}(\lambda)\,\exp\big\{\chi_1(\lambda)\,x\big\}+\widehat{B}(\lambda)\,\exp\big\{\chi_2(\lambda)\,x\big\}\qquad(0<x<\widehat{k}).$$

Using (3.14) in the second and in the last equation of (3.13) we obtain:

$$\widehat{A}(\lambda)=\frac{\widehat{\gamma}\,(\widehat{\beta}-\widehat{\alpha})}{(\lambda+\xi)^2}\,\widehat{U}_{-1}(\lambda)\,\frac{1-\exp\big\{\chi_2(\lambda)\,\widehat{k}\big\}}{\exp\big\{\chi_2(\lambda)\,\widehat{k}\big\}-\exp\big\{\chi_1(\lambda)\,\widehat{k}\big\}}\,\chi_1(\lambda),$$
$$\widehat{B}(\lambda)=\frac{\widehat{\gamma}\,(\widehat{\beta}-\widehat{\alpha})}{(\lambda+\xi)^2}\,\widehat{U}_{-1}(\lambda)\,\frac{\exp\big\{\chi_1(\lambda)\,\widehat{k}\big\}-1}{\exp\big\{\chi_2(\lambda)\,\widehat{k}\big\}-\exp\big\{\chi_1(\lambda)\,\widehat{k}\big\}}\,\chi_2(\lambda).$$

Substituting $\widehat{A}(\lambda)$ and $\widehat{B}(\lambda)$ in (3.14) we have:

$$\varphi(x,\lambda)=\frac{\widehat{\gamma}}{\lambda+\xi}\,\widehat{U}_{-1}(\lambda)+\frac{\widehat{\gamma}\,(\widehat{\beta}-\widehat{\alpha})}{\lambda+\xi}\,\widehat{U}_{-1}(\lambda)\,R(x,\lambda) \tag{3.15}$$

where $R(x,\lambda)$ is given in (3.12). Recalling that $\widehat{\gamma}=\widehat{\nu}/\widehat{k}$, by virtue of (3.9) and (3.10) one has:

$$\frac{\widehat{\gamma}}{\lambda+\xi}\,\widehat{U}_{-1}(\lambda)=\frac{1}{\widehat{k}}\,\frac{\widehat{\nu}\,(\lambda+\varrho)}{\lambda\,\widehat{D}(\lambda)}=\frac{1}{\widehat{k}}\,\widehat{U}_0(\lambda).$$

Hence, (3.11) immediately follows from (3.15). ◇

**Proposition 3.3** *For all* $t>0$ *we have:*

$$f(x,t)=\frac{\widehat{u}_0(t)}{\widehat{k}}+(\widehat{\beta}-\widehat{\alpha})\int_0^t\frac{\widehat{u}_0(\tau)}{\widehat{k}}\,r(x,t-\tau)\,d\tau\qquad(0\le x\le\widehat{k}), \tag{3.16}$$

*where $r(x,t)$ denotes the inverse Laplace transform of $R(x,\lambda)$ and is given by:*

$$r(x,t) = e^{-\xi t}\left[-\exp\left\{-\frac{\widehat{\alpha}-\widehat{\beta}}{\sigma^2}(\widehat{k}-x)\right\} A_1(t) + \exp\left\{\frac{\widehat{\alpha}-\widehat{\beta}}{\sigma^2}x\right\} A_2(t)\right], \tag{3.17}$$

*with, for $t>0$*

$$\begin{aligned}
A_1(t) = \sum_{i=0}^{+\infty}\Bigg[&\frac{\sqrt{2}}{\sigma\sqrt{\pi t}}\exp\left\{-\frac{[x-(2i-1)\widehat{k}]^2}{2\sigma^2 t}-\frac{(\widehat{\alpha}-\widehat{\beta})^2 t}{2\sigma^2}\right\}\\
&+\frac{\widehat{\alpha}-\widehat{\beta}}{\sigma^2}\exp\left\{\frac{\widehat{\alpha}-\widehat{\beta}}{\sigma^2}[x-(2i-1)\widehat{k}]\right\}\mathrm{Erfc}\left(-\frac{x-(2i-1)\widehat{k}}{\sigma\sqrt{2t}}-\frac{\widehat{\alpha}-\widehat{\beta}}{\sigma\sqrt{2}}\sqrt{t}\right)\Bigg]\\
&+\sum_{i=0}^{+\infty}\Bigg[\frac{\sqrt{2}}{\sigma\sqrt{\pi t}}\exp\left\{-\frac{[x+(2i+1)\widehat{k}]^2}{2\sigma^2 t}-\frac{(\widehat{\alpha}-\widehat{\beta})^2 t}{2\sigma^2}\right\}\\
&-\frac{\widehat{\alpha}-\widehat{\beta}}{\sigma^2}\exp\left\{\frac{\widehat{\alpha}-\widehat{\beta}}{\sigma^2}[x+(2i+1)\widehat{k}]\right\}\mathrm{Erfc}\left(\frac{x+(2i+1)\widehat{k}}{\sigma\sqrt{2t}}+\frac{\widehat{\alpha}-\widehat{\beta}}{\sigma\sqrt{2}}\sqrt{t}\right)\Bigg],\\
A_2(t) = \sum_{i=0}^{+\infty}\Bigg[&\frac{\sqrt{2}}{\sigma\sqrt{\pi t}}\exp\left\{-\frac{[x-2i\widehat{k}]^2}{2\sigma^2 t}-\frac{(\widehat{\alpha}-\widehat{\beta})^2 t}{2\sigma^2}\right\}\\
&+\frac{\widehat{\alpha}-\widehat{\beta}}{\sigma^2}\exp\left\{\frac{\widehat{\alpha}-\widehat{\beta}}{\sigma^2}(x-2i\widehat{k})\right\}\mathrm{Erfc}\left(-\frac{x-2i\widehat{k}}{\sigma\sqrt{2t}}-\frac{\widehat{\alpha}-\widehat{\beta}}{\sigma\sqrt{2}}\sqrt{t}\right)\Bigg]\\
&+\sum_{i=0}^{+\infty}\Bigg[\frac{\sqrt{2}}{\sigma\sqrt{\pi t}}\exp\left\{-\frac{[x+2i\widehat{k}]^2}{2\sigma^2 t}-\frac{(\widehat{\alpha}-\widehat{\beta})^2 t}{2\sigma^2}\right\}\\
&-\frac{\widehat{\alpha}-\widehat{\beta}}{\sigma^2}\exp\left\{\frac{\widehat{\alpha}-\widehat{\beta}}{\sigma^2}(x+2i\widehat{k})\right\}\mathrm{Erfc}\left(\frac{x+2i\widehat{k}}{\sigma\sqrt{2t}}+\frac{\widehat{\alpha}-\widehat{\beta}}{\sigma\sqrt{2}}\sqrt{t}\right)\Bigg],
\end{aligned}$$

*with* $\mathrm{Erfc}(x) = (2/\sqrt{\pi})\int_x^{+\infty} e^{-z^2}dz$.

**Proof.** From (3.11) we immediately obtain (3.16). By virtue of Proposition 3.2,

$$\chi_1(\lambda)\,\chi_2(\lambda) = -\frac{2\,(\lambda+\xi)}{\sigma^2}\cdot$$

Using such identities in (3.12) we are led to

$$\begin{aligned}R(x,\lambda) = \frac{1}{\sigma^2}\,\exp\Big\{\frac{\widehat{\beta}-\widehat{\alpha}}{\sigma^2}\,x\Big\}\Big[&\frac{e^{-\chi_2(\lambda)\,\widehat{k}}-1}{\chi_2(\lambda)}\,e^{-(\widehat{k}-x)\,\sqrt{\Lambda}/\sigma^2}\\
&+\frac{1-e^{-\chi_1(\lambda)\,\widehat{k}}}{\chi_1(\lambda)}\,e^{(\widehat{k}-x)\,\sqrt{\Lambda}/\sigma^2}\Big]\,\mathrm{cosech}\Big(\frac{\widehat{k}\,\sqrt{\Lambda}}{\sigma^2}\Big),\end{aligned} \tag{3.18}$$

where $\mathrm{cosech}(z) = 1/\sinh(z)$ and

$$\Lambda = (\widehat{\alpha}-\widehat{\beta})^2 + 2\,\sigma^2(\lambda+\xi) = 2\,\sigma^2\,(\lambda+d), \qquad d = \xi + \frac{(\widehat{\alpha}-\widehat{\beta})^2}{2\,\sigma^2}\cdot \tag{3.19}$$

Recalling that (see, for instance, Eq. (1.232.3) of [11], p. 27)

$$\mathrm{cosech}(z) = 2\sum_{i=0}^{+\infty} e^{-(2\,i+1)z} \qquad (z>0),$$

by virtue of (3.19), from (3.18) one has:

(3.20)

$$R(x,\lambda)=\frac{\sqrt{2}}{\sigma}\,\exp\Big\{\frac{\widehat{\alpha}-\widehat{\beta}}{\sigma^2}\,x\Big\}\Bigg[\sum_{i=0}^{+\infty}\Big[\sqrt{\lambda+d}-\frac{\widehat{\alpha}-\widehat{\beta}}{\sigma\sqrt{2}}\Big]^{-1}\exp\Big\{\frac{\sqrt{2}}{\sigma}\,\sqrt{\lambda+d}\,[x-2\,i\,\widehat{k}]\Big\}$$
$$-\exp\Big\{-\frac{\widehat{\alpha}-\widehat{\beta}}{\sigma^2}\,\widehat{k}\Big\}\sum_{i=0}^{+\infty}\Big[\sqrt{\lambda+d}-\frac{\widehat{\alpha}-\widehat{\beta}}{\sigma\sqrt{2}}\Big]^{-1}\exp\Big\{\frac{\sqrt{2}}{\sigma}\,\sqrt{\lambda+d}\,[x-(2\,i-1)\,\widehat{k}]\Big\}$$
$$+\sum_{i=0}^{+\infty}\Big[\sqrt{\lambda+d}+\frac{\widehat{\alpha}-\widehat{\beta}}{\sigma\sqrt{2}}\Big]^{-1}\exp\Big\{\frac{\sqrt{2}}{\sigma}\,\sqrt{\lambda+d}\,[-x-2\,i\,\widehat{k}]\Big\}$$
$$-\exp\Big\{-\frac{\widehat{\alpha}-\widehat{\beta}}{\sigma^2}\,\widehat{k}\Big\}\sum_{i=0}^{+\infty}\Big[\sqrt{\lambda+d}+\frac{\widehat{\alpha}-\widehat{\beta}}{\sigma\sqrt{2}}\Big]^{-1}\exp\Big\{\frac{\sqrt{2}}{\sigma}\,\sqrt{\lambda+d}\,[-x-(2\,i+1)\,\widehat{k}]\Big\}\Bigg].$$

Since (cf. Eq. 12 of [10] p. 246)

$$\int_0^{+\infty}e^{-\lambda t}\,e^{-dt}\left[\frac{1}{\sqrt{\pi\,t}}\,e^{-a^2/(4t)}-\beta\,e^{a\beta+\beta^2 t}\,\mathrm{Erfc}\Big(\frac{a}{2\sqrt{t}}+\beta\sqrt{t}\Big)\right]=\frac{\exp\{-a\sqrt{\lambda+d}\}}{\beta+\sqrt{\lambda+d}}\,,$$

taking the inverse Laplace transform of (3.20) we finally come to (3.17). ♢

We remark that in the symmetry case $\widehat{\alpha}=\widehat{\beta}$ Eq. (3.16) becomes:

$$f(x,t)=\frac{\widehat{u}_0(t)}{\widehat{k}}\qquad(0\le x\le\widehat{k}).$$

In order to discuss the goodness of the continuous approximation, we denote by $\Pi_n^{(\varepsilon)}(\lambda)$ the Laplace transform of $p_n(t)$ of the discrete process $N(t)$ when $k,\alpha,\beta,\gamma$ are given in (3.1).

**Proposition 3.4** *One has:*

$$\lim_{\varepsilon\downarrow 0}\frac{\Pi_n^{(\varepsilon)}(\lambda)}{\varepsilon}=\varphi(x,\lambda)\qquad(0\le x\le\widehat{k}).\tag{3.21}$$

**Proof.** Recalling the expressions of $\Psi_1(\lambda)$ and $\Psi_2(\lambda)$, by virtue of (3.1) one obtains:

$$\lim_{\varepsilon\downarrow 0}\Psi_i^{(\varepsilon)}(\lambda)=1\qquad(i=1,2),$$
$$\lim_{\varepsilon\downarrow 0}\frac{\Psi_1^{(\varepsilon)}(\lambda)-1}{\varepsilon}=-\chi_2(\lambda),\qquad\lim_{\varepsilon\downarrow 0}\frac{1-\Psi_2^{(\varepsilon)}(\lambda)}{\varepsilon}=\chi_1(\lambda)$$
$$\lim_{\varepsilon\downarrow 0}[\Psi_1^{(\varepsilon)}(\lambda)]^{k+1}=\exp\big\{-\widehat{k}\chi_2(\lambda)\big\},\qquad\lim_{\varepsilon\downarrow 0}[\Psi_2^{(\varepsilon)}(\lambda)]^{k+1}=\exp\big\{-\widehat{k}\chi_1(\lambda)\big\},$$
$$\lim_{\varepsilon\downarrow 0}[\Psi_1^{(\varepsilon)}(\lambda)]^{k-n}=\exp\big\{-(\widehat{k}-x)\chi_2(\lambda)\big\},\qquad\lim_{\varepsilon\downarrow 0}[\Psi_2^{(\varepsilon)}(\lambda)]^{k-n}=\exp\big\{-(\widehat{k}-x)\chi_1(\lambda)\big\},$$

where $\Psi_i^{(\varepsilon)}(\lambda)$ is equal to $\Psi_i(\lambda)$ with $k,\alpha,\beta,\gamma$ given in (3.1). Hence, (3.21) immediately follows from (2.12). ♢

In conclusion, denoting by $p_n^{(\varepsilon)}(t)$ the probabilities of the discrete process $N(t)$ when $k,\alpha,\beta,\gamma$ are given in (3.1), from (3.21) for $t\ge 0$ it follows:

$$\lim_{\varepsilon\downarrow 0}\frac{p_n^{(\varepsilon)}(t)}{\varepsilon}=f(x,t)\qquad(0\le x\le\widehat{k}).$$

This confirms the validity of the continuous approximation.

**4 Concluding remarks** Inspired by recent innovative experimental results on the sliding of myosin heads along actin filaments, in this paper we have proposed a novel Markov chain-based model to describe the dynamics of the actomyosin system during an ATP cycle. Our approach leads to the state probabilities of the locations occupied by the myosin head during its motion. A diffusive approximation of the process describing the myosin rising phase has also been proposed in order to provide more manageable expressions for a quantitative description of such a phenomenon.

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