ON PAIRWISE ALMOST REGULAR-LINDELÖF SPACES

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Received November 13, 2008; Revised July 9, 2009

ABSTRACT. In this work, we introduce and study the pairwise almost regular-Lindelöf bitopological spaces, their subspaces and subsets, and investigate some of their characterizations. We also show that a pairwise almost regular-Lindelöf property is not a hereditary property.

1. Introduction

The study of bitopological spaces was first initiated by J. C. Kelly [7] in 1963 and thereafter a large number of papers have been done to generalize the topological concepts to bitopological setting. In literature there are several generalizations of the notion of Lindelöf spaces and these are studied separately for different reasons and purposes. In 1984, Willard and Dissanayake [18] introduced the notion of almost k-Lindelöf spaces, that for $k = \aleph_0$ we call almost Lindelöf, and in 1996 Cammaroto and Santoro [2] studied and gave further new results about these spaces followed by Kılıçman and Fawakhreh [9]. In the same paper Cammaroto and Santoro introduced the notion of almost regular-Lindelöf spaces and gave some of their characterizations. In 2001, Fawakhreh and Kılıçman [5], and Kılıçman and Fawakhreh [10] studied and gave further new results about these spaces and their subspaces.

Recently the authors in [13] introduced and studied the notion of pairwise almost Lindelöf spaces in bitopological spaces and extended some results due to Cammaroto and Santoro [2], Kılıçman and Fawakhreh [9] and Fawakhreh [4]. The purpose of this paper is to define and extend the notion of almost regular-Lindelöf property in bitopological spaces, which we will call pairwise almost regular-Lindelöf spaces and investigate some of their characterizations. Further we also study the pairwise almost regular-Lindelöf subspaces and subsets and investigate some of their characterizations.

In section 3, we shall introduce the concept of pairwise almost regular-Lindelöf bitopological spaces by using pairwise regular cover. This study begin by investigating the (i, j)-almost regular-Lindelöf property and some results obtained. Furthermore, we study the relation between (i, j)-nearly Lindelöf, (i, j)-almost Lindelöf and (i, j)-almost regular-Lindelöf spaces, where $i, j = 1$ or 2, $i \neq j$.

In section 4, we shall define the concept of pairwise almost regular-Lindelöf subspaces and subsets. We shall define the concept of pairwise almost regular-Lindelöf relative to a bitopological space by investigating the (i, j)-almost regular-Lindelöf property and obtain some results. Some counterexamples are provided in order to show that the pairwise almost regular-Lindelöf property is not a hereditary property.

2. Preliminaries

Throughout this paper, all spaces (X, τ) and (X, τ_1, τ_2) (or simply X) are always meant topological spaces and bitopological spaces, respectively unless explicitly stated. We always

2000 *Mathematics Subject Classification.* 54B05, 54D20, 54E55, 54G20 .

Key words and phrases. Bitopological space, (i, j)-almost regular-Lindelöf, pairwise almost regular-Lindelöf, (i, j)-almost regular-Lindelöf relative, pairwise almost regular-Lindelöf relative, (i, j)-regular open, (i, j)-regular cover.

use (i, j)- to denote the certain property for τ_i has this property with respect to topology τ_j, where $i, j \in \{1, 2\}$ and $i \neq j$. By i-int (A) and i-cl (A), we shall mean the interior and the closure of a subset A of X with respect to topology τ_i, respectively. If $S \subseteq A \subseteq X$, then i-$\mathrm{int_A}$ (S) and i-$\mathrm{cl_A}$ (S) will be used to denote the interior and closure of S with respect to topology τ_i in the subspace A, respectively. By i-open cover of X, we mean that the cover of X by i-open sets in X; similar for the (i, j)-regular open cover of X etc. In this paper always $i, j \in \{1, 2\}$ and $i \neq j$.

Definition 2.1. [8, 14] *A subset S of a bitopological space (X, τ_1, τ_2) is said to be (i, j)-regular open* (*resp. (i, j)-regular closed*) *if i-int (j-cl (S)) = S* (*resp. i-cl (j-int (S)) = S*). *S is called pairwise regular open* (*resp. pairwise regular closed*) *if it is both $(1, 2)$-regular open and $(2, 1)$-regular open* (*resp. $(1, 2)$-regular closed and $(2, 1)$-regular closed*).

We note that, the complement of an (i, j)-regular open (resp. pairwise regular open) set is (i, j)-regular closed (resp. pairwise regular closed) and vice versa. It is always true that i-int (j-cl (S)) is an (i, j)-regular open set since i-int (j-cl (i-int (j-cl (S)))) = i-int (j-cl (S)) .

Definition 2.2. *Let (X, τ_1, τ_2) be a bitopological space. A subset F of X is said to be*
(*i*) *i-open if F is open with respect to τ_i in X, F is called open in X if it is both 1-open and 2-open in X, or equivalently, $F \in (\tau_1 \cap \tau_2)$ in X;*
(*ii*) *i-closed if F is closed with respect τ_i in X, F is called closed in X if it is both 1-closed and 2-closed in X, or equivalently, $X \setminus F \in (\tau_1 \cap \tau_2)$ in X;*
(*iii*) *i-clopen if F is both i-closed and i-open set in X, F is called clopen in X if it is both 1-clopen and 2-clopen in X;*
(*iv*) *(i, j)-clopen if F is i-closed and j-open set in X, F is called clopen if it is both $(1, 2)$-clopen and $(2, 1)$-clopen in X.*

Definition 2.3. [6, 11] *A bitopological space (X, τ_1, τ_2) is said to be i-Lindelöf if the topological space (X, τ_i) is Lindelöf. X is called Lindelöf* (*or p-Lindelöf in* [11]) *if it is i-Lindelöf for each $i = 1, 2$. Equivalently, (X, τ_1, τ_2) is Lindelöf if every i-open cover of X has a countable subcover for each $i = 1, 2$.*

Definition 2.4. [7, 8] *A bitopological space (X, τ_1, τ_2) is said to be (i, j)-regular if for each point $x \in X$ and for each i-open set V of X containing x, there exists an i-open set U such that $x \in U \subseteq j$-cl (U) $\subseteq$ V. X is said to be pairwise regular if it is both $(1, 2)$-regular and $(2, 1)$-regular.*

Definition 2.5. [8, 14] *A bitopological space X is said to be (i, j)-almost regular if for each $x \in X$ and for each (i, j)-regular open set V of X containing x, there is an (i, j)-regular open set U such that $x \in U \subseteq j$-cl (U) $\subseteq$ V. X is said to be pairwise almost regular if it is both $(1, 2)$-almost regular and $(2, 1)$-almost regular.*

Definition 2.6. [8, 14] *A bitopological space X is said to be (i, j)-semiregular if for each $x \in X$ and for each i-open set V of X containing x, there is an i-open set U such that $x \in U \subseteq i$-int (j-cl (U)) $\subseteq$ V. X is called pairwise semiregular if it is both $(1, 2)$-semiregular and $(2, 1)$-semiregular.*

In other words, we can say that a space (X, τ_1, τ_2) is (i, j)-semiregular if and only if the topology τ_i is generated by the (i, j)-regular open subsets of (X, τ_1, τ_2) where $i, j \in \{1, 2\}, i \neq j$.

Now we give a relation between (i, j)-regularity with (i, j)-almost regularity and (i, j)-semiregularity.

Theorem 2.1. *A space X is (i, j)-regular if and only if it is (i, j)-almost regular and (i, j)-semiregular.*

Proof. Let X be an (i, j)-almost regular and (i, j)-semiregular space. Let $x \in X$ and let M be any i-open set in X containing x. Since X is (i, j)-semiregular, there exists an i-open set U such that $x \in U \subseteq i$-int (j-cl (U)) $\subseteq$ M. Again, since i-int (j-cl (U)) is an (i, j)-regular open neighbourhood of x and X is (i,j)-almost regular, there is an (i,j)-regular open set V such that $x \in V \subseteq j\text{-cl}(\mathrm{V}) \subseteq \text{i-int}(\text{j-cl}(\mathrm{U}))$. Hence $x \in V \subseteq j\text{-cl}(\mathrm{V}) \subseteq \mathrm{M}$. Thus X is (i,j)-regular since an (i,j)-regular open set V is also i-open set in X. Conversely, let X be an (i,j)-regular space. It is obvious by the definitions that X is (i,j)-semiregular. Let $x \in X$ and let N be any (i,j)-regular open set in X containing x. Since X is (i,j)-regular, there exists an i-open set U such that $x \in U \subseteq j\text{-cl}(\mathrm{U}) \subseteq \mathrm{N}$. Thus $x \in U \subseteq i\text{-int}(\text{j-cl}(\mathrm{U})) \subseteq \text{j-cl}(\mathrm{U}) \subseteq \mathrm{N}$. So there exists an (i,j)-regular open set $i\text{-int}(\text{j-cl}(\mathrm{U}))$ such that $x \in i\text{-int}(\text{j-cl}(\mathrm{U})) \subseteq \text{j-cl}(\text{i-int}(\text{j-cl}(\mathrm{U}))) \subseteq \mathrm{N}$. Therefore X is (i,j)-almost regular and completes the proof. ☐

Corollary 2.1. *A space X is pairwise regular if and only if it is pairwise almost regular and pairwise semiregular.*

Definition 2.7. [12, 13] *A bitopological space X is said to be (i,j)-nearly Lindelöf (resp. (i,j)-almost Lindelöf) if for every i-open cover $\{U_\alpha : \alpha \in \Delta\}$ of X, there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of Δ such that $X = \bigcup_{n\in\mathbb{N}} i\text{-int}(\text{j-cl}(\mathrm{U}_{\alpha_n}))$ (resp. $X = \bigcup_{n\in\mathbb{N}} j\text{-cl}(\mathrm{U}_{\alpha_n})$). X is called pairwise nearly Lindelöf (resp. pairwise almost Lindelöf) if it is both $(1,2)$-nearly Lindelöf (resp. $(1,2)$-almost Lindelöf) and $(2,1)$-nearly Lindelöf (resp. $(2,1)$-almost Lindelöf).*

Theorem 2.2. *A bitopological space X is (i,j)-nearly Lindelöf if and only if every (i,j)-regular open cover of X has a countable subcover.*

Proof. Let X be an (i,j)-nearly Lindelöf space and let $\{U_\alpha : \alpha \in \Delta\}$ be an (i,j)-regular open cover of X. Then $X = \bigcup_{\alpha\in\Delta} U_\alpha = \bigcup_{\alpha\in\Delta} i\text{-int}(\text{j-cl}(\mathrm{U}_\alpha))$. Hence $\{U_\alpha : \alpha \in \Delta\}$ is also an i-open cover of X since $i\text{-int}(\text{j-cl}(\mathrm{U}_\alpha))$ is also i-open set. So there is a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of Δ such that $X = \bigcup_{n\in\mathbb{N}} i\text{-int}(\text{j-cl}(\mathrm{U}_{\alpha_n})) = \bigcup_{n\in\mathbb{N}} \mathrm{U}_{\alpha_n}$. Thus $\{U_\alpha : \alpha \in \Delta\}$ has a countable subfamily which cover X. Conversely, let $\{U_\alpha : \alpha \in \Delta\}$ be an i-open cover of X. Since $U_\alpha \subseteq i\text{-int}(\text{j-cl}(\mathrm{U}_\alpha))$, then $X = \bigcup_{\alpha\in\Delta} U_\alpha \subseteq \bigcup_{\alpha\in\Delta} i\text{-int}(\text{j-cl}(\mathrm{U}_\alpha))$. Hence $\{i\text{-int}(\text{j-cl}(\mathrm{U}_\alpha)) : \alpha \in \Delta\}$ is an (i,j)-regular open cover of X. So there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of Δ such that $X = \bigcup_{n\in\mathbb{N}} i\text{-int}(\text{j-cl}(\mathrm{U}_{\alpha_n}))$. Thus X is (i,j)-nearly Lindelöf. ☐

Corollary 2.2. *A bitopological space X is pairwise nearly Lindelöf if and only if every (i,j)-regular open cover of X has a countable subcover for each $i,j \in \{1,2\}, i \neq j$.*

Definition 2.8. [13] *A subset S of a bitopological space X is said to be (i,j)-almost Lindelöf relative to X if for every cover $\{U_\alpha : \alpha \in \Delta\}$ of S by i-open subsets of X such that $S \subseteq \bigcup_{\alpha\in\Delta} U_\alpha$, there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of Δ such that $S \subseteq \bigcup_{n\in\mathbb{N}} j\text{-cl}(\mathrm{U}_{\alpha_n})$. S is called pairwise almost Lindelöf relative to X if it is both $(1,2)$-almost Lindelöf relative to X and $(2,1)$-almost Lindelöf relative to X.*

Locally finite is the well known concept in topology. When we extend this concept to a bitopological space (X, τ_1, τ_2), the term i-locally finite apear as in the following definition.

Definition 2.9. *A family $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ of subsets of a space (X, τ_1, τ_2) is i-locally finite if for every point $x \in X$, there exists an i-neighbourhood U_x of x such that the set $\{\alpha \in \Delta : U_x \cap U_\alpha \neq \emptyset\}$ is finite, i.e., each $x \in X$ has an i-neighbourhood U_x meeting only finitely many $U_\alpha \in \mathcal{U}$.*

In 1969, Singal and Arya [15] introduced the notion of nearly paracompact spaces in topological spaces. We extend this notion to bitopological spaces as follows.

Definition 2.10. *A bitopological space X is said to be (i,j)-nearly paracompact if every cover of X by (i,j)-regular open sets admits an i-open refinement which is i-locally finite. X is called pairwise nearly paracompact if it is both $(1,2)$-nearly paracompact and $(2,1)$-nearly paracompact.*

3. Pairwise Almost Regular-Lindelöf Spaces

In 1981, Cammaroto and Faro [1] introduced the notion of regular cover in topological spaces. We extend this concept to bitopological spaces as in the following definition.

Definition 3.1. *An i-open cover $\{U_\alpha : \alpha \in \Delta\}$ of a bitopological space X is said to be (i,j)-regular cover if for every $\alpha \in \Delta$, there exists a (j,i)-regular closed subset C_α of X such that $C_\alpha \subseteq U_\alpha$ and $X = \bigcup_{\alpha\in\Delta} i\text{-int}\,(C_\alpha)$. $\{U_\alpha : \alpha \in \Delta\}$ is called pairwise regular cover if it is both $(1,2)$-regular cover and $(2,1)$-regular cover.*

In the following definition we extend the notion of almost regular-Lindelöf spaces due to Cammaroto and Santoro [2] to the bitopological spaces.

Definition 3.2. *A bitopological space X is said to be (i,j)-almost regular-Lindelöf if for every (i,j)-regular cover $\{U_\alpha : \alpha \in \Delta\}$ of X, there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of Δ such that $X = \bigcup_{n\in\mathbb{N}} j\text{-cl}\,(U_{\alpha_n})$. X is called pairwise almost regular-Lindelöf if it is both $(1,2)$-almost regular-Lindelöf and $(2,1)$-almost regular-Lindelöf.*

Now we give some characterizations of (i,j)-almost regular-Lindelöf spaces.

Theorem 3.1. *Let X be a bitopological space. The following conditions are equivalent:*
(i) X is (i,j)-almost regular-Lindelöf;
(ii) for every family $\{C_\alpha : \alpha \in \Delta\}$ of i-closed subsets of X such that for each $\alpha \in \Delta$ there exists a (j,i)-regular open subset A_α of X with $A_\alpha \supseteq C_\alpha$ and $\bigcap_{\alpha\in\Delta} i\text{-cl}\,(A_\alpha) = \emptyset$, there exists a countable subfamily $\{C_{\alpha_n} : n \in \mathbb{N}\}$ such that $\bigcap_{n\in\mathbb{N}} j\text{-int}\,(C_{\alpha_n}) = \emptyset$;
(iii) for every family $\{C_\alpha : \alpha \in \Delta\}$ of i-closed subsets of X for which every countable subfamily $\{C_{\alpha_n} : n \in \mathbb{N}\}$ satisfies $\bigcap_{n\in\mathbb{N}} j\text{-int}\,(C_{\alpha_n}) \neq \emptyset$, the intersection $\bigcap_{\alpha\in\Delta} i\text{-cl}\,(A_\alpha) \neq \emptyset$ for each (j,i)-regular open subset A_α of X with $A_\alpha \supseteq C_\alpha$.

Proof. $(i) \Leftrightarrow (ii)$: Let $\{C_\alpha : \alpha \in \Delta\}$ be a family of i-closed subsets of X such that for each $\alpha \in \Delta$ there exists a (j,i)-regular open subset A_α of X with $A_\alpha \supseteq C_\alpha$ and $\bigcap_{\alpha\in\Delta} i\text{-cl}\,(A_\alpha) = \emptyset$. It follows that $X = X \setminus \left(\bigcap_{\alpha\in\Delta} i\text{-cl}\,(A_\alpha)\right) = \bigcup_{\alpha\in\Delta} i\text{-int}\,(X \setminus A_\alpha)$. Since $C_\alpha \subseteq A_\alpha = j\text{-int}\,(i\text{-cl}\,(A_\alpha)) \subseteq i\text{-cl}\,(A_\alpha)$, then $X \setminus i\text{-cl}\,(A_\alpha) \subseteq X \setminus A_\alpha \subseteq X \setminus C_\alpha$, i.e., $i\text{-int}\,(X \setminus A_\alpha) \subseteq X \setminus A_\alpha \subseteq X \setminus C_\alpha$. Therefore $X = \bigcup_{\alpha\in\Delta} i\text{-int}\,(X \setminus A_\alpha) \subseteq \bigcup_{\alpha\in\Delta} (X \setminus C_\alpha)$. So, the family $\{X \setminus C_\alpha : \alpha \in \Delta\}$ is an (i,j)-regular cover of X because for each $\alpha \in \Delta$, the (j,i)-regular closed subset $X \setminus A_\alpha$ of X satisfies the conditions $X \setminus A_\alpha \subseteq X \setminus C_\alpha$ and $X = \bigcup_{\alpha\in\Delta} i\text{-int}\,(X \setminus A_\alpha)$. By (i), there exists a countable subfamily $\{X \setminus C_{\alpha_n} : n \in \mathbb{N}\}$ such that $X = \bigcup_{n\in\mathbb{N}} j\text{-cl}\,(X \setminus C_{\alpha_n}) = X \setminus \left(\bigcap_{n\in\mathbb{N}} j\text{-int}\,(C_{\alpha_n})\right)$. Therefore $\bigcap_{n\in\mathbb{N}} j\text{-int}\,(C_{\alpha_n}) = \emptyset$. Conversely, let $\{U_\alpha : \alpha \in \Delta\}$ be an (i,j)-regular cover of X. By Definition 3.1, for each $\alpha \in \Delta$, U_α is i-open set in X and there exists a (j,i)-regular closed subset C_α of X such that $C_\alpha \subseteq U_\alpha$ and $X = \bigcup_{\alpha\in\Delta} i\text{-int}\,(C_\alpha)$. The family $\{X \setminus U_\alpha : \alpha \in \Delta\}$ of i-closed subsets of X satisfies the conditions, for each $\alpha \in \Delta$ there exists a (j,i)-regular open subset $X \setminus C_\alpha$ of X such that $X \setminus C_\alpha \supseteq X \setminus U_\alpha$ and $\bigcap_{\alpha\in\Delta} i\text{-cl}\,(X \setminus C_\alpha) = \bigcap_{\alpha\in\Delta} (X \setminus i\text{-int}\,(C_\alpha)) = X \setminus \left(\bigcup_{\alpha\in\Delta} i\text{-int}\,(C_\alpha)\right) = X \setminus X = \emptyset$. By (ii), there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of Δ such that $\bigcap_{n\in\mathbb{N}} j\text{-int}\,(X \setminus U_{\alpha_n}) = \emptyset$, i.e., $X \setminus \left(\bigcup_{n\in\mathbb{N}} j\text{-cl}\,(U_{\alpha_n})\right) = \emptyset$ and therefore $X = \bigcup_{n\in\mathbb{N}} j\text{-cl}\,(U_{\alpha_n})$.

$(ii) \Leftrightarrow (iii)$: Straightforward. $\square$

Corollary 3.1. *Let X be a bitopological space. The following conditions are equivalent:*
(i) X is pairwise almost regular-Lindelöf;
(ii) for every family $\{C_\alpha : \alpha \in \Delta\}$ of i-closed subsets of X such that for each $\alpha \in \Delta$ there exists a (j,i)-regular open subset A_α of X with $A_\alpha \supseteq C_\alpha$ and $\bigcap_{\alpha\in\Delta} i\text{-cl}(\mathrm{A}_\alpha) = \emptyset$, there exists a countable subfamily $\{C_{\alpha_n} : n \in \mathbb{N}\}$ such that $\bigcap_{n\in\mathbb{N}} j\text{-int}(\mathrm{C}_{\alpha_n}) = \emptyset$ for each $i, j \in \{1,2\}, i \neq j$;
(iii) for every family $\{C_\alpha : \alpha \in \Delta\}$ of i-closed subsets of X for which every countable subfamily $\{C_{\alpha_n} : n \in \mathbb{N}\}$ satisfies $\bigcap_{n\in\mathbb{N}} j\text{-int}(\mathrm{C}_{\alpha_n}) \neq \emptyset$, the intersection $\bigcap_{\alpha\in\Delta} i\text{-cl}(\mathrm{A}_\alpha) \neq \emptyset$ for each(j,i)-regular open subset A_α of X with $A_\alpha \supseteq C_\alpha$ for each $i, j \in \{1,2\}, i \neq j$.

Suppose that $\{U_\alpha : \alpha \in \Delta\}$ is an (i,j)-regular cover (resp. pairwise regular cover) of a bitopological space X. If for every $\alpha \in \Delta$, U_α is an (i,j)-regular open (resp. pairwise regular open) subset of X, then $\{U_\alpha : \alpha \in \Delta\}$ is called (i,j)-regular cover (resp. pairwise regular cover) of X by (i,j)-regular open (resp. pairwise regular open) subsets of X.

Theorem 3.2. *Let (X, τ_1, τ_2) be a bitopological space. For the following conditions*
(i) X is (i,j)-almost regular-Lindelöf;
(ii) for every (i,j)-regular cover $\{U_\alpha : \alpha \in \Delta\}$ of X by (i,j)-regular open subsets of X, there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of Δ such that $X = \bigcup_{n\in\mathbb{N}} j\text{-cl}(\mathrm{U}_{\alpha_n})$;
(iii) for every family $\{C_\alpha : \alpha \in \Delta\}$ of (i,j)-regular closed subsets of X such that for each $\alpha \in \Delta$ there exists a (j,i)-regular open subset A_α of X with $A_\alpha \supseteq C_\alpha$ and $\bigcap_{\alpha\in\Delta} i\text{-cl}(\mathrm{A}_\alpha) = \emptyset$, there exists a countable subfamily $\{C_{\alpha_n} : n \in \mathbb{N}\}$ such that $\bigcap_{n\in\mathbb{N}} j\text{-int}(\mathrm{C}_{\alpha_n}) = \emptyset$;
(iv) for every family $\{C_\alpha : \alpha \in \Delta\}$ of (i,j)-regular closed subsets of X for which every countable subfamily $\{C_{\alpha_n} : n \in \mathbb{N}\}$ satisfies $\bigcap_{n\in\mathbb{N}} j\text{-int}(\mathrm{C}_{\alpha_n}) \neq \emptyset$, the intersection $\bigcap_{\alpha\in\Delta} i\text{-cl}(\mathrm{A}_\alpha) \neq \emptyset$ for each (j,i)-regular open subset A_α of X with $A_\alpha \supseteq C_\alpha$;
we have that (i) $\Rightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftrightarrow$ (iv) and if X is (i,j)-semiregular, then (ii) $\Rightarrow$ (i).

Proof. (*i*) $\Rightarrow$ (*ii*): It is obvious by the Definition 3.1 and Definition 3.2 since every (i,j)-regular open set in X is i-open.

(*ii*) $\Leftrightarrow$ (*iii*): Let $\{C_\alpha : \alpha \in \Delta\}$ be a family of (i,j)-regular closed subsets of X such that for each $\alpha \in \Delta$ there exists a (j,i)-regular open subset A_α of X with $A_\alpha \supseteq C_\alpha$ and $\bigcap_{\alpha\in\Delta} i\text{-cl}(\mathrm{A}_\alpha) = \emptyset$. It follows that $X = X \setminus \left(\bigcap_{\alpha\in\Delta} i\text{-cl}(\mathrm{A}_\alpha)\right) = \bigcup_{\alpha\in\Delta} i\text{-int}(\mathrm{X} \setminus \mathrm{A}_\alpha)$. Since $C_\alpha \subseteq A_\alpha = j\text{-int}(\text{i-cl}(\mathrm{A}_\alpha)) \subseteq \text{i-cl}(\mathrm{A}_\alpha)$, then $i\text{-int}(\mathrm{X} \setminus \mathrm{A}_\alpha) \subseteq \mathrm{X} \setminus \mathrm{A}_\alpha \subseteq \mathrm{X} \setminus \mathrm{C}_\alpha$. Therefore $X = \bigcup_{\alpha\in\Delta} i\text{-int}(\mathrm{X} \setminus \mathrm{A}_\alpha) \subseteq \bigcup_{\alpha\in\Delta} (\mathrm{X} \setminus \mathrm{C}_\alpha)$. So, the family $\{X \setminus C_\alpha : \alpha \in \Delta\}$ is an (i,j)-regular cover of X by (i,j)-regular open subsets of X. By (*ii*) there exists a countable subfamily $\{X \setminus C_{\alpha_n} : n \in \mathbb{N}\}$ such that $X = \bigcup_{n\in\mathbb{N}} j\text{-cl}(\mathrm{X} \setminus \mathrm{C}_{\alpha_n}) = \mathrm{X} \setminus \left(\bigcap_{\mathrm{n}\in\mathbb{N}} \text{j-int}(\mathrm{C}_{\alpha_n})\right)$. Therefore $\bigcap_{n\in\mathbb{N}} j\text{-int}(\mathrm{C}_{\alpha_n}) = \emptyset$. Conversely, let $\{U_\alpha : \alpha \in \Delta\}$ be an (i,j)-regular cover of X by (i,j)-regular open subsets of X. Then for each $\alpha \in \Delta$, there exists a (j,i)-regular closed subset C_α of X such that $C_\alpha \subseteq U_\alpha$ and $X = \bigcup_{\alpha\in\Delta} i\text{-int}(\mathrm{C}_\alpha)$. The family $\{X \setminus U_\alpha : \alpha \in \Delta\}$ of (i,j)-regular closed subsets of X satisfies the conditions, for each $\alpha \in \Delta$ there exists a (j,i)-regular open subset $X \setminus C_\alpha$ of X such that $X \setminus C_\alpha \supseteq X \setminus U_\alpha$ and $\bigcap_{\alpha\in\Delta} i\text{-cl}(\mathrm{X} \setminus \mathrm{C}_\alpha) = X \setminus \left(\bigcup_{\alpha\in\Delta} i\text{-int}(\mathrm{C}_\alpha)\right) = X \setminus X = \emptyset$. By (*iii*), there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of Δ such that $\bigcap_{n\in\mathbb{N}} j\text{-int}(\mathrm{X} \setminus \mathrm{U}_{\alpha_n}) = \emptyset$, i.e., $X \setminus \left(\bigcup_{n\in\mathbb{N}} j\text{-cl}(\mathrm{U}_{\alpha_n})\right) = \emptyset$. Therefore $X = \bigcup_{n\in\mathbb{N}} j\text{-cl}(\mathrm{U}_{\alpha_n})$ and (*ii*) proved.

(*iii*) $\Leftrightarrow$ (*iv*): Straightforward.

$(ii) \Rightarrow (i)$: Let $\{U_\alpha : \alpha \in \Delta\}$ be an (i,j)-regular cover of X satisfies the conditions of Definition 3.1 above. Since X is (i,j)-semiregular, by definition the topology τ_i is generated by the (i,j)-regular open subsets of X, so we can assume that U_α is (i,j)-regular open set in X for each $\alpha \in \Delta$. Hence $\{U_\alpha : \alpha \in \Delta\}$ is (i,j)-regular cover of X by (i,j)-regular open subsets of X. By (ii), there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of Δ such that $X = \bigcup_{n\in\mathbb{N}} j\text{-cl}\,(\mathrm{U}_{\alpha_n})$. This shows that X is (i,j)-almost regular-Lindelöf. □

Corollary 3.2. *Let (X, τ_1, τ_2) be a bitopological space. For the following conditions*
(i) X is pairwise almost regular-Lindelöf;
(ii) for every (i,j)-regular cover $\{U_\alpha : \alpha \in \Delta\}$ of X by (i,j)-regular open subsets of X, there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of Δ such that $X = \bigcup_{n\in\mathbb{N}} j\text{-cl}\,(\mathrm{U}_{\alpha_n})$ for each $i,j \in \{1,2\}, i \neq j$;
(iii) for every family $\{C_\alpha : \alpha \in \Delta\}$ of (i,j)-regular closed subsets of X such that for each $\alpha \in \Delta$ there exists a (j,i)-regular open subset A_α of X with $A_\alpha \supseteq C_\alpha$ and $\bigcap_{\alpha\in\Delta} i\text{-cl}\,(\mathrm{A}_\alpha) = \emptyset$, there exists a countable subfamily $\{C_{\alpha_n} : n \in \mathbb{N}\}$ such that $\bigcap_{n\in\mathbb{N}} j\text{-int}\,(\mathrm{C}_{\alpha_n}) = \emptyset$ for each $i,j \in \{1,2\}, i \neq j$;
(iv) for every family $\{C_\alpha : \alpha \in \Delta\}$ of (i,j)-regular closed subsets of X for which every countable subfamily $\{C_{\alpha_n} : n \in \mathbb{N}\}$ satisfies $\bigcap_{n\in\mathbb{N}} j\text{-int}\,(\mathrm{C}_{\alpha_n}) \neq \emptyset$, the intersection $\bigcap_{\alpha\in\Delta} i\text{-cl}\,(\mathrm{A}_\alpha) \neq \emptyset$ for each (j,i)-regular open subset A_α of X with $A_\alpha \supseteq C_\alpha$ for each $i,j \in \{1,2\}, i \neq j$;
we have that $(i) \Rightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv)$ and if X is pairwise semiregular, then $(ii) \Rightarrow (i)$.

It is clear that, every (i,j)-almost Lindelöf space is (i,j)-almost regular-Lindelöf.

Question 3.1. *Does (i,j)-almost regular-Lindelöf spaces imply (i,j)-almost Lindelöf?*

The authors conjecture that the answer of this question is no. We can answer this question with some restriction on the space as in the following proposition. Before that we need to prove the following lemma.

Lemma 3.1. *Let X be an (i,j)-almost regular space. Then for each $x \in X$ and for each (i,j)-regular open subset W of X containing x, there exist two (i,j)-regular open subsets U and V of X such that $x \in U \subseteq j\text{-cl}\,(\mathrm{U}) \subseteq \mathrm{V} \subseteq \mathrm{j\text{-}cl}\,(\mathrm{V}) \subseteq \mathrm{W}$.*

Proof. Let $x \in X$ and let W be an (i,j)-regular open subset of X containing x. Since X is (i,j)-almost regular, there is an (i,j)-regular open subset V such that $x \in V \subseteq j\text{-cl}\,(\mathrm{V}) \subseteq \mathrm{W}$. Again, since V is an (i,j)-regular open set in X containing x and X is (i,j)-almost regular, there exists an (i,j)-regular open subset U such that $x \in U \subseteq j\text{-cl}\,(\mathrm{U}) \subseteq \mathrm{V}$. So, $x \in U \subseteq j\text{-cl}\,(\mathrm{U}) \subseteq \mathrm{V} \subseteq \mathrm{j\text{-}cl}\,(\mathrm{V}) \subseteq \mathrm{W}$. Thus U and V are two (i,j)-regular open subsets of X that we are required. This completes the proof. □

Corollary 3.3. *Let X be a pairwise almost regular space. Then for each $x \in X$ and for each (i,j)-regular open subset W of X containing x, there exist two (i,j)-regular open subsets U and V of X such that $x \in U \subseteq j\text{-cl}\,(\mathrm{U}) \subseteq \mathrm{V} \subseteq \mathrm{j\text{-}cl}\,(\mathrm{V}) \subseteq \mathrm{W}$ for each $i,j \in \{1,2\}, i \neq j$.*

Proposition 3.1. *An (i,j)-almost regular-Lindelöf and (i,j)-almost regular space X is (i,j)-nearly Lindelöf.*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be an (i,j)-regular open cover of X. For each $x \in X$, there exists $\alpha_x \in \Delta$ such that $x \in U_{\alpha_x}$. Since X is (i,j)-almost regular, there exist two (i,j)-regular open subsets V_{α_x} and W_{α_x} of X such that $x \in V_{\alpha_x} \subseteq j\text{-cl}\,(\mathrm{V}_{\alpha_x}) \subseteq \mathrm{W}_{\alpha_x} \subseteq \mathrm{j\text{-}cl}\,(\mathrm{W}_{\alpha_x}) \subseteq \mathrm{U}_{\alpha_x}$ by Lemma 3.1. Since for each $\alpha \in \Delta$, there exists a (j,i)-regular closed set $j\text{-cl}\,(\mathrm{V}_{\alpha_x})$ in X such that $j\text{-cl}\,(\mathrm{V}_{\alpha_x}) \subseteq \mathrm{W}_{\alpha_x}$ and $X = \bigcup_{x\in X} V_{\alpha_x} = \bigcup_{x\in X} i\text{-int}\,(\mathrm{j\text{-}cl}\,(\mathrm{V}_{\alpha_x}))$, the family $\{W_{\alpha_x} : x \in X\}$ is an (i,j)-regular cover of X by (i,j)-regular open subsets of X. Since X is (i,j)-almost regular-Lindelöf, there exists a countable subset of points $x_1, x_2, \dots, x_n, \dots$ of X such that $X = \bigcup_{n\in\mathbb{N}} j\text{-cl}\left(\mathrm{W}_{\alpha_{x_n}}\right) \subseteq \bigcup_{n\in\mathbb{N}} \left(\mathrm{U}_{\alpha_{x_n}}\right)$ by Theorem 3.2. Therefore X is (i,j)-nearly Lindelöf. □

Corollary 3.4. *A pairwise almost regular-Lindelöf and pairwise almost regular space X is pairwise nearly Lindelöf.*

Remark 3.1. *The Proposition* 3.1 *immediately answers the restriction of Question* 3.1 *since (i,j)-nearly Lindelöf spaces is also (i,j)-almost Lindelöf.*

Proposition 3.1 implies the following corollary.

Corollary 3.5. *Let X be an (i,j)-almost regular space. Then X is (i,j)-almost regular-Lindelöf if and only if it is (i,j)-nearly Lindelöf.*

Corollary 3.6. *Let X be a pairwise almost regular space. Then X is pairwise almost regular-Lindelöf if and only if it is pairwise nearly Lindelöf.*

Lemma 3.2. *An (i,j)-semiregular space X is (i,j)-nearly Lindelöf if and only if it is i-Lindelöf.*

Proof. Let $\mathcal{U} = \{U_\alpha : \alpha \in \Delta\}$ be an i-open cover of X. For each $x \in X$, there exists $\alpha_x \in \Delta$ such that $x \in U_{\alpha_x}$. Since X is (i,j)-semiregular and $U_{\alpha_x} \in \mathcal{U}$, there is an i-open set V_{α_x} such that $x \in V_{\alpha_x} \subseteq i\text{-int}\,(\mathrm{j\text{-}cl(V_{\alpha_x})}) \subseteq \mathrm{U_{\alpha_x}}$. So, $X = \bigcup\limits_{x\in X} V_{\alpha_x} \subseteq \bigcup\limits_{x\in X} i\text{-int}\,(\mathrm{j\text{-}cl(V_{\alpha_x})})$. Now $\{i\text{-int}\,(\mathrm{j\text{-}cl\,(V_{\alpha_x})}) : \mathrm{x \in X}\}$ forms an (i,j)-regular open cover of X. Since X is (i,j)-nearly Lindelöf, there exists a countable subset of points $x_1, x_2, \dots, x_n, \dots$ of X such that

$$X = \bigcup_{n\in\mathbb{N}} i\text{-int}\,(\mathrm{j\text{-}cl\,(V_{\alpha_{x_n}})}) \subseteq \bigcup_{\mathrm{n}\in\mathbb{N}} \mathrm{U_{\alpha_{x_n}}}.$$

Therefore X is i-Lindelöf. The converse is obvious. □

Corollary 3.7. *A pairwise semiregular space X is pairwise nearly Lindelöf if and only if it is Lindelöf.*

Theorem 2.1, Proposition 3.1 and Lemma 3.2 yield the following proposition.

Proposition 3.2. *An (i,j)-regular and (i,j)-almost regular-Lindelöf space X is i-Lindelöf.*

Corollary 3.8. *A pairwise regular and pairwise almost regular-Lindelöf space X is Lindelöf.*

Proposition 3.2 implies the following corollary.

Corollary 3.9. *Let X be an (i,j)-regular space. Then X is (i,j)-almost regular-Lindelöf if and only if it is i-Lindelöf.*

Corollary 3.10. *Let X be a pairwise regular space. Then X is pairwise almost regular-Lindelöf if and only if it is Lindelöf.*

Proposition 3.3. *Let X be an (i,j)-almost regular and (i,j)-nearly Lindelöf space. Then X is (i,j)-nearly paracompact.*

Proof. Let $\mathcal{V} = \{V_\alpha : \alpha \in \Delta\}$ be an (i,j)-regular open cover of X. For each $x \in X$, there exists $\alpha_x \in \Delta$ such that $x \in V_{\alpha_x}$. Since X is (i,j)-almost regular, there exists an (i,j)-regular open neighbourhood U_{α_x} of x such that $x \in U_{\alpha_x} \subseteq j\text{-cl}\,(\mathrm{U_{\alpha_x}}) \subseteq \mathrm{V_{\alpha_x}}$. So $\{U_{\alpha_x} : x \in X\}$ is an (i,j)-regular open cover of X. Since X is (i,j)-nearly Lindelöf, there exists a countable subset of points $x_1, x_2, \dots, x_n, \dots$ of X such that $X = \bigcup\limits_{n\in\mathbb{N}} U_{\alpha_{x_n}}$. For each $n \in \mathbb{N}$, put $G_n = V_{\alpha_{x_n}} \setminus \left(\bigcup\limits_{k=1}^{n-1} i\text{-cl}\left(\mathrm{U_{\alpha_{x_k}}}\right)\right)$. By construction $\{G_n : n \in \mathbb{N}\}$ is an i-locally finite family of $\mathcal{V}$. In fact, if $x \in X$ then there exist $U_{\alpha_{x_p}}$ (since $\left\{U_{\alpha_{x_n}} : n \in \mathbb{N}\right\}$ is a cover of X) and $V_{\alpha_{x_p}}$ such that $x \in U_{\alpha_{x_p}} \subseteq V_{\alpha_{x_p}}$. We will prove that $U_{\alpha_{x_p}}$ intersects at most finitely many members of the family $\{G_n : n \in \mathbb{N}\}$. Since $G_1 = V_{\alpha_{x_1}}, G_2 = V_{\alpha_{x_2}} \setminus i\text{-cl}\left(\mathrm{U_{\alpha_{x_1}}}\right), \dots, \mathrm{G_p} = \mathrm{V_{\alpha_{x_p}}} \setminus \left(\mathrm{i\text{-}cl}\left(\mathrm{U_{\alpha_{x_1}}}\right) \cup \dots \cup \mathrm{i\text{-}cl}\left(\mathrm{U_{\alpha_{x_{p-1}}}}\right)\right), \mathrm{G_{p+1}} = \mathrm{V_{\alpha_{x_{p+1}}}} \setminus \left(\mathrm{i\text{-}cl}\left(\mathrm{U_{\alpha_{x_1}}}\right) \cup \dots \cup \mathrm{i\text{-}cl}\left(\mathrm{U_{\alpha_{x_p}}}\right)\right)$, then $U_{\alpha_{x_p}} \cap G_r = \emptyset$ for each $r \geq p+1$. Therefore $U_{\alpha_{x_p}}$ intersects at most a finite number of sets in the family $\{G_n : n \in \mathbb{N}\}$. Next we assert that $\{G_n : n \in \mathbb{N}\}$ is the required i-open refinement of $\mathcal{V}$. Let x be any point of X. We wish to prove that x lies in an element of $\{G_n : n \in \mathbb{N}\}$. Consider the cover $\left\{V_{\alpha_{x_n}} : n \in \mathbb{N}\right\}$ of X; let N be the smallest integer such that x lies in $V_{\alpha_{x_N}}$. Observe that the point x is not lies in G_k for $k < N$ but x lies in G_N since it is not lies in $\bigcup\limits_{k=1}^{N-1} i\text{-cl}\left(\mathrm{U}_{\alpha_{\mathrm{x_k}}}\right)$. Therefore $x \in \bigcup\limits_{n\in\mathbb{N}} G_n$ which implies that $\{G_n : n \in \mathbb{N}\}$ covers X. This shows that X is (i, j)-nearly paracompact. □

Corollary 3.11. *Let X be a pairwise almost regular and pairwise nearly Lindelöf space. Then X is pairwise nearly paracompact.*

By using Proposition 3.1 and Proposition 3.3, we have the following proposition.

Proposition 3.4. *An (i, j)-almost regular and (i, j)-almost regular-Lindelöf space X is (i, j)-nearly paracompact.*

Corollary 3.12. *A pairwise almost regular and pairwise almost regular-Lindelöf space X is pairwise nearly paracompact.*

Definition 3.3. [3] *A bitopological space X is said to be (i, j)-extremally disconnected if the i-closure of every j-open set is j-open. X is called pairwise extremally disconnected if it is both $(1, 2)$-extremally disconnected and $(2, 1)$-extremally disconnected.*

Proposition 3.5. *If (X, τ_1, τ_2) is (j, i)-extremally disconnected, then it is (i, j)-almost regular.*

Proof. Let $x \in X$ and let V be an (i, j)-regular open subset of X containing x. Hence V is also an i-open subset of X containing x. Since X is (j, i)-extremally disconnected, j-cl (V) is an i-open set in X and so j-cl (V) = i-int (j-cl (V)) = V. Thus V is an (i, j)-regular open subset of X such that $x \in V \subseteq j$-cl (V) $\subseteq$ V. This shows that X is (i, j)-almost regular. □

Corollary 3.13. *If (X, τ_1, τ_2) is pairwise extremally disconnected, then it is pairwise almost regular.*

The converse of Proposition 3.5 is not true by the following counterexample.

Example 3.1. *Consider $(\mathbb{R}, \tau_1, \tau_2)$ where $\mathbb{R}$ is the real line, τ_1 is usual topology and τ_2 is a right order topology* [16] *(or right ray topology or right hand topology), i.e., a topology generated by $\{(x, \infty) : x \in \mathbb{R}\}$. Observe that $\mathbb{R}$ is $(2, 1)$-almost regular since the only $(2, 1)$-regular open subsets of $\mathbb{R}$ are sets of the form $(x, \infty), x \in \mathbb{R}$. But it is not $(1, 2)$-extremally disconnected since $(3, \infty)$ is a 2-open set in $\mathbb{R}$ and 1-cl $((3, \infty)) = [3, \infty)$ is not 2-open set in $\mathbb{R}$.*

It is easy to prove the following proposition.

Proposition 3.6. *Let (X, τ_1, τ_2) be a (j, i)-extremally disconnected and (i, j)-almost regular-Lindelöf space, then it is (i, j)-nearly Lindelöf.*

Proof. This is a direct consequence of Proposition 3.5 and Proposition 3.1 above. □

Corollary 3.14. *Let (X, τ_1, τ_2) be a pairwise extremally disconnected and pairwise almost regular-Lindelöf space, then it is pairwise nearly Lindelöf.*

Proposition 3.6 implies the following corollary.

Corollary 3.15. *Let (X, τ_1, τ_2) be a (j, i)-extremally disconnected space. Then X is (i, j)-almost regular-Lindelöf if and only if it is (i, j)-nearly Lindelöf.*

Corollary 3.16. *Let (X, τ_1, τ_2) be a pairwise extremally disconnected space. Then X is pairwise almost regular-Lindelöf if and only if it is pairwise nearly Lindelöf.*

4. **Pairwise Almost Regular-Lindelöf Subspaces and Subsets**

A subset S of a bitopological space X is said to be (i,j)-almost regular-Lindelöf (resp. pairwise almost regular-Lindelöf) if S is (i,j)-almost regular-Lindelöf (resp. pairwise almost regular-Lindelöf) as a subspace of X, i.e., S is (i,j)-almost regular-Lindelöf (resp. pairwise almost regular-Lindelöf) with respect to the inducted bitopology from the bitopology of X.

Definition 4.1. *Let S be a subset of a bitopological space X. A cover $\{U_\alpha : \alpha \in \Delta\}$ of S by i-open subsets of X such that $S \subseteq \bigcup_{\alpha\in\Delta} U_\alpha$ is said to be (i,j)-regular cover of S by i-open subsets of X if for each $\alpha \in \Delta$, there exists a (j,i)-regular closed subset C_α of X such that $C_\alpha \subseteq U_\alpha$ and $S \subseteq \bigcup_{\alpha\in\Delta} i\text{-int}(\mathrm{C}_\alpha)$. $\{U_\alpha : \alpha \in \Delta\}$ is called pairwise regular cover by open subsets of X if it is both $(1,2)$-regular cover of S by 1-open subsets of X and $(2,1)$-regular cover of S by 2-open subsets of X.*

Definition 4.2. *A subset S of a bitopological space X is said to be (i,j)-almost regular-Lindelöf relative to X if for every (i,j)-regular cover $\{U_\alpha : \alpha \in \Delta\}$ of S by i-open subsets of X, there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of Δ such that $S \subseteq \bigcup_{n\in\mathbb{N}} j\text{-cl}(\mathrm{U}_{\alpha_n})$. S is called pairwise almost regular-Lindelöf relative to X if it is both $(1,2)$-almost regular-Lindelöf relative to X and $(2,1)$-almost regular-Lindelöf relative to X.*

It is obvious by the Definitions 4.1 and 4.2 that every (i,j)-almost Lindelöf relative to the space is (i,j)-almost regular-Lindelöf relative to the space.

Question 4.1. *Is (i,j)-almost regular-Lindelöf relative to the space implies (i,j)-almost Lindelöf relative to the space?*

The authors conjecture that the answer is no.

Theorem 4.1. *Let X be a bitopological space and $S \subseteq X$. The following are equivalent:*
(i) S is (i,j)-almost regular-Lindelöf relative to X;
(ii) for every family $\{C_\alpha : \alpha \in \Delta\}$ of i-closed subsets of X such that for each $\alpha \in \Delta$ there exists a (j,i)-regular open subset A_α of X with $A_\alpha \supseteq C_\alpha$ and $\left(\bigcap_{\alpha\in\Delta} i\text{-cl}(\mathrm{A}_\alpha)\right) \cap S = \emptyset$, there exists a countable subfamily $\{C_{\alpha_n} : n \in \mathbb{N}\}$ such that $\left(\bigcap_{n\in\mathbb{N}} j\text{-int}(\mathrm{C}_{\alpha_n})\right) \cap S = \emptyset$;
(iii) for every family $\{C_\alpha : \alpha \in \Delta\}$ of i-closed subsets of X for which every countable subfamily $\{C_{\alpha_n} : n \in \mathbb{N}\}$ satisfies $\left(\bigcap_{n\in\mathbb{N}} j\text{-int}(\mathrm{C}_{\alpha_n})\right) \cap S \neq \emptyset$, the intersection $\left(\bigcap_{\alpha\in\Delta} i\text{-cl}(\mathrm{A}_\alpha)\right) \cap S \neq \emptyset$ for each (j,i)-regular open subset A_α of X with $A_\alpha \supseteq C_\alpha$.

Proof. The proof of this theorem is similar to the proof of the Theorem 3.1 above in fact the Theorem 3.1 is a special case of this theorem for $S = X$. It can be done by replacing $X = X \setminus \left(\bigcap_{\alpha\in\Delta} i\text{-cl}(\mathrm{A}_\alpha)\right)$ with $S \subseteq X \setminus \left(\bigcap_{\alpha\in\Delta} i\text{-cl}(\mathrm{A}_\alpha)\right)$. □

Corollary 4.1. *Let X be a bitopological space and $S \subseteq X$. The following are equivalent:*
(i) S is pairwise almost regular-Lindelöf relative to X;
(ii) for every family $\{C_\alpha : \alpha \in \Delta\}$ of i-closed subsets of X such that for each $\alpha \in \Delta$ there exists a (j,i)-regular open subset A_α of X with $A_\alpha \supseteq C_\alpha$ and $\left(\bigcap_{\alpha\in\Delta} i\text{-cl}(\mathrm{A}_\alpha)\right) \cap S = \emptyset$, there exists a countable subfamily $\{C_{\alpha_n} : n \in \mathbb{N}\}$ such that $\left(\bigcap_{n\in\mathbb{N}} j\text{-int}(\mathrm{C}_{\alpha_n})\right) \cap S = \emptyset$ for each $i,j \in \{1,2\}, i \neq j$;
(iii) for every family $\{C_\alpha : \alpha \in \Delta\}$ of i-closed subsets of X for which every countable subfamily $\{C_{\alpha_n} : n \in \mathbb{N}\}$ satisfies $\left(\bigcap_{n\in\mathbb{N}} j\text{-int}(\mathrm{C}_{\alpha_n})\right) \cap S \neq \emptyset$, the intersection $\left(\bigcap_{\alpha\in\Delta} i\text{-cl}(\mathrm{A}_\alpha)\right) \cap S \neq \emptyset$ for each (j,i)-regular open subset A_α of X with $A_\alpha \supseteq C_\alpha$ for each $i,j \in \{1,2\}, i \neq j$.

Definition 4.3. *Let S be a subset of a bitopological space X. A cover $\{U_\alpha : \alpha \in \Delta\}$ of S by (i,j)-regular open subsets of X such that $S \subseteq \bigcup_{\alpha\in\Delta} U_\alpha$ is said to be (i,j)-regular cover of S by (i,j)-regular open subsets of X if for each $\alpha \in \Delta$, there exists a (j,i)-regular closed subset C_α of X such that $C_\alpha \subseteq U_\alpha$ and $S \subseteq \bigcup_{\alpha\in\Delta} i\text{-int}(\mathrm{C}_\alpha)$. $\{U_\alpha : \alpha \in \Delta\}$ is called pairwise regular cover by pairwise regular open subsets of X if it is both $(1,2)$-regular cover of S by $(1,2)$-regular open subsets of X and $(2,1)$-regular cover of S by $(2,1)$-regular open subsets of X.*

Theorem 4.2. *Let X be a bitopological space and $S \subseteq X$. For the following conditions*:
(i) S is (i,j)-almost regular-Lindelöf relative to X;
(ii) for every (i,j)-regular cover $\{U_\alpha : \alpha \in \Delta\}$ of S by (i,j)-regular open subsets of X, there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of Δ such that $S \subseteq \bigcup_{n\in\mathbb{N}} j\text{-cl}(\mathrm{U}_{\alpha_n})$;
(iii) for every family $\{C_\alpha : \alpha \in \Delta\}$ of (i,j)-regular closed subsets of X such that for each $\alpha \in \Delta$ there exists a (j,i)-regular open subset A_α of X with $A_\alpha \supseteq C_\alpha$ and $\left(\bigcap_{\alpha\in\Delta} i\text{-cl}(\mathrm{A}_\alpha)\right) \cap S = \emptyset$, there exists a countable subfamily $\{C_{\alpha_n} : n \in \mathbb{N}\}$ such that $\left(\bigcap_{n\in\mathbb{N}} j\text{-int}(\mathrm{C}_{\alpha_n})\right) \cap S = \emptyset$;
(iv) for every family $\{C_\alpha : \alpha \in \Delta\}$ of (i,j)-regular closed subsets of X for which every countable subfamily $\{C_{\alpha_n} : n \in \mathbb{N}\}$ satisfies $\left(\bigcap_{n\in\mathbb{N}} j\text{-int}(\mathrm{C}_{\alpha_n})\right) \cap S \neq \emptyset$, the intersection $\left(\bigcap_{\alpha\in\Delta} i\text{-cl}(\mathrm{A}_\alpha)\right) \cap S \neq \emptyset$ for each (j,i)-regular open subset A_α of X with $A_\alpha \supseteq C_\alpha$;
we have that $(i) \Rightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv)$ and if X is (i,j)-semiregular, then $(ii) \Rightarrow (i)$.

Proof. The proof of this theorem is similar to the proof of the Theorem 3.2 above and Theorem 3.2 is a special case of this theorem. So we omit the details. □

Corollary 4.2. *Let X be a bitopological space and $S \subseteq X$. For the following conditions*:
(i) S is pairwise almost regular-Lindelöf relative to X;
(ii) for every (i,j)-regular cover $\{U_\alpha : \alpha \in \Delta\}$ of S by (i,j)-regular open subsets of X, there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of Δ such that $S \subseteq \bigcup_{n\in\mathbb{N}} j\text{-cl}(\mathrm{U}_{\alpha_n})$ for each $i,j \in \{1,2\}, i \neq j$;
(iii) for every family $\{C_\alpha : \alpha \in \Delta\}$ of (i,j)-regular closed subsets of X such that for each $\alpha \in \Delta$ there exists a (j,i)-regular open subset A_α of X with $A_\alpha \supseteq C_\alpha$ and $\left(\bigcap_{\alpha\in\Delta} i\text{-cl}(\mathrm{A}_\alpha)\right) \cap S = \emptyset$, there exists a countable subfamily $\{C_{\alpha_n} : n \in \mathbb{N}\}$ such that $\left(\bigcap_{n\in\mathbb{N}} j\text{-int}(\mathrm{C}_{\alpha_n})\right) \cap S = \emptyset$ for each $i,j \in \{1,2\}, i \neq j$;
(iv) for every family $\{C_\alpha : \alpha \in \Delta\}$ of (i,j)-regular closed subsets of X for which every countable subfamily $\{C_{\alpha_n} : n \in \mathbb{N}\}$ satisfies $\left(\bigcap_{n\in\mathbb{N}} j\text{-int}(\mathrm{C}_{\alpha_n})\right) \cap S \neq \emptyset$, the intersection $\left(\bigcap_{\alpha\in\Delta} i\text{-cl}(\mathrm{A}_\alpha)\right) \cap S \neq \emptyset$ for each (j,i)-regular open subset A_α of X with $A_\alpha \supseteq C_\alpha$ for each $i,j \in \{1,2\}, i \neq j$;
we have that $(i) \Rightarrow (ii) \Leftrightarrow (iii) \Leftrightarrow (iv)$ and if X is pairwise semiregular, then $(ii) \Rightarrow (i)$.

Proposition 4.1. *If $\{A_k : k \in \mathbb{N}\}$ is a countable family of subsets of a space X such that each A_k is (i,j)-almost regular-Lindelöf relative to X, then $\bigcup\{A_k : k \in \mathbb{N}\}$ is (i,j)-almost regular-Lindelöf relative to X.*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be an (i,j)-regular cover of $\bigcup\{A_k : k \in \mathbb{N}\}$ by i-open subsets of X. Then for each $\alpha \in \Delta$, there exists a (j,i)-regular closed subset C_α of X such that $C_\alpha \subseteq U_\alpha$ and $\bigcup_{k\in\mathbb{N}} A_k \subseteq \bigcup_{\alpha\in\Delta} i\text{-int}(\mathrm{C}_\alpha)$. Let $\Delta_k = \{\alpha \in \Delta : U_\alpha \cap A_k \neq \emptyset\}$, then for each $\alpha_k \in \Delta_k \subseteq$

Δ there exists a (j,i)-regular closed subset C_{α_k} of X such that $C_{\alpha_k} \subseteq U_{\alpha_k}$ and $A_k \subseteq \bigcup_{\alpha_k \in \Delta_k} i\text{-int}\left(\mathrm{C}_{\alpha_k}\right)$. So $\{U_{\alpha_k} : \alpha_k \in \Delta_k\}$ is an (i,j)-regular cover of A_k by i-open subsets of X. Since A_k is (i,j)-almost regular-Lindelöf relative to X, there exists a countable subfamily $\left\{U_{\alpha_{k_n}} : n \in \mathbb{N}\right\}$ such that $A_k \subseteq \bigcup_{n\in\mathbb{N}} j\text{-cl}\left(\mathrm{U}_{\alpha_{k_n}}\right)$. But a countable union of countable sets is countable, so $\bigcup_{k\in\mathbb{N}} A_k \subseteq \bigcup_{k\in\mathbb{N}}\left(\bigcup_{n\in\mathbb{N}} j\text{-cl}\left(\mathrm{U}_{\alpha_{k_n}}\right)\right) = \bigcup_{k_n\in\mathbb{N}} j\text{-cl}\left(\mathrm{U}_{\alpha_{k_n}}\right)$. This implies that $\bigcup\{A_k : k \in \mathbb{N}\}$ is (i,j)-almost regular-Lindelöf relative to X. $\square$

Corollary 4.3. *If $\{A_k : k \in \mathbb{N}\}$ is a countable family of subsets of a space X such that each A_k is pairwise almost regular-Lindelöf relative to X, then $\bigcup\{A_k : k \in \mathbb{N}\}$ is pairwise almost regular-Lindelöf relative to X.*

Theorem 4.3. *If S is an (i,j)-almost regular-Lindelöf subspace of a bitopological space X, then S is (i,j)-almost regular-Lindelöf relative to X.*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be an (i,j)-regular cover of S by i-open subsets of X. Then for each $\alpha \in \Delta$, there exists a (j,i)-regular closed subset C_α of X such that $C_\alpha \subseteq U_\alpha$ and $S \subseteq \bigcup_{\alpha\in\Delta} i\text{-int}_{\mathrm{X}}\left(\mathrm{C}_\alpha\right)$. For each $\alpha \in \Delta$, we have $i\text{-int}_{\mathrm{X}}\left(\mathrm{C}_\alpha\right) \cap \mathrm{S}$ and $U_\alpha \cap S$ are i-open sets in S, and $C_\alpha \cap S$ is j-closed set in S. Since for each $\alpha \in \Delta$, there exists a (j,i)-regular closed set $j\text{-cl}_{\mathrm{S}}\left(\mathrm{i\text{-}int_X}\left(\mathrm{C}_\alpha\right) \cap \mathrm{S}\right)$ in S such that $j\text{-cl}_{\mathrm{S}}\left(\mathrm{i\text{-}int_X}\left(\mathrm{C}_\alpha\right) \cap \mathrm{S}\right) \subseteq \mathrm{C}_\alpha \cap \mathrm{S} \subseteq \mathrm{U}_\alpha \cap \mathrm{S}$ and $S = \left(\bigcup_{\alpha\in\Delta} i\text{-int}_{\mathrm{X}}\left(\mathrm{C}_\alpha\right)\right) \cap S = \bigcup_{\alpha\in\Delta}\left(i\text{-int}_{\mathrm{X}}\left(\mathrm{C}_\alpha\right) \cap \mathrm{S}\right) \subseteq \bigcup_{\alpha\in\Delta} i\text{-int}_{\mathrm{S}}\left(\mathrm{j\text{-}cl_S}\left(\mathrm{i\text{-}int_X}\left(\mathrm{C}_\alpha\right) \cap \mathrm{S}\right)\right)$, i.e., $S = \bigcup_{\alpha\in\Delta} i\text{-int}_{\mathrm{S}}\left(\mathrm{j\text{-}cl_S}\left(\mathrm{i\text{-}int_X}\left(\mathrm{C}_\alpha\right) \cap \mathrm{S}\right)\right)$, then the family $\{U_\alpha \cap S : \alpha \in \Delta\}$ is an (i,j)-regular cover of S. Since S is an (i,j)-almost regular-Lindelöf subspace of X, there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of Δ such that $S = \bigcup_{n\in\mathbb{N}} j\text{-cl}_{\mathrm{S}}\left(\mathrm{U}_{\alpha_\mathrm{n}} \cap \mathrm{S}\right)$. Since for each $n \in \mathbb{N}$, $j\text{-cl}_{\mathrm{S}}\left(\mathrm{U}_{\alpha_\mathrm{n}} \cap \mathrm{S}\right) \subseteq \mathrm{j\text{-}cl_X}\left(\mathrm{U}_{\alpha_\mathrm{n}}\right)$, we obtain that $S \subseteq \bigcup_{n\in\mathbb{N}} j\text{-cl}_{\mathrm{X}}\left(\mathrm{U}_{\alpha_\mathrm{n}}\right)$. This shows that S is (i,j)-almost regular-Lindelöf relative to X. $\square$

Corollary 4.4. *If S is a pairwise almost regular-Lindelöf subspace of a bitopological space X, then S is pairwise almost regular-Lindelöf relative to X.*

Question 4.2. *Is the converse of Theorem* 4.3 *above true?*

The authors conjecture that the answer is no.

Theorem 4.4. *If every (i,j)-regular closed proper subset of a bitopological space X is (i,j)-almost regular-Lindelöf relative to X, then X is (i,j)-almost regular-Lindelöf.*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be an (i,j)-regular cover of X. Then for each $\alpha \in \Delta$, there exists a (j,i)-regular closed subset C_α of X such that $C_\alpha \subseteq U_\alpha$ and $X = \bigcup_{\alpha\in\Delta} i\text{-int}\left(\mathrm{C}_\alpha\right)$. Choose an arbitrary $\alpha_0 \in \Delta$ so that $i\text{-int}\left(\mathrm{C}_{\alpha_0}\right)$ is nonempty and let $\Delta^* = \Delta \setminus \{\alpha_0\}$. Put $K = X \setminus \left(i\text{-int}\left(\mathrm{C}_{\alpha_0}\right)\right)$, then K is an (i,j)-regular closed proper subset of X and $K \subseteq \bigcup_{\alpha\in\Delta^*} i\text{-int}\left(\mathrm{C}_\alpha\right)$. Therefore $\{U_\alpha : \alpha \in \Delta^*\}$ is an (i,j)-regular cover of K by i-open subsets of X by Definition 4.1. By hypothesis, K is (i,j)-almost regular-Lindelöf relative to X, hence there exists a countable subset $\{\alpha_n : n \in \mathbb{N}^*\}$ of Δ^* such that $K \subseteq \bigcup_{n\in\mathbb{N}^*} j\text{-cl}\left(\mathrm{U}_{\alpha_\mathrm{n}}\right)$. So we have

$$X = K \cup \left(i\text{-int}\left(\mathrm{C}_{\alpha_0}\right)\right) \subseteq K \cup \left(j\text{-cl}\left(\mathrm{U}_{\alpha_0}\right)\right) \subseteq \left(\bigcup_{n\in\mathbb{N}^*} j\text{-cl}\left(\mathrm{U}_{\alpha_\mathrm{n}}\right)\right) \cup \left(j\text{-cl}\left(\mathrm{U}_{\alpha_0}\right)\right) = \bigcup_{n\in\mathbb{N}} j\text{-cl}\left(\mathrm{U}_{\alpha_\mathrm{n}}\right).$$

This shows that X is (i,j)-almost regular-Lindelöf. $\square$

Corollary 4.5. *If every (i,j)-regular closed proper subset of a bitopological space X is (i,j)-almost regular-Lindelöf relative to X for each $i,j \in \{1,2\}, i \neq j$, then X is pairwise almost regular-Lindelöf.*

Theorem 4.5. *Let X be a bitopological space and let A a (j,i)-clopen subspace of X. Then A is (i,j)-almost regular-Lindelöf if and only if it is (i,j)-almost regular-Lindelöf relative to X.*

Proof. The necessity can be obtained from Theorem 4.3. For the sufficiency, let $\{U_\alpha : \alpha \in \Delta\}$ be an (i,j)-regular cover of A. Then for each $\alpha \in \Delta, U_\alpha$ is i-open set in A and there exists a (j,i)-regular closed subset C_α of A such that $C_\alpha \subseteq U_\alpha$ and $A = \bigcup_{\alpha\in\Delta} i\text{-int}_A(C_\alpha)$. Since i-open and (j,i)-regular closed subsets of a (j,i)-clopen subspace of X is i-open and (j,i)-regular closed sets in X respectively, then $\{U_\alpha : \alpha \in \Delta\}$ is an (i,j)-regular cover of A by i-open subsets of X. Since A is (i,j)-almost regular-Lindelöf relative to X, there exists a countable subset $\{\alpha_n : n \in \mathbb{N}\}$ of Δ such that $A \subseteq \bigcup_{n\in\mathbb{N}} j\text{-cl}_X(U_{\alpha_n})$. Then $A = \left(\bigcup_{n\in\mathbb{N}} j\text{-cl}_X(U_{\alpha_n})\right) \cap A = \bigcup_{n\in\mathbb{N}} (j\text{-cl}_X(U_{\alpha_n}) \cap A) = \bigcup_{n\in\mathbb{N}} j\text{-cl}_A(U_{\alpha_n})$. Therefore A is (i,j)-almost regular-Lindelöf. □

Corollary 4.6. *Let X be a bitopological space and let A a clopen subspace of X. Then A is pairwise almost regular-Lindelöf if and only if it is pairwise almost regular-Lindelöf relative to X.*

Note that, the space X in above propositions is any bitopological space. If we consider X itself is an (i,j)-almost regular-Lindelöf, we have the following results.

Theorem 4.6. *Let X be an (i,j)-almost regular-Lindelöf space. If A is a proper (i,j)-clopen subset of X, then A is (i,j)-almost regular-Lindelöf relative to X.*

Proof. Let $\{U_\alpha : \alpha \in \Delta\}$ be an (i,j)-regular cover of A by i-open subsets of X. Hence the family $\{U_\alpha : \alpha \in \Delta\} \cup \{X \setminus A\}$ is an (i,j)-regular cover of X since $X \setminus A$ is a proper (j,i)-clopen subset of X is also a (j,i)-regular closed subset of X. Since X is (i,j)-almost regular-Lindelöf, there exists a countable subfamily $\{X \setminus A, U_{\alpha_1}, U_{\alpha_2}, \dots\}$ such that $X = \left(\bigcup_{n\in\mathbb{N}} j\text{-cl}(U_{\alpha_n})\right) \cup j\text{-cl}(X \setminus A) = \left(\bigcup_{n\in\mathbb{N}} j\text{-cl}(U_{\alpha_n})\right) \cup (X \setminus A)$. But A and $X \setminus A$ are disjoint; therefore we have $A \subseteq \bigcup_{n\in\mathbb{N}} j\text{-cl}(U_{\alpha_n})$. This completes the proof. □

Corollary 4.7. *Let X be a pairwise almost regular-Lindelöf space. If A is a proper clopen subset of X, then A is pairwise almost regular-Lindelöf relative to X.*

It is very clear that Theorem 4.6 implies the following corollary.

Corollary 4.8. *Let X be an (i,j)-almost regular-Lindelöf space. If A is an (i,j)-clopen subset of X, then A is (i,j)-almost regular-Lindelöf relative to X.*

Theorem 4.7. *Let X be an (i,j)-almost regular-Lindelöf space. If A is a clopen subset of X, then A is (i,j)-almost regular-Lindelöf.*

Proof. This is a direct consequence of Corollary 4.8 and Theorem 4.5 above. □

Corollary 4.9. *Let X be a pairwise almost regular-Lindelöf space. If A is a clopen subset of X, then A is pairwise almost regular-Lindelöf relative to X.*

Question 4.3. *Is i-closed subspace of an (i,j)-almost regular-Lindelöf space X (i,j)-almost regular-Lindelöf?*

Question 4.4. *Is (i,j)-regular closed subspace of an (i,j)-almost regular-Lindelöf space X (i,j)-almost regular-Lindelöf?*

The authors conjecture that the answers are no. Observe that the condition in Theorem 4.6 that a subset should be (i,j)-clopen is necessary and it is not sufficient to be only i-open or (i,j)-regular open as examples below show. Arbitrary subspaces of (i,j)-almost regular-Lindelöf spaces need not be (i,j)-almost regular-Lindelöf nor (i,j)-almost regular-Lindelöf relative to the spaces. An i-open or (i,j)-regular open subset of an (i,j)-almost regular-Lindelöf space is neither (i,j)-almost regular-Lindelöf nor (i,j)-almost regular-Lindelöf relative to the spaces as the following examples also show. We need the following lemma (see [17], 11).

Lemma 4.1. *If A is a countable subset of ordinals Ω not containing ω_1 where ω_1 being the first uncountable ordinal, then* $\sup A < \omega_1$.

Example 4.1. *Let Ω denotes the set of ordinals which are less than or equal to the first uncountable ordinal number ω_1, i.e., $\Omega = [0, \omega_1]$. This Ω is an uncountable well-ordered set with a largest element ω_1, having the property that if $\alpha \in \Omega$ with $\alpha < \omega_1$, then $\{\beta \in \Omega : \beta \leq \alpha\}$ is countable. Since Ω is a totally ordered space, it can be provided with its order topology. Let us denote this order topology by τ_1. Choose discrete topology as another topology for Ω denoted by τ_2. So (Ω, τ_1, τ_2) forms a bitopological space. Now it is known that Ω is a 1-Lindelöf space (see [17, 16]), so it is $(1,2)$-almost Lindelöf and thus $(1,2)$-almost regular-Lindelöf. The subspace $\Omega_0 = \Omega \setminus \{\omega_1\} = [0, \omega_1)$, however is not 1-Lindelöf (see [17, 16]). We notice that Ω_0 is $(2,1)$-clopen subset of Ω and also $(1,2)$-regular open subset of Ω. Observe that Ω_0 is not $(1,2)$-almost regular-Lindelöf by Proposition 3.2 since it is $(1,2)$-regular space. Moreover Ω_0 is not $(1,2)$-almost regular-Lindelöf relative to Ω by Theorem 4.5. By another way, the family $\{[0,\alpha) : \alpha \in \Omega_0\}$ of 1-open sets in Ω is $(1,2)$-regular cover of Ω_0 by 1-open subsets of Ω because $\Omega_0 \subseteq \bigcup_{\alpha \in \Omega_0} [0,\alpha)$ and for each $\alpha \in \Omega_0$, there exists a $(2,1)$-regular closed subset $[0,\alpha)$ of Ω such that $[0,\alpha) \subseteq [0,\alpha)$ and $\Omega_0 \subseteq \bigcup_{\alpha \in \Omega_0} [0,\alpha) = \bigcup_{\alpha \in \Omega_0} 1\text{-int}\,([0,\alpha))$. But the family $\{[0,\alpha) : \alpha \in \Omega_0\}$ has no countable subfamily $\{[0,\alpha_n) : n \in \mathbb{N}\}$ such that $\Omega_0 \subseteq \bigcup_{n \in \mathbb{N}} 2\text{-}\mathrm{cl}\,([0,\alpha_{\mathrm{n}})) = \bigcup_{\mathrm{n} \in \mathbb{N}} [0,\alpha_{\mathrm{n}})$. For if $\{[0,\alpha_1), [0,\alpha_2), \dots\}$ satisfy the condition: 2-closures of it elements cover Ω_0, then $\sup\{\alpha_1, \alpha_2, \dots\} = \omega_1$ which is impossible by Lemma 4.1.*

Example 4.2. *Let $\Omega = [0, \omega_1]$ and τ_1 is the order topology as in Example 4.1. Choose cocountable topology as another topology for Ω denoted by τ_2. Now Ω is a 1-Lindelöf space (see [17, 16]), so it is $(1,2)$-almost Lindelöf and thus $(1,2)$-almost regular-Lindelöf. The subspace $\Omega_0 = \Omega \setminus \{\omega_1\}$, however is not 1-Lindelöf (see [17, 16]). We notice that Ω_0 is $(2,1)$-clopen subset of Ω and also $(1,2)$-regular open subset of Ω. Observe that Ω_0 is not $(1,2)$-almost regular-Lindelöf by Proposition 3.2 since it is $(1,2)$-regular space. Moreover Ω_0 is not $(1,2)$-almost regular-Lindelöf relative to Ω by the similar argument as Example 4.1.*

So we conclude that an (i,j)-almost regular-Lindelöf property is not hereditary property and therefore pairwise almost regular-Lindelöf property is not so.

Acknowledgement: The authors gratefully acknowledge that this research partially supported by Ministry of Science, Technology and Innovations(MOSTI), Malaysia under the e-Science Grant 06-01-04-SF0115. The authors also wish to thank the referees for their constructive comments and suggestions.

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