PATH OF BREGMAN-PETZ OPERATOR DIVERGENCE

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Received June 19, 2007

ABSTRACT. For the Bregman operator divergence defined by D.Petz, we introduce two paths of operator divergences including this one as a terminal. This gives other explanations from the viewpoint of operator means or solidarities.

In [7], Petz introduced the Bregman operator divergence: For an operator convex function F and positive (invertible) operators A and B on a Hilbert space, put

$$\begin{aligned} D_{[F]}(A|B) &= F(A) - F(B) - \lim_{t\to+0} \frac{F(B+t(A-B)) - F(B)}{t} \\ &= \lim_{t\to+0} \frac{tF(A) + (1-t)F(B) - F(tA+(1-t)B)}{t} \\ &= \lim_{t\to+0} \frac{F(B)\ \nabla_t\ F(A) - F(B\ \nabla_t\ A)}{t} \geq 0. \end{aligned}$$

He gives a nice representation of $D_{[F]}$ by hard calculation, by which, for density matrices A and B and $F(x) = x\log x$,

$$\operatorname{Tr} D_{[x\log x]}(A,B) = \operatorname{Tr} A(\log A - \log B) = s(A,B),$$

the Umegaki relative entropy [8].

In [2, 3], we define the relative operator entropy $S(A|B)$ as

$$S(A|B) = A^{\frac{1}{2}}\left(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}},$$

where $-\operatorname{Tr} S(A|B)$ is the Belavkin-Staszewski entropy [1]. Petz also gives another operator version of the Bregman divergence by

$$S_{FK}(A|B) = B - A - S(A|B).$$

But unfortunately $S_{FK}(A|B)$ does not coincides with $D_{[x\log x]}(A,B)$ in general.

So we construct a class of operator divergences including $S_{FK}(A|B)$ from the viewpoint of operator means [6] or solidarity [4], which is based on operator monotone or operator concave functions. To see this, we extend $D_{[F]}(A|B)$ to fit to this viewpoint. Replacing F with an operator concave function $f = -F$, we define a path of divergences extending $D_{[F]}(A|B)$: For $0 \leq t \leq 1$, let

$$\begin{aligned} D_{f,t}(A,B) &= \frac{f(B\ \nabla_t\ A) - f(B)\ \nabla_t\ f(A)}{t(1-t)} \\ &= \frac{f(tA+(1-t)B) - tf(A) - (1-t)f(B)}{t(1-t)}. \end{aligned}$$

2000 *Mathematics Subject Classification.* 47A64, 47A63, 94A15.
Key words and phrases. Bregman divergence, operator divergence, operator mean.

Then we have a symmetric property of this path between $D_{[F]}(A,B)$ and $D_{[F]}(B,A)$:

$$\lim_{t\to 0} D_{f,t}(A,B) = D_{[F]}(A,B), \quad \lim_{t\to 1} D_{f,t}(A,B) = D_{[F]}(B,A).$$

Note that if f is an affine function $f(x) = a + bx$, then $D_{f,t}(A,B) = O$. So we assume that a function f is non-affine throughout this note. Then we have a basic property of this divergence:

Theorem 1. *For a non-affine operator concave function f, the divergence $D_{f,t}(A,B)$ is a positive operator and $D_{f,t}(A,B) = O$ holds if and only if $A = B$.*

For this, we need the following lemma which is easily obtained since it is reduced to the commutative case; $1 - t + tX = (1 - t + tX^{-1})^{-1}$ holds only when $X = I$:

Lemma 2. *For the harmonic mean $A!_tB = ((1-t)A^{-1} + tB^{-1})^{-1}$ for selfadjoint invertible operators A and B, the equation $A\nabla_tB = A!_tB$ holds if and only if $A = B$.*

Proof of Theorem 1. The positivity of $D_{f,t}(A,B)$ is merely the operator concavity of f. To show the extreme case, suppose $f(A\nabla_tB) = f(A)\nabla_tf(B)$. Since we may assume f is operator concave on $(-1,1)$, then f has an integral representation

$$f(x) = a + bx + \int_{-1}^{1} \frac{x^2}{tx-1} dm(t).$$

The essential part of the function is

$$f_0(x) = \frac{x^2}{x-s} = x + s + \frac{s^2}{x-s},$$

so that we have only to show $A = B$ when

$$(A\nabla_tB - s)^{-1} = (1-t)(A-s)^{-1} + t(B-s)^{-1}$$

for some $s \notin (-1,1)$. Taking inverse, we have

$$(A-s)\nabla_t(B-s) = A\nabla_tB - s = (A-s)!_t(B-s).$$

Thus, Lemma 2 shows $A - s = B - s$, that is, $A = B$. The converse is clear. □

Now we will define a path $\mathfrak{D}_{f,t}(A,B)$ including $S_{FK}(A|B)$. The following path of operator divergences is naturally defined, but the symmetric property does not hold, so we denote it by $\tilde{\mathfrak{D}}_{f,t}(A,B)$:

$$\tilde{\mathfrak{D}}_{f,t}(A,B) = A^{1/2}\frac{f(A^{-1/2}BA^{-1/2}\nabla_tI) - f(A^{-1/2}BA^{-1/2})\nabla_tf(I)}{t(1-t)}A^{1/2}.$$

In fact, if $f(x) = \eta(x) \equiv -x\log x$, then this path runs from $S_{FK}(A|B)$ to $S_{FK}(B|A)$:

$$\tilde{\mathfrak{D}}_{\eta,t}(A,B) = A^{\frac{1}{2}}\frac{\eta(X\ \nabla_t\ I) - \eta(X)\ \nabla_t\ \eta(I)}{t(1-t)}A^{\frac{1}{2}},$$

where $X = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$. Then

$$\begin{aligned}\tilde{\mathfrak{D}}_{\eta,0}(A,B) \equiv \lim_{t\to 0}\tilde{\mathfrak{D}}_{\eta,t}(A,B) &= \lim_{t\to 0} A^{\frac{1}{2}}\frac{\eta(X + t(I-X)) - \eta(X) + t\eta(X)}{t(1-t)}A^{\frac{1}{2}}\\ &= A^{\frac{1}{2}}(\eta'(X)(I-X) + \eta(X))A^{\frac{1}{2}}\\ &= A^{\frac{1}{2}}(-\log X - I + X)A^{\frac{1}{2}}\\ &= B - A - S(A|B) = S_{FK}(A|B)\end{aligned}$$

and

$$\begin{aligned}\tilde{\mathfrak{D}}_{\eta,1}(A,B) &\equiv \lim_{t\to 1}\tilde{\mathfrak{D}}_{\eta,t}(A,B) = \lim_{t\to 1} A^{\frac{1}{2}}\frac{\eta(I+(1-t)(X-I)) - \eta(I) - (1-t)\eta(X)}{t(1-t)}A^{\frac{1}{2}} \\ &= A^{\frac{1}{2}}(\eta'(I)(X-I) - \eta(X))A^{\frac{1}{2}} \\ &= A^{\frac{1}{2}}(-X+I+X\log X)A^{\frac{1}{2}} \\ &= A^{\frac{1}{2}}(-A^{-\frac{1}{2}}BA^{-\frac{1}{2}} + I + A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} \\ &= -B + A - B^{-\frac{1}{2}}\log(B^{\frac{1}{2}}AB^{-\frac{1}{2}})B^{\frac{1}{2}} \\ &= A - B - S(B|A) = S_{FK}(B|A).\end{aligned}$$

But a symmetric equation $\tilde{\mathfrak{D}}_{f,0}(A,B) = \tilde{\mathfrak{D}}_{f,1}(B,A)$ does not always hold: Putting $X = A^{-1/2}BA^{-1/2}$, we easily compute it as follows:

$$\begin{aligned}\tilde{\mathfrak{D}}_{f,0}(A,B) &= \lim_{t\to 0} A^{1/2}\big(f'(X\nabla_t I)(I-X) + f(X) - f(I)\big)A^{1/2} \\ &= A^{1/2}\big(f'(X)(I-X) + f(X) - f(I)\big)A^{1/2} \\ &= A^{1/2}f'(X)(I-X)A^{1/2} + A^{1/2}f(X)A^{1/2} - f(I)A\end{aligned}$$

and

$$\begin{aligned}\tilde{\mathfrak{D}}_{f,1}(A,B) &= \lim_{t\to 1}\tilde{\mathfrak{D}}_{f,t}(A,B) = \lim_{t\to 1} A^{\frac{1}{2}}\big(f'(X\nabla_t I)(X-I) - f(X) + f(I)\big)A^{\frac{1}{2}} \\ &= A^{1/2}\big(f'(I)(X-I) - f(X) + f(I)\big)A^{1/2} \\ &= f'(I)(B-A) - A^{1/2}f(X)A^{1/2} + f(I)A.\end{aligned}$$

Thus the symmetric equation is false in general.

To define a symmetric path of operator divergence, we recall the Kubo-Ando theory of operator means [6] in which they gave the one-to-one correspondence between operator means and positive operator monotone functions. For a positive operator monotone function f on $(0,\infty)$, the transpose f° of f, defined by $f^\circ(x) = xf(x^{-1})$, is also positive operator monotone and then $A\ m_{f^\circ}\ B = B\ m_f\ A$, where m_f is the operator mean corresponding to f:

$$A\ m_f B = A^{\frac{1}{2}} f\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)A^{\frac{1}{2}} = B^{\frac{1}{2}} f^\circ\left(B^{-\frac{1}{2}}AB^{-\frac{1}{2}}\right)B^{\frac{1}{2}}.$$

Moreover M.Fujii [5] showed the following equivalence for f which is not always positive:

Theorem F. *A real-valued function f on $(0,\infty)$ is operator concave if and only if its transpose f° is operator concave.*

For example, the entropy function $\eta(x) = -x\log x$ and $\log x$ are operator concave and these are the transpose each other.

Now we define a path of *Bregman-Petz operator divergences* $\mathfrak{D}_{f,t}(A,B)$ for f as

$$B^{\frac{1}{2}}\frac{(f(I\ \nabla_t\ Y) - f(I)\ \nabla_t\ f(Y))\ \nabla_t\ (f^\circ(Y\ \nabla_t\ I) - f^\circ(Y)\ \nabla_t\ f^\circ(I))}{t(1-t)}B^{\frac{1}{2}}$$

for $Y = B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$ and in particular we denote

$$\mathfrak{D}_f(A,B) = \lim_{t\to 0}\mathfrak{D}_{f,t}(A,B).$$

Since

$$B^{-\frac{1}{2}}\mathfrak{D}_{f,t}(A,B)B^{-\frac{1}{2}} = \frac{f(I \nabla_t Y) - f(I)\nabla_t f(Y)}{t} + \frac{f^\circ(Y \nabla_t I) - f^\circ(Y) \nabla_t f^\circ(I)}{1-t},$$

we have

$$\begin{aligned}\mathfrak{D}_f(A,B) &= \lim_{t\to 0}\mathfrak{D}_{f,t}(A,B) = \lim_{t\to 0} B^{\frac{1}{2}}\frac{f(I \nabla_t Y) - f(I) \nabla_t f(Y)}{t}B^{\frac{1}{2}} \\ &= \lim_{t\to 0} B^{\frac{1}{2}}(f'(I \nabla_t Y)(Y-I) + f(I) - f(Y))B^{\frac{1}{2}} \\ &= B^{\frac{1}{2}}(f'(I)(Y-I) + f(I) - f(Y))B^{\frac{1}{2}} \\ &= f'(I)(A-B) + f(I)B - B^{\frac{1}{2}}f(Y)B^{\frac{1}{2}}\end{aligned}$$

and also

$$\begin{aligned}\lim_{t\to 1}\mathfrak{D}_{f,t}(A,B) &= \lim_{t\to 1} B^{\frac{1}{2}}\frac{f^\circ(Y \nabla_t I) - f^\circ(Y) \nabla_t f^\circ(I)}{1-t}B^{\frac{1}{2}} \\ &= \lim_{t\to 0} B^{\frac{1}{2}}\frac{f^\circ(I \nabla_t Y) - f^\circ(I) \nabla_t f^\circ(Y)}{t}B^{\frac{1}{2}} \\ &= (f^\circ)'(I)(A-B) + f^\circ(I)B - B^{\frac{1}{2}}f^\circ(Y)B^{\frac{1}{2}} = \mathfrak{D}_{f^\circ}(A,B).\end{aligned}$$

Thus this path has a kind of symmetry between $\mathfrak{D}_f(A,B)$ and $\mathfrak{D}_{f^\circ}(A,B)$, which is more clarified by the following theorem:

Theorem 3. *Let f be an operator concave function and f° be the transpose of f. Then*

$$\mathfrak{D}_{f^\circ}(A,B) = \mathfrak{D}_f(B,A).$$

Proof. The equality $f^\circ(1) = f(1)$ holds and also $(f^\circ)'(1) = f(1) - f'(1)$ hods since

$$(f^\circ)'(x) = (xf(1/x))' = f(1/x) - \frac{1}{x}f'(1/x).$$

For the above $X = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ and $Y = B^{-\frac{1}{2}}AB^{-\frac{1}{2}}$, we have

$$\begin{aligned}\mathfrak{D}_{f^\circ}(A,B) &= (f(I) - f'(I))(A-B) + f(I)B - B^{\frac{1}{2}}Yf(X)B^{\frac{1}{2}} \\ &= f'(I)(B-A) + f(I)A - A^{\frac{1}{2}}A^{\frac{1}{2}}B^{-\frac{1}{2}}f(((A^{\frac{1}{2}}B^{-\frac{1}{2}})^*A^{\frac{1}{2}}B^{-\frac{1}{2}})^{-1})B^{\frac{1}{2}} \\ &= f'(I)(B-A) + f(I)A - A^{\frac{1}{2}}f((A^{\frac{1}{2}}B^{-\frac{1}{2}}(A^{\frac{1}{2}}B^{-\frac{1}{2}})^*)^{-1})A^{\frac{1}{2}}B^{-\frac{1}{2}}B^{\frac{1}{2}} \\ &= f'(I)(B-A) + f(I)A - A^{\frac{1}{2}}f(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} = \mathfrak{D}_f(B,A).\end{aligned}$$

by the above calculation for $\lim_{t\to 1}\mathfrak{D}_{f,t}(A,B)$. □

Thus $\mathfrak{D}_{f,t}(A,B)$ combines the Bregman-Petz divergences $\mathfrak{D}_f(A,B)$ with $\mathfrak{D}_f(B,A)$. In particular, for $f(x) = \eta(x) = -x\log x$, we have

$$\mathfrak{D}_\eta(A,B) = B - A - S(A|B) = S_{FK}(A|B)$$

and

$$\mathfrak{D}_\eta(B,A) = \mathfrak{D}_{\eta^\circ}(A,B) = A - B - S(B|A) = S_{FK}(B|A).$$

Similarly to Theorem 1, we have the basic property of these divergences:

Theorem 4. *For a non-affine operator concave function f, the Bregman-Petz divergence is positive and equals to zero if and only if the operators are equal:*

$$\begin{aligned}&\mathfrak{D}_{f,t}(A,B) \geq O; \qquad \mathfrak{D}_{f,t}(A,B) = O \iff A = B \\ &\big(\tilde{\mathfrak{D}}_{f,t}(A,B) \geq O; \qquad \tilde{\mathfrak{D}}_{f,t}(A,B) = O \iff A = B\big).\end{aligned}$$

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