IDEAL EXTENSIONS OF WEAKLY REDUCTIVE ORDERED SEMIGROUPS

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Abstract. Ideal extensions of semigroups (without order) have been first considered by Clifford and Preston in 1950. The main theorem of the ideal extensions of ordered semigroups has been studied by Kehayopulu and Tsingelis in 2003. As a continuation of our paper on the ideal extensions of ordered semigroups, we present here the ideal extensions of weakly reductive ordered semigroups.

Introduction. Ideal extensions (or just extensions) of semigroups (without order) have been first considered by Clifford and Preston in [1], with a detailed exposition of the theory in [2],[5]. For (ideal) extensions in the particular case of weakly reductive semigroups we also refer to [2],[5]. The main theorem of the (ideal) extensions of ordered semigroups has been given in [4]. This paper is a continuation of [4]. The aim of the present paper is to construct the (ideal) extensions of weakly reductive ordered semigroups. We start with an weakly reductive ordered semigroup S and an ordered semigroup Q having a zero such that $S \cap Q^* = \emptyset$ (where Q^* is the set of nonzero elements of Q). We construct all the ordered semigroups V having an ideal S' which is isomorphic to S and the Rees quotient V/S' is isomorphic to Q. Conversely, we prove that each ordered semigroup which is an extension of an weakly reductive ordered semigroup S by an ordered semigroup Q can be so constructed. As an application of our results mentioned above, we study the ideal extensions for the weakly reductive ordered semigroup of natural numbers. For the necessary definitions, notations, and prerequisites we refer to [4].

1. The main result. Our aim is to study the main theorem of the (ideal) extensions in the particular case of the weakly reductive ordered semigroups. In case of weakly reductive ordered semigroups the main theorem of the (ideal) extensions of ordered semigroups can be simplified in the way given in this section.

Definition 1. An ordered semigroup $(S, ., \leq)$ is called *weakly reductive* if for each $a, b \in S$ such that $ax \leq bx$ and $xa \leq xb$ for all $x \in S$, we have $a \leq b$.
In a weakly reductive ordered semigroup S, for each $a, b \in S$ such that $ax = bx$ and $xa = xb$ for all $x \in S$, we have $a = b$.

Proposition 2. *An ordered semigroup $(S, ., \leq)$ is weakly reductive if and only if the mapping $\pi : S \to \Omega(S) \mid s \to \pi_s$ is reverse isotone.*

Proof. $\Longrightarrow$. Let $a, b \in S$, $\pi_a \leq_\Omega \pi_b$. Since $\pi_a := (\lambda_a, \rho_a)$ and $\pi_b := (\lambda_b, \rho_b)$, we have $(\lambda_a, \rho_a) \leq_\Omega (\lambda_b, \rho_b)$, so $\lambda_a \leq_\Lambda \lambda_b$ and $\rho_a \leq_{\mathrm{P}} \rho_b$. Then $\lambda_a(x) \leq \lambda_b(x)$ and $\rho_a(x) \leq \rho_b(x)$ for all $x \in S$, from which $ax \leq bx$ and $xa \leq xb$ for all $x \in S$. Since S is weakly reductive, we

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have $a \le b$.

$\Longleftarrow$. Let $a, b \in S$, $ax \le bx$ and $xa \le xb$ for all $x \in S$. Since $\lambda_a(x) \le \lambda_b(x)$ and $\rho_a(x) \le \rho_b(x)$ for all $x \in S$, we have $\lambda_a \le_\Lambda \lambda_b$ and $\rho_a \le_{\mathrm{P}} \rho_b$, then $(\lambda_a, \rho_a) \le_\Omega (\lambda_b, \rho_b)$ i.e. $\pi_a \le_\Omega \pi_b$. Since the mapping π is reverse isotone, we get $a \le b$. □

Theorem 3. *Let $(S, . \le_S)$ be a weakly reductive ordered semigroup, $(Q, ., \le_Q)$ an ordered semigroup with 0, $S \cap Q^* = \emptyset$. Let*

$$\theta : Q^* \to \Omega(S) \mid a \to (\lambda^a, \rho^a)$$

be a partial homomorphism such that

$$\forall\ a, b \in Q^*,\ ab = 0,\ \textit{we have}\ \theta(a).\theta(b) \in \pi(S).$$

Moreover, let $r \subseteq S \times Q^*$ such that the following conditions are satisfied:

(D1) $x \le_S y$, $(y, z) \in r$, $z \le_Q t \Longrightarrow (x, t) \in r$.
(D2) $a, b, c \in Q^*$, $s \in S$, $a \le_Q b$, $bc \ne 0$, $ac = 0$, $\theta(a).\theta(c) = \pi_s \Longrightarrow (s, bc) \in r$.
(D3) $\pi_a \le_\Omega \theta(b)\ \ \forall\ (a, b) \in r$.
(D4) $b, c \in Q^*$, $(a, b) \in r$, $bc \ne 0 \Longrightarrow (\rho^c(a), bc) \in r$.
(D5) $a, b, c \in Q^*$, $s \in S$, $a \le_Q b$, $cb \ne 0$, $ca = 0$, $\theta(c).\theta(a) = \pi_s \Longrightarrow (s, cb) \in r$.
(D6) $b, c \in Q^*$, $(a, b) \in r$, $cb \ne 0 \Longrightarrow (\lambda^c(a), cb) \in r$.

Let $V := S \cup Q^*$ and "$\circ$", "$\le_V$" an operation and an order on V, respectively, defined by:

$$a \circ b := \begin{cases} ab & if\ \ a, b \in S \\ \rho^b(a) & if\ \ a \in S,\ b \in Q^* \\ \lambda^a(b) & if\ \ a \in Q^*,\ b \in S \\ c & where\ c \in S\ such\ that\ \theta(a).\theta(b) = \pi_c\ if\ \ a, b \in Q^*,\ ab = 0 \\ ab & if\ \ a, b, ab \in Q^* \end{cases}$$

and

$$\le_V := \le_S \cup r \cup \{(x, y) \mid x, y \in Q^*,\ x \le_Q y\}.$$

Then $(V, \circ, \le_V)$ is an ordered semigroup and it is an extension of S by Q. Conversely, every extension V of S by Q can be so constructed.

Proof. Since S is weakly reductive, for each $a, b \in Q^*$ the bitranslations $\theta(a)$ and $\theta(b)$ are permutable (cf. [5,3]). We consider the mapping:

$$f : \{(a, b) \mid a, b \in Q^*,\ ab = 0\} \to S \mid (a, b) \to c,\ \text{where } c \in S \text{ such that } \theta(a).\theta(b) = \pi_c.$$

The mapping f is a ramification mapping. In fact: Let $(a_1, b_1), (a_2, b_2) \in \{(a, b) \mid a, b \in Q^*,\ ab = 0\}$ such that $a_1 \le_Q a_2$ and $b_1 \le_Q b_2$. We have $f(a_1, b_1) := s_1$ for some $s_1 \in S$ such that $\theta(a_1).\theta(b_1) = \pi_{s_1}$ and $f(a_2, b_2) := s_2$ for some $s_2 \in S$ such that $\theta(a_2).\theta(b_2) = \pi_{s_2}$. Since $(\Omega(S), ., \le_\Omega)$ is an ordered semigroup, Q an ordered semigroup with 0, $\theta : Q^* \to \Omega(S)$ a partial homomorphism, $a_1, a_2 \in Q^*$ and $a_1 \le_Q a_2$, we have $\theta(a_1) \le_\Omega \theta(a_2)$. Similarly, $b_1, b_2 \in Q^*$, $b_1 \le_Q b_2$ implies $\theta(b_1) \le_\Omega \theta(b_2)$. Then $\theta(a_1).\theta(b_1) \le_\Omega \theta(a_2).\theta(b_2)$, so $\pi_{s_1} \le \pi_{s_2}$. By Proposition 2, we get $s_1 \le s_2$.

The mapping f satisfies the conditions (C1)–(C5) of the Theorem 1 in [4] (cf. [5], for a detailed proof we refer to [3]). Conditions (D1), (D3), (D4), (D6) are, respectively, the conditions (O1), (O3), (O5), (O8) of the Theorem 1 in [4]. Conditions (O2), (O4), (O6) and (O7) of the same theorem in [4] are satisfied as well. In fact,

(O2) Let Let $a, b, c \in Q^*$, $ac = 0$, $bc \neq 0$, $a \leq_Q b$. Then $(f(a,c), bc) \in r$. Indeed: Since $a, c \in Q^*$ and $ac = 0$, by (C1), we have $\theta(a).\theta(c) = \pi_{f(a,c)}$. Then, by (D2), $(f(a,c), bc) \in r$.

(O4) Let $b, c \in Q^*$, $bc = 0$, $(a, b) \in r$. Then $\rho^c(a) \leq_S f(b,c)$. In fact: Since $(a, b) \in r$, by (D3), we obtain $\pi_a \leq_\Omega \theta(b)$. Since $c \in Q^*$, we have $\theta(c) \in \Omega(S)$, then $\pi_a.\theta(c) \leq_\Omega \theta(b).\theta(c)$. Since $b, c \in Q^*$ and $bc = 0$, by (C1), we have $\theta(b).\theta(c) = \pi_{f(b,c)}$. Then $\pi_a.\theta(c) \leq_\Omega \pi_{f(b,c)}$. On the other hand, $\lambda_a.\lambda^c = \lambda_{\rho^c(a)}$ and $\rho_a.\rho^c = \rho_{\rho^c(a)}$. Indeed: Let $x \in S$. Since λ^c is a left translation and λ_a an inner translation (and so a left translation on S, as well), we have $(\lambda_a.\lambda^c)(x) = \lambda_a(\lambda^c(x)) = a\lambda^c(x)$. Since $\theta(c) := (\lambda^c, \rho^c) \in \Omega(S)$, the pair (λ^c, ρ^c) is a bitranslation on S, so $a\lambda^c(x) = \rho^c(a)x = \lambda_{\rho^c(a)}(x)$. Then $(\lambda_a.\lambda^c)(x) = \lambda_{\rho^c(a)}(x)$, and so $\lambda_a.\lambda^c = \lambda_{\rho^c(a)}$. Similarly, since ρ^c and ρ_a are right translations on S, we have $(\rho_a.\rho^c)(x) = \rho^c(\rho^a(x)) = \rho^c(xa) = x\rho^c(a) = \rho_{\rho^c(a)}(x)$, so $\rho_a.\rho^c = \rho_{\rho^c(a)}$. Hence we have $\pi_a.\theta(c) = (\lambda_a, \rho_a).(\lambda^c, \rho^c) = (\lambda_a.\lambda_c, \rho_a.\rho^c) = (\lambda_{\rho^c(a)}, \rho_{\rho^c(a)}) = \pi_{\rho^c(a)}$.Then $\pi_{\rho^c(a)} \leq_\Omega \pi_{f(b,c)}$ and, by Proposition 2, $\rho^c(a) \leq_S f(b,c)$. The proof of condition (O6) (resp. (O7)) is similar to that of (O2) (resp. (O4)).

By the first part of the Theorem 1 in [4], the set $V := S \cup Q^*$ with the operation " $\circ$ " and the order " $\leq_V$ " on V defined by:

$$a \circ b := \begin{cases} ab & if \;\; a, b \in S \\ \rho^b(a) & if \;\; a \in S, \; b \in Q^* \\ \lambda^a(b) & if \;\; a \in Q^*, \; b \in S \\ f(a,b) & if \;\; a, b \in Q^*, \;\; ab = 0 \\ ab & if \;\; a, b, ab \in Q^* \end{cases}$$

and

$$\leq_V := \leq_S \cup r \cup \{(x, y) \mid x, y \in Q^*, \; x \leq_Q y\}$$

is an ordered semigroup and it is an extension of S by Q.

It remains to prove that if $a, b \in Q^*$, $ab = 0$, then $f(a,b)$ is the only element of S such that $\theta(a).\theta(b) = \pi_{f(a,b)}$. Now let $a, b \in Q^*$, $ab = 0$. First of all, by (C1), $\theta(a).\theta(b) = \pi_{f(a,b)}$. Now let $d \in S$ such that $\theta(a).\theta(b) = \pi_d$. Since $\pi_{f(a,b)} = \pi_d$, by Proposition 2, we have $f(a,b) = d$.

The converse statement: Let $(V, ., \leq_V)$ be an extension of $(S, ., \leq_S)$ by $(Q, ., \leq_Q)$. By the second part of the Theorem 1 in [4], there exists a partial homomorphism $\theta : Q^* \to \Omega(S)$ such that for each $a, b \in Q^*$ the bitranslations $\theta(a) := (\lambda^a, \rho^a)$ and $\theta(b) := (\lambda^b, \rho^b)$ are permutable. Moreover, there exists a ramification mapping $f : \{(a,b) \mid a, b \in Q^*, ab = 0\} \to S$ and an $r \subseteq S \times Q^*$ such that conditions (C1)–(C5) and (O1)–(O8) mentioned in Theorem 1 in [4] hold true. According to the first part of the Theorem 1 in [4], the set $V' := S \cup Q^*$ with the multiplication " $\circ$ " and the order " $\leq_{V'}$ " on V' defined by:

$$a \circ b := \begin{cases} ab & if \;\; a, b \in S \\ \rho^b(a) & if \;\; a \in S, \; b \in Q^* \\ \lambda^a(b) & if \;\; a \in Q^*, \; b \in S \\ f(a,b) & if \;\; a, b \in Q^*, \; ab = 0 \\ ab & if \;\; a, b, ab \in Q^* \end{cases}$$

and

$$\leq_{V'} := \leq_S \cup r \cup \{(x, y) \mid x, y \in Q^*, \; x \leq_Q y\}$$

is an ordered semigroup, it is an extension of S by Q, and $(V', \circ, \leq_{V'}) \approx (V, ., \leq_V)$.

Let $a, b \in Q^*, ab = 0$. Then $\theta(a).\theta(b) \in \pi(S)$. Indeed: By (C1), $\theta(a).\theta(b) = \pi_{f(a,b)}$. Since $f(a,b) \in S$, we have $\theta(a).\theta(b) = \pi_{f(a,b)} := \pi(f(a,b)) \in \pi(S)$.
Conditions (D1), (D3), (D4), (D6) being, respectively, the same as (O1), (O3), (O5) and (O8) are satisfied.

(D2) Let $a, b, c \in Q^*$, $s \in S$, $a \leq_Q b$, $bc \neq 0$, $ac = 0$, $\theta(a).\theta(c) = \pi_s$. Then $(s, bc) \in r$. Indeed: By (O2), we have $(f(a,c), bc) \in r$. Since $a, c \in Q^*$, $ac = 0$, by (C1), we have $\theta(a).\theta(c) = \pi_{f(a,c)}$. Since $\theta(a).\theta(c) = \pi_s$, we get $\pi_{f(a,c)} = \pi_s$ then, by Proposition 2, $f(a,c) = s$. Hence we have $(s, bc) \in r$.
Condition (D5) can be proved in a similar way.
Finally, if $a, b \in Q^*$, $ab = 0$, then $f(a,b) \in S$ and, by (C1), $\theta(a).\theta(b) = \pi_{f(a,b)}$, so $\theta(a).\theta(b) \in \pi(S)$. The proof of the theorem is complete. □

2. Ideal extensions of natural numbers. In this section, as an application of Theorem 3, we study the ideal extensions for the weakly reductive ordered semigroup of natural numbers. In the following N will stand for the set of natural numbers $\{1, 2, ..., n\}$ with the usual operation, order " + " and " ≤ ". Denote by 1_N the idempotent mapping on N.

Remark 4. For each $k \in N$, the inner left, right translation and the inner bitranslation of N are the mappings
$\lambda_k : N \to N \mid n \to k + n$
$\rho_k : N \to N \mid n \to n + k$
$\pi_k := (\lambda_k, \rho_k)$.
Moreover, λ_k (resp. ρ_k) is a left (resp. right) translation and π_k is a bitranslation on N. That is, $\lambda_k \in \Lambda(N)$, $\rho_k \in \mathrm{P(N)}$, $\pi_k \in \Omega(N)$.

Lemma 5. $\Lambda(N) = \{\lambda_k \mid k \in N\} \cup \{1_N\}$.

Proof. Let $\lambda \in \Lambda(N)$. Since $1 \in N$, $\lambda(1) \in N$. If $\lambda(1) = 1$, then $\lambda = 1_N$. Indeed: If $n = 1$, then $\lambda(n) = \lambda(1) = 1 = n$. Let $n > 1$. Since $n - 1 \in N$ and λ is a left translation on N, we have $\lambda(n) = \lambda(1 + n - 1) = \lambda(1) + n - 1 = 1 + n - 1 = n$.
Let $\lambda(1) > 1$. Then $\lambda = \lambda_{\lambda(1)-1}$, where $\lambda(1) - 1 \in N$. Indeed: Let $n \in N$. If $n = 1$, then $\lambda(n) = \lambda(1) = \lambda(1) - 1 + 1 = \lambda_{\lambda(1)-1}(1) = \lambda_{\lambda(1)-1}(n)$. Let $n > 1$. Since $n - 1 \in N$, we have $\lambda(n) = \lambda(1 + n - 1) = \lambda(1) + n - 1 = \lambda(1) - 1 + n = \lambda_{\lambda(1)-1}(n)$.
On the other hand, clearly $\{\lambda_k \mid k \in N\} \cup \{1_N\} \subseteq \Lambda(N)$. □
In a similar way we prove the following:

Lemma 6. $\mathrm{P(N)} = \{\rho_\mathrm{k} \mid \mathrm{k} \in \mathrm{N}\} \cup \{1_\mathrm{N}\}$.

Remark 7. $\Lambda(N) = \mathrm{P(N)}$. This is because $\lambda_k = \rho_k$ for each $k \in N$.

Lemma 8. $\Omega(N) = \{\pi_k \mid k \in N\} \cup \{(1_N, 1_N)\}$.

Proof. Let $(\lambda, \rho) \in \Omega(N)$. Since $1 \in N$ and (λ, ρ) is a bitranslation on N, we have $1 + \lambda(1) = \rho(1) + 1$, then $\lambda(1) = \rho(1)$. Moreover $\lambda = \rho$. In fact: Let $n \in N$, $n > 1$. Since $n - 1 \in N$, λ is a left translation and ρ a right translation on N, we have

$$\begin{aligned}\lambda(n) &= \lambda(1 + n - 1) = \lambda(1) + n - 1 = \rho(1) + n - 1 = n - 1 + \rho(1)\\ &= \rho(n - 1 + 1) = \rho(n).\end{aligned}$$

Since $\lambda \in \Lambda(N)$, by Lemma 5, $\lambda = \lambda_k$ for some $k \in N$ or $\lambda = 1_N$. If $\lambda = \lambda_k$ for some $k \in N$, then $\rho = \lambda_k = \rho_k$. Then $(\lambda, \rho) = (\lambda_k, \rho_k) = \pi_k$. If $\lambda = 1_N$, then $\rho = 1_N$, and $(\lambda, \rho) = (1_N, 1_N)$. Obviously, $\{\pi_k \mid k \in N\} \cup \{(1_N, 1_N)\} \subseteq \Omega(N)$. □

Lemma 9. $\Omega(N) \approx (N \cup \{0\}, +, \leq)$.

Proof. We consider the mapping:

$$f : \Omega(N) \to (N \cup \{0\}, +, \leq) \mid (\lambda, \rho) \to \lambda(1) - 1.$$

If $(\lambda, \rho) \in \Omega(N)$, then $\lambda(1) \in N$, so $\lambda(1) - 1 \in N \cup \{0\}$, so the mapping f is well defined.
1. The mapping f is a homomorphism. In fact: Let $(\lambda, \rho), (\lambda', \rho') \in \Omega(N)$. Then

$$\begin{aligned}f((\lambda, \rho).(\lambda', \rho')) &= f(\lambda.\lambda', \rho.\rho') = (\lambda.\lambda')(1) - 1\\ &= \lambda(\lambda'(1)) - 1\\ &= 1 + \lambda(\lambda'(1)) - 1 - 1\\ &= \rho(1) + \lambda'(1) - 1 - 1 \ \text{ (since } (\rho, \lambda) \in \Omega(S))\\ &= \rho(1) - 1 + \lambda'(1) - 1\\ &= \lambda(1) - 1 + \lambda'(1) - 1 \ \text{ (since } \rho = \lambda)\\ &= f(\lambda, \rho) + f(\lambda', \rho').\end{aligned}$$

Let now $(\lambda, \rho), (\lambda', \rho') \in \Omega(N)$ such that $(\lambda, \rho) \leq_\Omega (\lambda', \rho')$. Then $f(\lambda, \rho) \leq f(\lambda', \rho')$. Indeed: Since $\lambda \leq_\Lambda \lambda'$, we have $\lambda(1) \leq \lambda'(1)$. Hence we have

$$f(\lambda, \rho) := \lambda(1) - 1 \leq \lambda'(1) - 1 := f(\lambda', \rho').$$

2. The mapping f is reverse isotone: Let $(\lambda, \rho), (\lambda', \rho') \in \Omega(N)$, $f(\lambda, \rho) \leq f(\lambda', \rho')$. Then $(\lambda, \rho) \leq_\Omega (\lambda', \rho')$. Indeed: Since $\lambda(1) - 1 \leq \lambda'(1) - 1$, we have $\lambda(1) \leq \lambda'(1)$.
Let now $n \in N$, $n > 0$. Since $n - 1 \in N$ and $\lambda, \lambda' \in \Lambda(N)$, we have

$$\begin{aligned}\lambda(n) = \lambda(1 + n - 1) = \lambda(1) + n - 1 &\leq \lambda'(1) + n - 1\\ &= \lambda'(1 + n - 1) = \lambda'(n).\end{aligned}$$

Since $\lambda(n) \leq \lambda'(n)\ \forall\, n \in N$, we have $\lambda \leq_\Lambda \lambda'$.
Since $(\lambda, \rho) \in \Omega(N)$, we have $\lambda = \rho$. Similarly $\lambda' = \rho'$. Hence we have

$$\rho(n) = \lambda(n) \leq \lambda'(n) = \rho'(n)\ \forall\, n \in N,$$

so $\rho \leq_{\mathrm{P}} \rho'$. Since $\lambda \leq_\Lambda \lambda'$ and $\rho \leq_{\mathrm{P}} \rho'$, we have $(\lambda, \rho) \leq_\Omega (\lambda', \rho')$.
3. f is onto: Let $n \in N \cup \{0\}$. If $n = 0$ then, for the bitranslation $(1_N, 1_N)$ of N we have $f(1_N, 1_N) = 1 - 1 = 0$. If $n \in N$ then, for the bitranslation $\pi_n \in \Omega(N)$ we have $f(\pi_N) = f(\lambda_n, \rho_n) = \lambda_n(-1) = n + 1 - 1 = n$. □

Theorem 10. *Let $(Q, ., \leq_Q)$ be an ordered semigroup having a zero 0_Q, let $N \cap Q^* = \emptyset$ and $\theta : Q^* \to (N \cup \{0\}, ., \leq)$ be a partial homomorphism such that*

For each $a, b \in Q^*$, $ab = 0_Q$, we have $\theta(a) + \theta(b) > 0$.

Suppose $r \subseteq \{(a, b) \in N \times Q^* \mid a \leq \theta(b)\}$ having the following properties:

(P1) $x, y \in N,\ z, t \in Q^*,\ x \leq y,\ (y, z) \in r,\ z \leq_Q t \implies (x, t) \in r.$

(P2) $a, b, c \in Q^*,\ a \leq_Q b,\ ac = 0_Q,\ bc \neq 0_Q \implies (\theta(a) + \theta(c), bc) \in r.$

(P3) $a \in N,\ b, c \in Q^*,\ (a, b) \in r,\ bc \neq 0_Q \implies (a + \theta(c), bc) \in r.$

(P4) $a, b, c \in Q^*,\ a \leq_Q b,\ ca = 0_Q,\ cb \neq 0_Q \implies (\theta(c) + \theta(a), cb) \in r.$

(P5) $a \in N,\ b, c \in Q^*,\ (a, b) \in r,\ cb \neq 0_Q \implies (\theta(c) + a, cb) \in r.$

Define an operation " $\circ$ " and an order " $\leq_V$ " on $V := N \cup Q^*$ as follows:

$$a \circ b := \begin{cases} a + \theta(b) & if\ \ a \in N,\ b \in Q^* \\ \theta(a) + b & if\ \ a \in Q^*,\ \ b \in N \\ \theta(a) + \theta(b) & if\ \ a, b \in Q^*,\ \ ab = 0_Q \\ ab & if\ \ a, b \in Q^*,\ ab \neq 0_Q \\ a + b & if\ \ a, b \in N \end{cases}$$

and

$$\leq_V := \leq_S \cup r \cup \{(x, y) \mid x, y \in Q^*,\ x \leq_Q y\}.$$

Then $(V, \circ, \leq_V)$ is an ordered semigroup and it is an extension of N by Q. Conversely, every extension V of N by Q can be so constructed.

Proof. The mapping $f : \Omega(N) \to (N \cup \{0\}, +, \leq) \mid (\lambda, \rho) \to \lambda(1) - 1$ defined in Lemma 9 is an isomorphism. The mapping

$$g :\ N \cup \{0\} \to \Omega(N) \mid n \to \begin{cases} (1_N, 1_N) & if\ n = 0 \\ \ \ \pi_n & if\ n > 0 \end{cases}$$

being the inverse mapping of f is an isomorphism as well. We consider the mapping

$$h := g \circ \theta :\ Q^* \to \Omega(N) \mid a \to (\lambda^a, \rho^a).$$

The mapping h satisfies the conditions of Theorem 3. In fact: Since θ is a partial homomorphism and g a homomorphism, the mapping h is a partial homomorphism.

Let $a, b \in Q^*,\ ab = 0_Q$. Then $h(a).h(b) \in \pi(N)$. Indeed: Since g is a homomorphism, we have $h(a).h(b) = g(\theta(a)).g(\theta(b)) = g(\theta(a) + \theta(b))$. Since $\theta(a) + \theta(b) > 0$, we get $g(\theta(a) + \theta(b)) = \pi_{\theta(a)+\theta(b)} = \pi(\theta(a) + \theta(b)) \in \pi(N)$. Hence we have $h(a).\, h(b) \in \pi(N)$.

(D2) Let $a, b, c \in Q^*,\ s \in N,\ a \leq_Q b,\ bc \neq 0,\ ac = 0,\ h(a).\, h(c) = \pi_s$. Then $(s, bc) \in r$. Indeed: By (P2), we have $(\theta(a) + \theta(c), bc) \in r$. On the other hand, since g is a homomorphism, we have $\pi_s = h(a).\, h(c) = g(\theta(a)).\, g(\theta(c)) = g(\theta(a) + \theta(c))$. Since $a, b \in Q^*,\ ab = 0_Q$, we have $\theta(a) + \theta(c) > 0$. Then $g(\theta(a) + \theta(c)) = \pi_{\theta(a)+\theta(c)}$, therefore $\pi_s = \pi_{\theta(a)+\theta(c)}$. Since $(N, ., \leq)$ is weakly reductive, by Proposition 2, we have $\theta(a) + \theta(c) = s$. Then $(s, bc) \in r$.

(D3) Let $(a, b) \in r$. Then $\pi_a \leq_\Omega h(b)$. Indeed: Since $(a, b) \in r$, we have $N \ni a \leq \theta(b) \in N \cup \{0\}$, then $\theta(b) \in N$. Since the mapping $\pi : N \to \Omega(N) \mid n \to \pi_n$ is isotone, we have $\pi_a \leq_\Omega \pi_{\theta(b)} = g(\theta(b)) = h(b)$.

(D4) Let $b, c \in Q^*, (a, b) \in r, bc \neq 0$. Then $(\rho^c(a), bc) \in r$. Indeed: By (P3), we have $(a + \theta(c), bc) \in r$. Since $c \in Q^*$, $\theta(c) \in N \cup \{0\}$. If $\theta(c) = 0$, then $h(c) := g(\theta(c)) = (1_N, 1_N)$.

Since $h(c) := (\lambda^c, \rho^c)$, we have $\rho^c = 1_N$, then $\rho^c(a) = a = a + \theta(c)$, hence $(\rho^c(a), bc) \in r$. If $\theta(c) \in N$, then $h(c) := g(\theta(c)) = (\lambda_{\theta(c)}, \rho_{\theta(c)})$, $h(c) := (\lambda^c, \rho^c)$, so $\rho^c = \rho_{\theta(c)}$, hence $\rho^c(a) = \rho_{\theta(c)}a = a + \theta(c)$, and $(\rho^c(a), bc) \in r$.

Condition (D1) is satisfied and the proof of (D5), (D6) are similar to that of (D2), (D4), respectively. From the first part of Theorem 3, the set $V := N \cup Q^*$ with the operation and the order defined in Theorem 3, is an ordered semigroup and it is an extension of N by Q. Moreover we have the following:

1. If $a \in N$, $b \in Q^*$, then $\rho^b(a) = a + \theta(b)$. Indeed: Since $b \in Q^*$, $\theta(b) \in N \cup \{0\}$. If $\theta(b) = 0$, then $h(b) = g(\theta(b)) = (1_N, 1_N)$. Besides, $h(b) := (\lambda^b, \rho^b)$, so $\rho^b = 1_N$, and $\rho^b(a) = a = a + 0 = a + \theta(b)$. If $\theta(b) \in N$, then $h(b) = g(\theta(b)) = (\lambda_{\theta(b)}, \rho_{\theta(b)})$. Since $h(b) := (\lambda^b, \rho^b)$, we get $\rho^b = \rho_{\theta(b)}$, thus $\rho^b(a) = \rho_{\theta(b)}a = a + \theta(b)$.

2. If $a \in Q^*$, $b \in N$, then $\theta(a) + b = \lambda^a(b)$. The proof is similar to that of 1.

3. Let $a, b \in Q^*$, $ab = 0_Q$. Then the element $\theta(a) + \theta(b)$ is the only element of N such that $\theta(a).\,\theta(b) = \pi_{\theta(a)+\theta(b)}$. Indeed: Since $\theta(a) + \theta(b) > 0$ and g is a homomorphism, we obtain $h(a).\,h(b) = g(\theta(a)).\,g(\theta(b)) = g(\theta(a) + \theta(b)) = \pi_{\theta(a)+\theta(b))}$. Let now $d \in N$ such that $\theta(a).\,\theta(b) = \pi_d$. Since $\pi_d = \pi_{\theta(a)+\theta(b)}$, by Proposition 2, we get $d = \theta(a) + \theta(b)$.

The converse statement: Let $(V, ., \leq)$ be an extension of $(N, ., \leq)$ by $(Q, ., \leq_Q)$. Since N is weakly reductive, by the second part of the Theorem 3, there exists a partial homomorphism $\theta' : Q^* \to \Omega(N)$ and an $r \subseteq N \times Q^*$ satisfying the conditions given in Theorem 3. We consider the isomorphism

$$f : \Omega(N) \to (N \cup \{0\}, +, \leq) \mid (\lambda, \rho) \to \lambda(1) - 1$$

given in Lemma 9 and the composition mapping

$$\theta := f \circ \theta' \mid Q^* \to (N \cup \{0\}, +, \leq).$$

The mapping θ satisfies the conditions given in Theorem 10. In fact: Since f is a homomorphism and θ' a partial homomorphism, θ is a partial homomorphism.

Let $a, b \in Q^*$, $ab = 0_Q$. Then $\theta(a) + \theta(b) > 0$. Indeed: Suppose $\theta(a) + \theta(b) = 0$. By Theorem 3, we have $\theta'(a).\,\theta'(b) \in \pi(N)$. Then $\theta'(a).\,\theta'(b) = \pi_n := (\lambda_n, \rho_n)$ for some $n \in N$. Since f is a homomorphism, we have

$$\begin{aligned} \theta(a) + \theta(b) &= f(\theta'(a)) + f(\theta'(b)) = f(\theta'(a).\,\theta'(b)) \\ &= f(\lambda_n, \rho_n) = \lambda_n(1) - 1 = n + 1 - 1 = n \in N \end{aligned}$$

which is impossible.

$r \subseteq \{(a, b) \in N \times Q^* \mid a \leq \theta(b)\}$. Indeed: Let $(a, b) \in r$. By (D3), $\pi_a \leq_\Omega \theta'(b)$, then $f(\pi_a) \leq f(\theta'(b)) = \theta(b)$. Since $f(\pi_a) = f(\lambda_a, \rho_a) = \lambda_a(1) - 1 = 1 + a - 1 = a$, we get $a \leq \theta(b)$.

Condition (P1) is satisfied (cf. Theorem 3).

(P2) Let $a, b, c \in Q^*$, $a \leq_Q b$, $ac = 0_Q$, $bc \neq 0_Q$. Then $(\theta(a) + \theta(c), bc) \in r$. Indeed: Since $a, c \in Q^*$, $ac = 0_Q$, by Theorem 3, we have $\theta'(a).\,\theta'(c) \in \pi(N)$. Then $\theta'(a).\,\theta'(c) = \pi_n := (\lambda_n, \rho_n)$ for some $n \in N$. Since f is a homomorphism, we have

$$\begin{aligned} \theta(a) + \theta(c) &= f(\theta'(a)) + f(\theta'(c)) = f(\theta'(a).\,\theta'(c)) \\ &= f(\lambda_n, \rho_n) = \lambda_n(1) - 1 = 1 + n - 1 = n \in N. \end{aligned}$$

Since $a, b, c \in Q^*$, $n \in N$, $a \leq_Q b$, $bc \neq 0_Q$, $ac = 0_Q$, $\theta'(a).\,\theta'(c) = \pi_n$, by Theorem 3, we have $(n, bc) \in r$, therefore $(\theta(a) + \theta(c), bc) \in r$.
(P3) Let $b, c \in Q^*$, $(a, b) \in r$, $bc \neq 0_Q$ Then $(a + \theta(c), bc) \in r$. Indeed: By (D4), we have $(\rho^c(a), bc) \in r$. Since $\theta'(c) := (\lambda^c, \rho^c) \in \Omega(N)$, we have

$$\theta(c) = f(\theta'(c)) = f(\lambda^c, \rho^c) := \lambda^c(1) - 1.$$

By Lemma 8, $(\lambda^c, \rho^c) \in \{\pi_k \mid k \in N\} \cup \{(1_N, 1_N)\}$. If $(\lambda^c, \rho^c) = (1_N, 1_N)$, then $\lambda^c(a) = 1$, $\theta(c) = 0$, $\rho^c(a) = a$, $a + \theta(c) = \rho^c(a)$, thus $(a + \theta(c), bc) \in r$. Let $(\lambda^c, \rho^c) = \pi_k := (\lambda_k, \rho_k)$ for some $k \in N$. Then $\lambda^c = \lambda_k$ and $\rho^c = \rho_k$. Since $a \in N$, we have $\rho^c(a) = \rho_k(a) := a + k$. Besides, $\lambda^c(1) = \lambda_k(1) := k + 1$. Thus we get $\theta(c) = \lambda^c(1) - 1 = k$, $\rho^c(a) = a + \theta(c)$, and $(a + \theta(c), bc) \in r$.
Conditions (P4) and (P5) can be proved in a similar way as the (P2), (P3) using the conditions (D5), (D6), respectively. □

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References

[1] A.H. Clifford, *Extensions of semigroups*, Trans. Amer. Math. Soc. **68** (1950), 165–173.

[2] A.H. Clifford and G.B. Preston, *The Algebraic Theory of Semigroups*, Vol. **I**, Amer. Math. Soc., Math. Surveys Number 7, Providence, Rhode Island 1964. xv+224 pp.

[3] N. Kehayopulu, *Extensions of Semigroups*, University of Athens, 1997 (in Greek). iv+87pp.

[4] N. Kehayopulu, M. Tsingelis, *Ideal extensions of ordered semigroups*, Comm. Algebra **31**, no. 10 (2003), 4939–4969.

[5] M. Petrich, *Introduction to Semigroups*, Merrill Research and Lecture Series. Charles E. Merrill Publishing Co., Columbus, Ohio 1973. viii+198 pp.

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