CONSTRUCTING SQUARE ROOTS IN A BANACH ALGEBRA

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Abstract. The primary aim of the paper is to prove, within Bishop-style constructive mathematics and by elementary means (that is, without employing the full functional calculus), that under suitable conditions on the Banach algebra, its positive elements have positive square roots. This requires careful definition of such notions as positivity for Banach-algebra elements. The results are then applied to a problem about principal ideals in a Banach algebra.

1 Introduction Throughout this paper, the term *Banach algebra* refers to a separable commutative Banach algebra with an identity element. Our primary aim is to prove, *within Bishop-style constructive mathematics*[1] *and by elementary means* (that is, without employing the full functional calculus), that under suitable conditions on the Banach algebra, its positive elements have positive square roots. Of course, this requires us to define carefully what we mean by *positive* for Banach-algebra elements. We then apply our results to a problem about principal ideals in a Banach algebra.

In passing, we observe that the typical classical proof of the existence of square roots of positive elements of a Banach algebra uses either the (non-elementary) functional calculus or else some kind of iteration which involves the (non-constructive) monotone convergence theorem for real sequences.

We shall need some background in the constructive theory of Banach algebras. Let B be such an object, with identity e, and let B' denote the dual of B, which we endow with the weak* topology. The multiplicative linear functionals on B, also known as **characters** of B, form the **spectrum** Σ_B, or **character space**, of B. Constructively, the (classically valid) weak* compactness of Σ_B is not generally provable; essentially, this is a consequence of the fact (revealed by a recursive example due to Metakides et al. [13]) that when we carry out a Hahn-Banach extension of a linear functional, the best we can hope for is to increase the norm by an arbitrarily small positive quantity, rather than by 0. However, adapting proofs from Chapter 11 of [3], we see that if $(x_n)_{n\geqslant 1}$ is a dense sequence in B, then for all but countably many $t > 0$ the set

$$\Sigma_B^t = \{u \in B' : |u(x_i x_j) - u(x_i)u(x_j)| \leqslant t \quad (1 \leqslant i, j \leqslant n) \text{ and } |1 - u(e)| \leqslant t\}$$

is (inhabited and) compact (cf. (1.3) and (2.7) in Chapter 11 of [3]). Instead of working with Σ_B, we work with the sets Σ_B^t for carefully chosen t, as is exemplified by two important results from Chapter 11 of [3]:

Proposition 1 *If $a_1, \dots, a_n$ are elements of B, and t, ε are positive numbers such that*

$$|u(a_1)| + \cdots + |u(a_n)| \geqslant \varepsilon \quad \left(u \in \Sigma_B^t\right),$$

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[1] By **constructive mathematics** we mean mathematics carried out with intuitionistic logic and based on an appropriate corresponding foundation such as the Aczel-Myhill-Rathjen CST [1, 14]. Background material on constructive analysis can be found in [2, 3, 9, 16] or the more recent monograph [10].

then there exist $b_1, \ldots, b_n$ in B such that $a_1 b_1 + \cdots + a_n b_n = e$.

Proposition 2 *Let $(\varepsilon_n)_{n\geqslant 1}$ be a strictly decreasing sequence of positive numbers converging to 0, such that $\Sigma_B^{\varepsilon_n}$ is compact for each n. For each $x \in B$ and each n define*

$$\|x\|_{\Sigma_B^{\varepsilon_n}} = \sup\left\{|u(x)| : u \in \Sigma_B^{\varepsilon_n}\right\}.$$

Then the sequences $\left(\|x\|_{\Sigma_B^{\varepsilon_n}}\right)_{n\geqslant 1}$ and $\left(\|x^n\|^{1/n}\right)_{n\geqslant 1}$ are ***equiconvergent****, in the sense that for each term a_m of one sequence and each $\varepsilon > 0$ there exists N such that $b_n < a_m + \varepsilon$ whenever b_n is a term of the other sequence and $n \geqslant N$.*

In reading this last proposition, observe that the **spectral radius**

$$r(x) = \inf \|x^n\|^{1/n} = \lim_{n\to\infty} \|x^n\|^{1/n}$$

may not exist constructively. Nevertheless, it is convenient for us to adopt, for example, the notation $r(x) < t$ to signify that $\|x^n\|^{1/n} < t$ for all sufficiently large n.

By a **state** of B we mean a linear functional f on B such that $f(e) = 1 = \|f\|$. As was the case for the spectrum, there is no guarantee that the **state space**

$$V_B = \{f \in B' : f(e) = 1 = \|f\|\}.$$

of B is weak* compact (or even inhabited) as it is classically. For this reason we introduce, for each $t > 0$, the approximation

$$V_B^t = \{f \in B' : \|f\| \leqslant 1, \ \ |1 - f(e)| \leqslant t\}$$

to V_B. Applying the Hahn-Banach theorem with $X = B, Y = \mathbf{C}e$, and $v(\lambda e) = \lambda$, we see that V_B^t is inhabited (that is, contains a constructible element). In fact, since the mapping $f \rightsquigarrow |1 - f(e)|$ is weak* uniformly continuous on the unit ball B_1' of B' ([3], page 351, (6.3)), V_B^t is weak* compact for all but countably many $t > 0$ ([3], page 98, (4.9)). We say that $t > 0$ is **admissible** if V_B^t is weak* compact. If $(\varepsilon_n)_{n\geqslant 1}$ is a strictly decreasing sequence of admissible numbers converging to 0, then

$$V_B = \bigcap_{n\geqslant 1} V_B^{\varepsilon_n},$$

the intersection of a descending sequence of inhabited, weak* compact sets. Hence V_B is weak* complete. We say that V_B is **firm** if[2]

- ▶ it is weak*-compact and
- ▶ for each $\varepsilon > 0$ and each $x \in B$, there exists $t > 0$ such that if $0 < t' \leqslant t$ and $f \in V_B$, then there exists $g \in V_B^{t'}$ with $|f(x) - g(x)| < \varepsilon$.

Classically, V_B is always firm.

We say that $x \in B$ is

- ▶ **Hermitian** if for each $\varepsilon > 0$ there exists $t > 0$ such that $|\mathrm{Im}\, f(x)| < \varepsilon$ for all $f \in V_B^t$;

[2] In [8], firmness is expressed in terms of the so-called *double norm* on B'. Our current notion of firmness is equivalent to the one therein.

- ▶ **positive** if for each $\varepsilon > 0$ there exists $t > 0$ such that $\operatorname{Re} f(x) \geqslant -\varepsilon$ and $|\operatorname{Im} f(x)| \leqslant \varepsilon$ for every $f \in V_B^t$.

Every positive element is Hermitian. The identity e of B is positive: for each $t \in (0,1)$ and each $f \in V_B^t$ we have $|\operatorname{Re} f(e)| \leqslant |f(e)| \leqslant 1$ and

$$\operatorname{Re}(1 - f(e))^2 + \operatorname{Im} f(e)^2 = |1 - f(e)|^2 \leqslant t^2,$$

from which it follows that $\operatorname{Re} f(e) > 1 - t > 0$ and $|\operatorname{Im} f(e)| \leqslant t$. We write $x \geqslant y$, or equivalently $y \leqslant x$, to denote that $x - y$ is positive.

Every element f of the state space of B is positive, in the sense that $f(x) \geqslant 0$ whenever $x \geqslant 0$. If B is generated by Hermitian elements and has firm state space, then the character space of every separable commutative unital Banach subalgebra of B is inhabited ([8], Corollary 3.5).

We will need **Sinclair's theorem**: *If x is Hermitian, then $\|x^n\|^{1/n} = \|x\|$ for each positive integer n.* A constructive proof of this theorem is found in [7] and [12].

Consider the relation between our constructive notion of positivity and the classical one, in which $x \geqslant 0$ means that $f(x) \geqslant 0$ for all $f \in V_B$. If x is positive in our sense, then since $V_B \subset V_B^t$ for each $t > 0$, x is certainly classically positive (although, of course, we have no constructive guarantee that V_B is inhabited). Suppose, conversely, that x is classically, but not constructively, positive. Arguing with classical logic, we see that there exists $\alpha > 0$ such that for each positive integer n there exists $f_n \in V_B^{1/n}$ such that either $\operatorname{Re} f_n(x) < -\alpha$ or $|\operatorname{Im} f_n(x)| > \alpha$. Since B_1' is weak* compact, there exists a subsequence $(f_{n_k})_{k \geqslant 1}$ of $(f_n)_{n \geqslant 1}$ that converges to a limit $f \in B_1'$. Passing to a subsequence, if necessary, we may assume that either $\operatorname{Re} f_{n_k}(x) < -\alpha$ for all k or else $|\operatorname{Im} f_{n_k}(x)| > \alpha$ for all k. In the first case,

$$0 \leqslant f(x) = \operatorname{Re} f(x) = \lim_{k \to \infty} \operatorname{Re} f_{n_k}(x) \leqslant -\alpha,$$

contrary to the classical positivity of f; in the second case we have

$$|\operatorname{Im} f(x)| = \lim_{k \to \infty} |\operatorname{Im} f_{n_k}(x)| \geqslant \alpha,$$

which is absurd since $f(x)$ is real. Thus the two notions of positivity are classically equivalent.

2 Extracting square roots in B We say that B is **semi-simple** if it has the following property: for each $x \in B$, if for each $\varepsilon > 0$ there exists $t > 0$ such that $|u(x)| < \varepsilon$ for every $u \in \Sigma_B^t$, then $x = 0$. In symbols, semi-simplicity reads like this:

$$\forall_{x \in B} \left(\forall_{\varepsilon > 0} \exists_{t > 0} \forall_{u \in \Sigma_B^t} \; (|u(x)| \leqslant \varepsilon) \Rightarrow x = 0 \right).$$

An argument like that used in the final paragraph of the preceding section shows that our notion of semi-simplicity is classically equivalent to the standard classical one (namely, that if $u(x) = 0$ for all $u \in \Sigma_B$, then $x = 0$). Our aim in this section is to prove

Theorem 3 *Let B be a semi-simple Banach algebra that has firm state space, and let a be a positive element of B such that $\|a\| \leqslant 1$. Then there exists a unique positive element b of B such that $b^2 = a$. Moreover, b is in the closed ideal of B generated by a.*

We first establish some basic results on positivity.

Lemma 4 *If $x \geqslant 0$, then $\|x\| e \geqslant x$ and $e - x \leqslant e$.*

Proof. Fix ε_n and $\Sigma_B^{\varepsilon_n}$ as in Proposition 2. Given $\varepsilon > 0$, choose N such that

▷ $0 < \varepsilon_N < (1 + \|x\|)^{-1} \varepsilon/2$ and

▷ $\operatorname{Re} f(x) \geqslant -\varepsilon/2$ and $|\operatorname{Im} f(x)| \leqslant \varepsilon/2$ whenever $n \geqslant N$ and $f \in V_B^{\varepsilon_n}$.

Then for $n \geqslant N$ and $f \in V_B^{\varepsilon_n}$ we have

$$\begin{aligned}\operatorname{Re} f(\|x\| e - x) &= \|x\| \operatorname{Re} f(e) - \operatorname{Re} f(x) \geqslant \|x\| (1 - \varepsilon_n) - \|x\| \\ &= -\varepsilon_n \|x\| \geqslant -\varepsilon_N \|x\| > -\varepsilon\end{aligned}$$

and

$$|\operatorname{Im} f(\|x\| e - x)| \leqslant \|x\| |\operatorname{Im} f(e)| + |\operatorname{Im} f(x)| < \|x\| \varepsilon_n + \frac{\varepsilon}{2} < \varepsilon.$$

Thus $\|x\| e - x \geqslant 0$ and therefore $\|x\| e \geqslant x$. On the other hand, since $e - (e - x) = x \geqslant 0$, we have $e \geqslant e - x$. ■

Lemma 5 *Let x be a positive element of B such that $\|x\| \leqslant 1$. Then $e - x$ is positive and $\|e - x\| \leqslant 1$.*

Proof. By Sinclair's theorem, $r(x) = \|x\|$. With ε_n and Σ_{B_n} as in Proposition 2, it follows from that proposition that for each $\varepsilon > 0$ there exists N_1 such that

$$\|x\|_{\Sigma_{B_n}} < \|x\| + \varepsilon \leqslant 1 + \varepsilon \quad (n \geqslant N_1).$$

On the other hand, since $x \geqslant 0$, there exists N_2 such that for each $n \geqslant N_2$,

- $\varepsilon_n < \varepsilon$ and
- $\operatorname{Re} f(x) \geqslant -\varepsilon_n$ and $|\operatorname{Im} f(x)| < \varepsilon_n$ for all $f \in V_B^{\varepsilon_n}$.

If $n \geqslant \max\{N_1, N_2\}$ and $f \in V_B^{\varepsilon_n}$, then

$$\begin{aligned}\operatorname{Re} f(e - x) &= \operatorname{Re} f(e) - \operatorname{Re} f(x) \\ &\geqslant (1 - \varepsilon_n) - |f(x)| \\ &\geqslant 1 - \varepsilon_n - 1 > -\varepsilon\end{aligned}$$

and

$$\begin{aligned}|\operatorname{Im} f(e - x)| &\leqslant |\operatorname{Im} f(e)| + |\operatorname{Im} f(x)| \\ &< |\operatorname{Im}(1 - f(e))| + \varepsilon_n \leqslant 2\varepsilon_n < 2\varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, it follows that $e - x \geqslant 0$. Also, for each $n \geqslant \max\{N_1, N_2\}$ and each $u \in \Sigma_{B_n}$, since $u \in V_B^{\varepsilon_n}$ we have

$$\begin{aligned}-\varepsilon < \operatorname{Re} u(e - x) &= \operatorname{Re} u(e) - \operatorname{Re} u(x) \\ &\leqslant 1 + \varepsilon_n - (-\varepsilon_n) = 1 + 2\varepsilon_n < 1 + 2\varepsilon,\end{aligned}$$

the first inequality coming from the first estimate in the sentence before last. Hence

$$|u(e - x)| \leqslant |\operatorname{Re} u(e - x)| + |\operatorname{Im} u(e - x)| < 1 + 4\varepsilon.$$

Thus $\|e - x\|_{\Sigma_{B_n}} < 1 + 4\varepsilon$ for all $n \geqslant \max\{N_1, N_2\}$. It follows from Sinclair's theorem and Proposition 2 that $\|e - x\| \leqslant 1$. ■

In order to extract square roots in B, we need a few more preliminaries. Surprisingly, the first of these does not appear explicitly in the constructive literature.

Proposition 6 *Let $U \subset C$ be a connected open set, $f : U \to C$ a differentiable function, and K an infinite compact set well contained in U, such that $f(z) = 0$ for each $z \in K$. Then $f(z) = 0$ for all $z \in U$.*

Proof. Suppose that $f(\zeta) \neq 0$ for some $\zeta \in U$. By Theorem (5.13) on page 159 of [3], either

$$\inf_{z \in K} |f(z)| > 0,$$

which is absurd, or else, as must be the case, there exist finitely many points $z_1, \dots, z_n$ of U and an analytic function g on U such that

$$f(z) = (z - z_1) \cdots (z - z_n)\, g(z) \quad (z \in U)$$

and g is nonvanishing on K. Since K is an infinite set, we can find $z_0 \in K$ such that $z_0 \neq z_j$ $(1 \leqslant j \leqslant n)$. Then $f(z_0) \neq 0$, a contradiction. We conclude that $f(z) = 0$ for all $z \in U$. ∎

Lemma 7 *Let $c_0 = 1$ and, for $n \geqslant 1$,*

$$c_n = -\left(2^n n!\right)^{-1} 1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-3)\,.$$

Then the series $\sum_{n=0}^{\infty} c_n z^n$ converges absolutely and uniformly on the closed unit disc D of C, and

$$\left(\sum_{n=0}^{\infty} c_n z^n \right)^2 = 1 - z$$

for each $z \in D$.

Proof. It is shown on pages 104–105 of [3] that the real series $\sum_{n=0}^{\infty} c_n t^n$ converges to $(1-t)^{1/2}$ on the interval $[-1, 1]$. For each $z \in B$ and for $N \geqslant 1$ we have

$$\sum_{n=N+1}^{\infty} |c_n|\, |z|^n \leqslant \sum_{n=N+1}^{\infty} |c_n| = -\sum_{n=N+1}^{\infty} c_n \to 0 \text{ as } N \to \infty.$$

Hence the series $\sum_{n=0}^{\infty} c_n z^n$ converges absolutely and uniformly on D. It follows that

$$f(z) \equiv \left(\sum_{n=0}^{\infty} c_n z^n \right)^2 - (1 - z)$$

defines an analytic function on B. Since $f(z) = 0$ for all z in the infinite compact set $\left[-\frac{1}{2}, \frac{1}{2}\right]$, which is well contained in the open unit disc D°, we see from Lemma 6 that $f(z) = 0$ for all $z \in D^\circ$ and hence, by continuity, for all $z \in D$. ∎

Our next proposition is closely related to Theorem 4.2 of [8].

Proposition 8 *If B has firm state space, then $a^n \geqslant 0$ for each positive element a of B.*

Proof. Let a be a positive (and hence Hermitian) element of B, and A the commutative Banach algebra generated by e and a. By Proposition 2.4 of [8], the state space V_A of A is firm. By Corollary 3.5 of that same paper, for each $f \in V_A$ there exist characters $u_1, \ldots, u_m$ of A, and nonnegative numbers $\lambda_1, \ldots, \lambda_m$, such that $\sum_{i=1}^{m} \lambda_i = 1$ and

$$\left| f(a^n) - \sum_{i=1}^{m} \lambda_i u_i(a^n) \right| < \varepsilon.$$

Since $u_i \in \Sigma_A \subset V_A$, Lemma 4.1 of [8] shows that $u_i(a^n) = u_i(a)^n \geqslant 0$. Hence

$$\begin{aligned} \operatorname{Re} f(a^n) &\geqslant \operatorname{Re} \sum_{i=1}^{m} \lambda_i u_i(a^n) - \left| f(a^n) - \sum_{i=1}^{m} \lambda_i u_i(a^n) \right| \\ &> \operatorname{Re} \sum_{i=1}^{m} \lambda_i u_i(a)^n - \varepsilon \geqslant -\varepsilon \end{aligned}$$

and therefore $\operatorname{Re} f(a^n) \geqslant 0$. It follows from Lemma 4.1 of [8] that $a^n \geqslant 0$. ∎

Proposition 9 *Let B be semi-simple with firm state space. Let a be a positive element of B such that $\|a\| \leqslant 1$, and for each $n \geqslant 0$ let c_n be as in* Lemma 7. *Then $\sum_{n=0}^{\infty} c_n a^n$ is the unique positive element of B whose square equals $e - a$.*

Proof. By Lemma 7, the series $\sum_{n=0}^{\infty} c_n \|a\|^n$ converges absolutely. Noting Sinclair's theorem, we see that

$$\begin{aligned} \sum_{n=0}^{\infty} \|c_n a^n\| = \sum_{n=0}^{\infty} |c_n| \, \|a^n\| &= \sum_{n=0}^{\infty} |c_n| \, \|a\|^n \\ &= 1 - \sum_{n=1}^{\infty} c_n \|a\|^n = 2 - (1 - \|a\|)^{1/2} \leqslant 2. \end{aligned}$$

Hence the series $\sum_{n=0}^{\infty} c_n a^n$ converges absolutely to an element x of A with $\|x\| \leqslant 2$.

Given $\varepsilon > 0$, choose N such that

$$\sum_{n=N+1}^{\infty} |c_n| < \varepsilon. \tag{1}$$

We first consider the case $\|a\| < 1$. Writing

$$\alpha \equiv \frac{1}{2}(1 + \|a\|),$$

we see from Sinclair's theorem that $\|a^n\| = \|a\|^n < \alpha^n < 1$ for each n. By Proposition 2, there exists $t > 0$ such that

- Σ_B^t is compact and inhabited,
- $\|a\|_{\Sigma_B^t} < \alpha$, and
- for each $u \in \Sigma_B^t$,

$$\begin{aligned} \left| u(x^2) - u(x)^2 \right| &< \varepsilon, \\ |u(e-a) - (1 - u(a))| &< \varepsilon, \end{aligned}$$

  and

$$|u(a^n) - u(a)^n| < \frac{\varepsilon}{(N+1)\,|c_n|} \quad (0 \leqslant n \leqslant N).$$

For each $u \in \Sigma_B^t$ we have $|u(a)| < \alpha < 1$; whence, by Lemma 7, the series $\sum_{n=0}^{\infty} c_n u(a)^n$ converges absolutely. Moreover,

$$\left|\sum_{n=0}^{\infty} c_n u(a)^n\right| = (1 - u(a))^{1/2} \leqslant 2.$$

For each $u \in \Sigma_B^t$, since (by Lemma 7)

$$\left(\sum_{n=0}^{\infty} c_n u(a)^n\right)^2 = 1 - u(a),$$

we have

$$\begin{aligned}
\left|u(x^2 - (e-a))\right| &\leqslant \left|u(x)^2 - (1 - u(a))\right| + \left|u\left(x^2\right) - u(x)^2\right| \\
&\qquad + \left|u(e-a) - (1 - u(a))\right| \\
&< \left|\left(u\left(\sum_{n=0}^{\infty} c_n a^n\right)\right)^2 - \left(\sum_{n=0}^{\infty} c_n u(a)^n\right)^2\right| + 2\varepsilon \qquad (2)
\end{aligned}$$

But

$$\begin{aligned}
&\left|\left(u\left(\sum_{n=0}^{\infty} c_n a^n\right)\right)^2 - \left(\sum_{n=0}^{\infty} c_n u(a)^n\right)^2\right| \\
&= \left|u\left(\sum_{n=0}^{\infty} c_n a^n\right) + \sum_{n=0}^{\infty} c_n u(a)^n\right| \left|u\left(\sum_{n=0}^{\infty} c_n a^n\right) - \sum_{n=0}^{\infty} c_n u(a)^n\right| \\
&\leqslant \left(\left\|\sum_{n=0}^{\infty} c_n a^n\right\| + \left|\sum_{n=0}^{\infty} c_n u(a)^n\right|\right) \left|u\left(\sum_{n=0}^{\infty} c_n a^n\right) - \sum_{n=0}^{\infty} c_n u(a)^n\right| \\
&\leqslant 4\left|\sum_{n=0}^{\infty} c_n u\left(a^n\right) - \sum_{n=0}^{\infty} c_n u(a)^n\right| \\
&\leqslant 4\sum_{n=0}^{\infty} |c_n| \left|u(a^n) - u(a)^n\right| \\
&= 4\sum_{n=0}^{N} |c_n| \left|u(a^n) - u(a)^n\right| + 4\sum_{n=N+1}^{\infty} |c_n| \left|u(a^n) - u(a)^n\right| \\
&\leqslant 4\sum_{n=0}^{N} |c_n| \frac{\varepsilon}{(N+1)|c_n|} + 4\sum_{n=N+1}^{\infty} 2|c_n| < 12\varepsilon.
\end{aligned}$$

It follows from this and (2) that

$$\left|u(x^2 - (e-a))\right| < 14\varepsilon$$

for each $u \in \Sigma_b^t$. Since $\varepsilon > 0$ is arbitrary and B is semi-simple, we conclude that $x^2 = e - a$.

To prove that x is positive, fix $\varepsilon > 0$ and let N be as in (1). By Proposition 8, a^n is positive for each positive integer n. We can therefore choose $t \in (0, \varepsilon)$ such that $|\operatorname{Im} u(a^n)| < \varepsilon$ for all $u \in V_B^t$ and all $n \in \{1, \ldots, N\}$. For each $u \in V_B^t$ we have

$$t^2 > |1 - u(e)|^2 = (1 - \operatorname{Re} u(e))^2 + (\operatorname{Im} u(e))^2.$$

Noting that $\left(\sum_{n=0}^{\infty} c_n\right)^2 = 1 - 1 = 0$, we compute

$$\begin{aligned}\operatorname{Re} u(x) &= \operatorname{Re} u(e) + \sum_{n=1}^{\infty} c_n \operatorname{Re} u(a^n) \\ &> 1 - t - \sum_{n=1}^{\infty} |c_n| = 1 - t + \left(\sum_{n=0}^{\infty} c_n - 1\right) = -t > -\varepsilon.\end{aligned}$$

Also,so

$$\begin{aligned}|\operatorname{Im} u(x)| &\leqslant |\operatorname{Im} u(e)| + \sum_{n=1}^{\infty} |c_n| \, |\operatorname{Im} u(a^n)| \\ &< t + \sum_{n=1}^{N} |c_n| \, \varepsilon + \sum_{n=N+1}^{\infty} |c_n| \\ &< \left(2 + \sum_{n=1}^{\infty} |c_n|\right) \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we conclude from all these computations that $x \geqslant 0$.

Now consider the general case where $\|a\| \leqslant 1$. With N as at (1) pick $\delta \in (0,1)$ such that

$$1 - \delta^N < \frac{\varepsilon}{\sum_{n=1}^{N} |c_n|}.$$

Then $\delta a \geqslant 0$ and $\|\delta a\| < 1$, so, by the foregoing,

$$x_\delta \equiv \sum_{n=0}^{\infty} c_n \, (\delta a)^n$$

is positive and satisfies $x_\delta^2 = e - \delta a$. Also,

$$\begin{aligned}\|x - x_\delta\| &\leqslant \sum_{n=1}^{N} |c_n| \, (1 - \delta^n) \, \|a\|^n + \sum_{n=N+1}^{\infty} |c_n| \\ &\leqslant \left(1 - \delta^N\right) \sum_{n=1}^{N} |c_n| + \varepsilon < 2\varepsilon\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we conclude that

$$x = \lim_{\delta \to 1^-} x_\delta.$$

By the continuity of the squaring function on B,

$$x^2 = \lim_{\delta \to 1^-} x_\delta^2 = \lim_{\delta \to 1^-} (e - \delta a) = e - a,$$

as required.

It remains to prove the uniqueness of x. Suppose, then, that $y^2 = e - a$ for some positive element y of B. Given $\varepsilon > 0$, this time pick $t > 0$ such that for all $u \in \Sigma_B^t$,

$$\left|u(x^2) - u(x)^2\right| < \varepsilon^2 \text{ and } \left|u(y^2) - u(y)^2\right| < \varepsilon^2,$$

and such that for all $f \in V_B^t$,

$$\min\{\operatorname{Re} f(x), \operatorname{Re} f(y)\} \geqslant -\varepsilon \text{ and } \max\{|\operatorname{Im} f(x)|, |\operatorname{Im} f(y)|\} \leqslant \varepsilon.$$

Then for $u \in \Sigma_B^t$ we have

$$\begin{aligned}
&|u(x-y)|\,|u(x)+u(y)| \\
&= \left|u(x)^2 - u(y)^2\right| \\
&\leqslant \left|u(x^2-y^2)\right| + \left|u(x^2) - u(x)^2\right| + \left|u(y^2) - u(y)^2\right| < 2\varepsilon^2.
\end{aligned}$$

Either $|u(x-y)| < 2\varepsilon$ or $|u(x-y)| > \varepsilon$. In the latter case, $|u(x+y)| < 2\varepsilon$, so $|\operatorname{Re} u(x) + \operatorname{Re} u(y)| < 2\varepsilon$. Suppose that $\operatorname{Re} u(x) > 3\varepsilon$. Then

$$\operatorname{Re} u(y) < 2\varepsilon - \operatorname{Re} u(x) < -\varepsilon,$$

which, since $u \in V_B^t$, contradicts our choice of t. Hence $-\varepsilon \leqslant \operatorname{Re} u(x) \leqslant 3\varepsilon$ and therefore $|\operatorname{Re} u(x)| \leqslant 3\varepsilon$; whence

$$|u(x)|^2 = (\operatorname{Re} u(x))^2 + (\operatorname{Im} u(x))^2 \leqslant 10\varepsilon^2$$

and therefore $|u(x)| \leqslant \sqrt{10}\varepsilon$. Likewise, $|u(y)| \leqslant \sqrt{10}\varepsilon$. It follows that

$$|u(x-y)| \leqslant 2\sqrt{10}\varepsilon,$$

an inequality that holds also in the case $|u(x-y)| < 2\varepsilon$. Since $\varepsilon > 0$ and $u \in \Sigma_B^t$ are arbitrary, we conclude from the semi-simplicity of B that $x - y = 0$. ■

We now give the proof of Theorem 3.

Proof. To prove the existence of the square root, first take $\|a\| \leqslant 1$. By Lemma 5, $e - a \geqslant 0$ and $\|e-a\| \leqslant 1$. It follows from Proposition 9 that

$$b \equiv \sum_{n=0}^{\infty} c_n (e-a)^n$$

is the unique positive $b \in B$ such that $b^2 = e - (e-a) = a$. For each N write

$$b_N \equiv \sum_{n=0}^{N} c_n (e-a)^n = \sum_{n=0}^{N} c_n e + a p_N(a),$$

where $p_N(a)$ is a polynomial in a of degree $N-1$. Fix $\varepsilon > 0$. Choose N such that $\|b - b_N\| < \varepsilon/2$ and $\left|\sum_{n=0}^{N} c_n\right| < \varepsilon/2$ (remember, $\sum_{n=0}^{\infty} c_n = 0$). Then

$$\begin{aligned}
\|b - a p_N(a)\| &\leqslant \|b - b_N\| + \left\| \left(\sum_{n=0}^{N} c_n\right) e \right\| \\
&< \frac{\varepsilon}{2} + \left|\sum_{n=0}^{N} c_n\right| < \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we conclude that b lies in the closed ideal of B generated by a.

For the general case, choose $\delta > 0$ such that $\|\delta a\| < 1$. By the first part of the proof, there exists a unique $b_\delta \geqslant 0$ in B such that $b_\delta^2 = \delta a$; moreover, b_δ belongs to the closed ideal of B generated by δa and hence by a. Then $\delta^{-1/2} b_\delta$ is in the closed ideal generated by a, $\delta^{-1/2} b_\delta \geqslant 0$, and $\left(\delta^{-1/2} b_\delta\right)^2 = a$; moreover, if $b \geqslant 0$ and $b^2 = a$, then $\delta^{1/2} b \geqslant 0$ and $\left(\delta^{1/2} b\right)^2 = \delta a$, so (by the uniqueness of the positive square root of δa) $\delta^{1/2} b = b_\delta$ and therefore $b = \delta^{-1/2} b_\delta$. ■

3 An application: principal ideals The following result—classically vacuous, but constructively nontrivial—appears in [6] (Corollary 13):

> (*) *If the principal ideal generated by the element a of a Banach algebra is closed, then either $a = 0$ or $a \neq 0$.*

Here, $a \neq 0$ means that $\|a\| > 0$, a constructively stronger statement than $\neg (a = 0)$. Fred Richman, in a private communication, pointed out to us that the only closed principal ideals of the Banach algebra $C(X)$, where X is a compact metric space, are those generated by an idempotent—in other words, a continuous mapping of X into $\{0, 1\}$; and that since, by the classical Gelfand-Naimark theorem ([15], page 289), $C(X)$ is the generic B^*-algebra, there may be little else to say constructively about closed principal ideals in a B^*-algebra.

From a constructive viewpoint the Gelfand-Naimark theorem is rather problematic, since we cannot prove the weak* compactness of the spectrum. Nevertheless, as we now show at the end of this section, for a certain class of Banach algebras we can prove a nice generalisation of (*). First, though, we define firmness for the spectrum: we say that Σ_B is **firm** if

- it is weak*-compact and
- for each $\varepsilon > 0$ and each $x \in B$, there exists $t > 0$ such that if $0 < t' \leqslant t$ and $v \in \Sigma_B^{t'}$, then there exists $u \in \Sigma_B$ with $|u(x) - v(x)| < \varepsilon$.

The second of these conditions is classically equivalent to the statement $\rho\left(\Sigma_B^t, \Sigma_B\right) \to 0$ as $t \to 0$, where ρ denotes the Hausdorff metric on the set of weak*-compact subsets of the unit ball of the dual B'.

Proposition 10 *Suppose that Σ_B is firm. If $a_1, \dots, a_n$ are elements of B,and ε is a positive numbers such that*

$$|u(a_1)| + \cdots + |u(a_n)| \geqslant \varepsilon \quad (u \in \Sigma_B),$$

then there exist $b_1, \dots, b_n$ in B such that $a_1 b_1 + \cdots + a_n b_n = e$.

Proof. By firmness, there exists $t > 0$ such that for each $u \in \Sigma_B$, there exists $v \in \Sigma_B^t$ with $|u(a) - v(a)| < \varepsilon/2$. Then

$$|u(a_1)| + \cdots + |u(a_n)| \geqslant \frac{\varepsilon}{2} \quad \left(u \in \Sigma_B^t\right),$$

so the required elements b_k of B exists, by Proposition 1. ■

The **radical** of B is the set

$$\mathrm{rad}(B) \equiv \bigcap \{\ker(u) : u \in \Sigma_B\}.$$

If Σ_B is inhabited, then $\mathrm{rad}(B)$ is an ideal.

Proposition 11 *Suppose that Σ_B is firm. Then B is semi-simple if and only if* $\mathrm{rad}(B) = \{0\}$.

Proof. Consider any $x \in \mathrm{rad}(B)$. Given $\varepsilon > 0$, we use the firmness of Σ_B to obtain $t > 0$ such that if $v \in \Sigma_B^t$, then there exists $u \in \Sigma_B$ with

$$|v(x)| = |u(x) - v(x)| < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, it follows that if B is semi-simple, then $x = 0$ for each $x \in \mathrm{rad}\,(B)$.

Conversely, suppose that $\mathrm{rad}(B) = \{0\}$. Consider any $x \in B$ with the following property: for each $\varepsilon > 0$ there exists $t > 0$ such that $|u(x)| < \varepsilon$ for every $u \in \Sigma_B^t$. Then, since $\Sigma_B \subset \Sigma_B^t$, it follows that for each $u \in \Sigma_B$, $|u(x)| < \varepsilon$; whence, $\varepsilon > 0$ being arbitrary, we have $u(x) = 0$. Thus $x \in \mathrm{rad}(B)$ and so $x = 0$. ■

The argument in the proof of our final theorem is lifted from that in the special case proved by Richman (see page 107 of [6]).

Theorem 12 *Let B be a commutative, separable, semi-simple Banach algebra. Suppose that the spectrum of B is firm and connected, and that the state space is firm. Let a be a positive element of B, and let I be the principal ideal I of B generated by a. If I is closed, then either a is invertible or $a \in \mathrm{rad}(B)$.*

Proof. By Proposition 9 and Theorem 3,

$$b \equiv \sum_{n=0}^{\infty} c_n \,(e - a)^n$$

is the unique positive square root of a in B, and b lies in the closed ideal I. Hence there exists $g \in B$ such that $b = ga$. Let $M \equiv 1 + \|g\|$. For each $u \in \Sigma_B$, either $|u(a)| < 1/M^2$ or $|u(a)| > 1/2M^2$. In the former case, if $u(a) \neq 0$, then since

$$u(a) = u(b^2) = u(b)^2 = u(g)^2 u(a)^2,$$

we have

$$|u(g)|^2 = \frac{1}{|u(a)|} > M^2,$$

which is absurd; whence $u(a) = 0$. It follows that

$$\Sigma_B = \{u \in \Sigma_B : u(a) = 0\} \cup \left\{ u \in \Sigma_B : u(a) > \frac{1}{2M^2} \right\},$$

where the two constituent subsets of Σ_B are open and clearly disjoint. Since Σ_B is connected, either $u(a) = 0$ for all $u \in \Sigma_B$ and therefore $a \in \mathrm{rad}(B)$; or else $|u(a)| > 1/2M^2$ for all $u \in \Sigma_B$, and therefore, by Proposition 10, a is invertible. ■

Corollary 13 *Under the hypotheses of* Theorem 12*, either $a = 0$ or a is invertible.*

Proof. This follows from Theorem 12 and Proposition 11. ■

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