THE HIAI-PETZ GEODESIC FOR STRONGLY CONVEX NORM IS THE UNIQUE SHORTEST PATH

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Abstract. Recently Hiai-Petz [7] introduced two types of interesting geometries of positive-definite matrices whose geodesics are paths of operator means and then the author [5] showed these geometries have Finsler structures for all unitarily invariant norms. Though the geodesic is of the shortest length between fixed two matrices, the shortest paths are not unique in general as pointed out in [7]. In this paper, we show that their geodesic is the unique shortest path in each Hiai-Petz geometry for all strongly convex unitarily invariant norms. As counter examples, we show that this uniqueness is false for Ky Fan norms.

1 Introduction. Let $\mathcal{M}$ (resp. $\mathcal{M}^+$) be the $n \times n$ (complex) matrices (resp. positive definite matrices). Throughout this paper, a *path* $\gamma(t)$ in $\mathcal{M}^+$ means a smooth curve for $t \in [0,1]$. Recently Hiai and Petz [7] introduced a new geometry for $\mathcal{M}^+$ parametrized by each real number r with a pull-back metric for a diffeomorphism $A \mapsto \ln_r A$ to the Euclidian space where

$$\ln_r(x) = \begin{cases} \frac{x^r-1}{r} & (r \neq 0) \\ \log x & (r = 0). \end{cases}$$

In this geometry, the geodesic is

$$A\,\mathfrak{m}_{r,t}B = \ln_r^{-1}\left((1-t)\ln_r(A) + t\ln_r(B)\right) = \left((1-t)A^r + tB^r\right)^{\frac{1}{r}},$$

which we call in [4] a *chaotically quasi-arithmetic mean* for $r \in [-1,1]$. Though the above means do not have monotonicity any longer for $|r| > 1$, we use the same symbols for the sake of convenience in this paper.

On the other hand, the Hiai-Kosaki mean [6] for A and B is defined by the left (resp. right) multiplication operator $\mathbf{L}_A$(resp. $\mathbf{R}_B$) and the Hadamard product $\circ$:

$$\varphi(\mathbf{L}_A, \mathbf{R}_B)X = U\Big(\big(\varphi(d_i, e_j)\big) \circ U^*XV\Big)V^*$$

where φ is a mean function and the matrices U and V are unitaries which diagonalize $A = U\mathrm{diag}\,(d_i)U^*$, $B = V\mathrm{diag}\,(e_j)V^*$. Here we use the case $A = B$. Throughout this paper, we use a fixed diagonalization of a path γ as $D(t) = \mathrm{diag}\,(d_i(t)) = U_t^*\gamma(t)U_t$. Then

$$\varphi(\mathbf{L}_{\gamma(t)}, \mathbf{R}_{\gamma(t)})X = U_t\left(\left(\varphi(d_i(t), d_j(t))\right) \circ U_t^*XU_t\right)U_t^*.$$

For a continuously differentiable function f, we put

$$\varphi(x,y) = f^{[1]}(x,y) = \begin{cases} \frac{f(x)-f(y)}{x-y} & (x \neq y) \\ f'(x) & (x = y). \end{cases}$$

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Then, as in [5] for the action

$$\Phi_{A,r}(X) \equiv \Phi_{A,\ln_r}(X) = U\left(\left(\ln_r^{[1]}(d_i, d_j)\right) \circ U^*XU\right)U^*,$$

the first Hiai-Petz metric is defined by

$$L_r(X;A) \equiv L_{r,|||\ |||}(X;A) \equiv \left|\left|\left|\Phi_{A,r}(X)\right|\right|\right| = \left|\left|\left|\left(\ln_r^{[1]}(d_i, d_j)\right) \circ U^*XU\right|\right|\right|$$

which is a Finsler one [5]. In this case, they showed the distance between A and B is $|||\ln_r B - \ln_r A|||$.

Hiai-Petz [7, Theorem 3.3] also introduced another parametrized geometry for $\alpha > 0$ whose geodesic is $(A^\alpha \#_t B^\alpha)^{\frac{1}{\alpha}}$ where the path of the geometric operator means in the sense of Kubo-Ando [8] is defined as

$$A \#_t B = A^{\frac{1}{2}}\left(A^{-\frac{1}{2}}BA^{-\frac{1}{2}}\right)^t A^{\frac{1}{2}}.$$

This is an extension of the geometry of Corach-Porta-Recht [2, 3]. For the action for a function $f_\alpha(x) = x^\alpha$

$$\Phi_A(X) \equiv \Phi_A^{[\alpha]}(X) = U\left[\left(f_\alpha^{[1]}(d_i, d_j)\right) \circ U^*XU\right]U^*,$$

the second Hiai-Petz metric is defined by

$$L(X;A) \equiv L_{[\alpha]}(X;A) = \frac{1}{\alpha}|||A^{-\frac{\alpha}{2}}\Phi_A(X)A^{-\frac{\alpha}{2}}|||$$

which is also a Finsler one [5].

Now, recall that the norm $\|\ \|$ is *strictly convex* if

$$\|(1-t)x + ty\| < 1 \qquad \text{for all } t \in (0,1) \text{ and all distinct unit vectors } x \text{ and } y.$$

Then the strict triangle inequality holds

$$\|x + y\| < \|x\| + \|y\|$$

unless one of the vectors is a nonnegative multiple of the other. In particular, for nonzero vectors, triangle equality shows one is a positive multiple of the other.

In this paper we show that the geodesic is the unique shortest path for each Hiai-Petz geometry for all strongly convex unitarily invariant norms (Bhatia mentioned without proof in [1] that it holds for all uniformly convex norms which are strongly convex). Conversely we show Ky Fan norms give counterexamples for such uniqueness.

2 Geometry with the geodesic $A\,\mathfrak{m}_{r,t}B$.

Theorem 1. *If a unitarily invariant norm is strictly convex, the chaotically quasi-arithmetic mean*

$$A\,\mathfrak{m}_{r,t}B = \ln_r^{-1}\left((1-t)\ln_r(A) + t\ln_r(B)\right) = \left((1-t)A^r + tB^r\right)^{\frac{1}{r}}$$

is the unique shortest geodesic for the first Hiai-Petz metric

$$L_r(X;A) = L_{r,|||\ |||}(X;A) = \left|\left|\left|\left(\ln_r^{[1]}(d_i, d_j)\right) \circ U^*XU\right|\right|\right|$$

where the shortest length is $|||\ln_r B - \ln_r A|||$.

Proof. Suppose γ attains the shortest length. Since

$$L_r(\dot{\gamma};\gamma) = \left|\left|\left| \left(\ln_r^{[1]}(d_i(t), d_j'(t))\right) \circ U_t^*\dot{\gamma}U_t \right|\right|\right| = \left|\left|\left| \frac{d\ln_r \gamma}{dt}(t) \right|\right|\right|$$

for a parametrized diagonalization $U_t^*\gamma(t)U_t = \mathrm{diag}\,(d_j(t))$, the length $\ell(\gamma)$ satisfies

$$|||\ln_r B - \ln_r A||| = \ell(\gamma) \equiv \int_0^1 L_r(\dot{\gamma};\gamma)dt \geqq \left|\left|\left| \int_0^1 \frac{d\ln_r \gamma}{dt}(t)dt \right|\right|\right| = |||\ln_r B - \ln_r A|||$$

For $H(t) = \ln_r \gamma(t)$, it must satisfy

$$\int_0^1 |||H'(t)|||dt = \left|\left|\left| \int_0^1 H'(t)dt \right|\right|\right| = |||\ln_r B - \ln_r A|||.$$

Here we use the broken line approximation to obtain the length of $H(t)$:

$$\int_0^1 |||H'(t)|||dt = \lim_{|\Delta|\to 0} \sum_{t_n\in\Delta} |||H(t_{n+1}) - H(t_n)|||.$$

Take the following monotone increasing sequence converging to $\int_0^1 \|H'(t)\|dt$:

$$\sum_{k=1}^{2^n} \left|\left|\left| H\left(\frac{k}{2^n}\right) - H\left(\frac{k-1}{2^n}\right) \right|\right|\right| \quad \uparrow \quad \int_0^1 |||H'(t)|||dt.$$

Then all the triangle inequalities

$$\begin{aligned} &\left|\left|\left| H\left(\frac{k}{2^n}\right) - H\left(\frac{k-1}{2^n}\right) \right|\right|\right| \\ &\qquad \leqq \left|\left|\left| H\left(\frac{2k}{2^{n+1}}\right) - H\left(\frac{2k-1}{2^{n+1}}\right) \right|\right|\right| + \left|\left|\left| H\left(\frac{2k-1}{2^{n+1}}\right) - H\left(\frac{2(k-1)}{2^{n+1}}\right) \right|\right|\right| \end{aligned}$$

are equal, so that there exists $s_{\frac{2k-1}{2^{n+1}}} > 0$ with

$$H\left(\frac{2k}{2^{n+1}}\right) - H\left(\frac{2k-1}{2^{n+1}}\right) = s_{\frac{2k-1}{2^{n+1}}}\left(H\left(\frac{2k-1}{2^{n+1}}\right) - H\left(\frac{2(k-1)}{2^{n+1}}\right)\right),$$

that is, $H\left(\frac{2k-1}{2^{n+1}}\right)$ at each binary fraction $\frac{2k-1}{2^{n+1}}$ in $[0,1]$ is the convex combination for $H\left(\frac{k-1}{2^n}\right)$ and $H\left(\frac{k}{2^n}\right)$;

$$H\left(\frac{2k-1}{2^{n+1}}\right) = \frac{s_{\frac{2k-1}{2^{n+1}}} H\left(\frac{k-1}{2^n}\right) + H\left(\frac{k}{2^n}\right)}{s_{\frac{2k-1}{2^{n+1}}} + 1}$$

holds for all n and $k = 1, \cdots, 2^n$. Thus, all the constants $s_{\frac{2k-1}{2^{n+1}}}$ are defined from the terminal points $H(0) = \ln_r A$ and $H(1) = \ln_r(B)$ with $s_0 = 0$ and $s_1 = 1$. Therefore we can define a function w on the binary fractions in $[0,1]$ inductively with the relation

$$H\left(\frac{2k-1}{2^{n+1}}\right) = \left(1 - w\left(\frac{2k-1}{2^{n+1}}\right)\right)\ln_r A + w\left(\frac{2k-1}{2^{n+1}}\right)\ln_r B.$$

In fact, restricting ourselves to the coefficient of $\ln_r B$, we have the recurrence equation

$$w\left(\frac{2k-1}{2^n}\right) = \frac{s_{\frac{2k-1}{2^n}} w\left(\frac{k-1}{2^{n-1}}\right) + w\left(\frac{k}{2^{n-1}}\right)}{s_{\frac{2k-1}{2^n}} + 1}.$$

So, by $w(0)=0$ and $w(1)=1$, we have inductively

$$\begin{aligned}
w\left(\frac{1}{2^n}\right) &= \frac{w\left(\frac{1}{2^{n-1}}\right)}{s_{\frac{1}{2^n}}+1} = \frac{w\left(\frac{1}{2^{n-2}}\right)}{\left(s_{\frac{1}{2^{n-1}}}+1\right)\left(s_{\frac{1}{2^n}}+1\right)} = \cdots = \frac{1}{\prod_{k=1}^{n}\left(s_{\frac{1}{2^k}}+1\right)}\\
w\left(\frac{3}{2^n}\right) &= \frac{s_{\frac{3}{2^n}}w\left(\frac{1}{2^{n-1}}\right)+w\left(\frac{1}{2^{n-2}}\right)}{s_{\frac{3}{2^n}}+1} = \frac{s_{\frac{3}{2^n}}+s_{\frac{1}{2^{n-1}}}+1}{\left(s_{\frac{3}{2^n}}+1\right)\prod_{k=1}^{n-1}\left(s_{\frac{1}{2^k}}+1\right)}\\
w\left(\frac{5}{2^n}\right) &= \frac{s_{\frac{5}{2^n}}w\left(\frac{1}{2^{n-2}}\right)+w\left(\frac{3}{2^{n-1}}\right)}{s_{\frac{5}{2^n}}+1} = \frac{s_{\frac{5}{2^n}}\left(s_{\frac{3}{2^{n-1}}}+1\right)+s_{\frac{3}{2^{n-1}}}+s_{\frac{1}{2^{n-2}}}+1}{\left(s_{\frac{5}{2^n}}+1\right)\left(s_{\frac{3}{2^{n-1}}}+1\right)\prod_{k=1}^{n-2}\left(s_{\frac{1}{2^k}}+1\right)}\\
w\left(\frac{7}{2^n}\right) &= \frac{s_{\frac{7}{2^n}}w\left(\frac{3}{2^{n-1}}\right)+w\left(\frac{1}{2^{n-3}}\right)}{s_{\frac{7}{2^n}}+1} = \frac{s_{\frac{7}{2^n}}\left(s_{\frac{3}{2^{n-1}}}+s_{\frac{1}{2^{n-2}}}+1\right)+\left(s_{\frac{3}{2^{n-1}}}+1\right)\left(s_{\frac{1}{2^{n-2}}}+1\right)}{\left(s_{\frac{7}{2^n}}+1\right)\left(s_{\frac{3}{2^{n-1}}}+1\right)\prod_{k=1}^{n-2}\left(s_{\frac{1}{2^k}}+1\right)}\\
&\vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad .
\end{aligned}$$

By the initial conditions $w(0)=0$ and $w(1)=1$, the function w is monotone increasing on the above fractions. The smoothness of γ implies that w is smoothly extended to a function on $[0,1]$ which is monotone increasing and satisfies

$$\ln_r(\gamma(t)) = H(t) = (1-w(t))\ln_r A + w(t)\ln_r B.$$

Thus

$$\gamma(t) = \ln_r^{-1}\left((1-w(t))\ln_r A + w(t)\ln_r B\right) = A\,\mathfrak{m}_{r,w(t)}B,$$

that is, γ can be identified with $A\,\mathfrak{m}_{r,t}B$. □

Typical unitarily invariant norms which are not strongly convex are Ky Fan's, that is $\|X\|_{(k)}$ means the sum of singular values for X from the largest to the k-th. In this case, the shortest paths are not uniquely determined for the Hiai-Petz geometry as in the following example:

Example 1. Let $B=(b_j)$ be a diagonal positive-definite matrix greater than I with b_j is (strictly) monotone decreasing. For a path from I to B, the shortest length is $\|\ln_r B\|_{(k)}$. Then, for two path of distinct means $\mathrm{m}_t \neq \mathrm{n}_t$, we have $I\mathrm{m}_t B$ is different from $I\mathrm{n}_t B$ as paths by the strict monotonicity for b_j.

case $r=1$**:** Let $\delta(t)=B^t$ which differs from the geodesic $(1-t)I+tB$. Then $\dot{\delta}(t)=B^t\log B \geq O$ and $x^t\log x$ is monotone increasing for $x>1$. Since $\ln_1'(x)=1$, we have

$$L_{1,\|\ \|_{(k)}}(\dot{\delta};\delta) = \|B^t\log B\|_{(k)} = \sum_{j=1}^{k} b_j^t \log b_j,$$

so that

$$\ell_{\|\ \|_{(k)}}(\delta) = \int_0^1 \sum_{j=1}^{k}(b_j^t\log b_j)dt = \sum_{j=1}^{k}\int_0^1 (b_j^t)'dt = \sum_{j=1}^{n}\left[b_j^t\right]_0^1 = \sum_{j=1}^{k}(b_j-1) = \|\ln_1 B\|_{(k)}.$$

case $r \geqq 0$ **and** $r\neq 1$**:** Let $\gamma(t)=(1-t)I+tB=I+t(B-I)$. Then $\dot{\gamma}(t)=B-I$ and

$\ln_r'(x) = x^{r-1}$. Hence

$$\begin{aligned} L_{r,\|\ \|_{(k)}}(\dot\gamma;\gamma) &= \|\ln_r'(\gamma(t))\circ(B-I)\|_{(k)} \\ &= \|\mathrm{diag}\ \left((1+t(b_j-1))^{r-1}(b_j-1)\right)\|_{(k)}. \end{aligned}$$

Here a function $f(x) = (1+tx)^{r-1}x$ is monotone increasing for $x > 0$ since

$$f'(x) = (1+tx)^{r-2}(rtx+1) \geqq 0.$$

By $\dot\gamma(t) \geq O$ in this case, we have

$$L_{r,\|\ \|_{(k)}}(\dot\gamma;\gamma) = \sum_{j=1}^{k}(1+t(b_j-1))^{r-1}(b_j-1) = \sum_{j=1}^{k}\left(\frac{(1+t(b_j-1))^r}{r}\right)',$$

so that,

$$\begin{aligned} \ell_{\|\ \|_{(k)}}(\gamma) &= \int_0^1 \sum_{j=1}^{k}\left(\frac{(1+t(b-1))^r}{r}\right)' dt \\ &= \sum_{j=1}^{k}\left[\frac{(1+t(b_j-1))^r}{r}\right]_0^1 = \sum_{j=1}^{k}\frac{b_j^r-1}{r} = \|\ln_r B\|_{(k)}. \end{aligned}$$

case $r < 0$: Let $1 < b_k < 1 - \frac{1}{r}$. Then for $0 < x < -\frac{1}{r}$, a function $f(x)$ in the above is also monotone increasing since

$$f'(x) = (1+tx)^{r-2}(rtx+1) > (1+tx)^{r-2}(-t+1) \geqq 0.$$

Similarly as the above, we have $\ell_{\|\ \|_{(k)}}(\gamma) = \|\ln_r(B)\|_{(k)}$.

3 Geometry with the geodesic $(A^\alpha \#_t B^\alpha)^{\frac{1}{\alpha}}$.

Theorem 2. *If a unitarily invariant norm is strictly convex, the path*

$$A\ p_{\alpha,t}B = (A^\alpha \#_t B^\alpha)^{\frac{1}{\alpha}}$$

is the unique shortest geodesic for the second Hiai-Petz metric

$$L_{[\alpha]}(X;A) = \frac{1}{\alpha}\left|\!\left|\!\left|\left(\frac{f_{[\alpha]}^{[1]}(d_i,d_j)}{(d_id_j)^{\frac{\alpha}{2}}}\right)\circ U^*XU\right|\!\right|\!\right| = \frac{1}{\alpha}|\!|\!|A^{-\frac{\alpha}{2}}U\left[\left(f_{[\alpha]}^{[1]}(d_i,d_j)\right)\circ U^*XU\right]U^*A^{-\frac{\alpha}{2}}|\!|\!|$$

for a diagonalization $U^*AU = diag(d_j)$ and a function $f_{[\alpha]}(x) = x^\alpha$ where the shortest length is $|\!|\!|\log(A^{-\frac{\alpha}{2}}B^\alpha A^{-\frac{\alpha}{2}})^{1/\alpha}|\!|\!|$.

Note that the differential formula shows

$$\begin{aligned} \alpha L_\alpha(\dot\gamma(t);\gamma(t)) &= \left|\!\left|\!\left|U_t^*\gamma(t)^{-\frac{\alpha}{2}}U_t\left[\left(f_{[\alpha]}^{[1]}(d_i(t),d_j(t))\right)\circ U_t^*\dot\gamma(t)U_t\right]U_t^*\gamma(t)^{-\frac{\alpha}{2}}U_t\right|\!\right|\!\right| \\ &= \left|\!\left|\!\left|\gamma(t)^{-\frac{\alpha}{2}}(\gamma(t)^\alpha)'\gamma(t)^{-\frac{\alpha}{2}}\right|\!\right|\!\right|. \end{aligned}$$

Here we use the following property of this metric instead of 'homogeneity': For a path γ and an invertible matrix Y, define the path γ_Y by

$$\gamma_Y(t) \equiv \gamma_{Y,\alpha}(t) \equiv (Y^*\gamma^\alpha(t)Y)^{\frac{1}{\alpha}}.$$

Then we have an invariance property:

Lemma 3. $$L_{[\alpha]}(\dot{\gamma_Y};\gamma_Y) = L_{[\alpha]}(\dot{\gamma};\gamma).$$

Proof. Since $|||Z||| = |||\,|Z|\,||| = |||\sqrt{Z^*Z}||| = |||\sqrt{ZZ^*}|||$, we have

$$\begin{aligned}
\alpha L_\alpha(\dot{\gamma_Y};\gamma_Y) &= \left|\left|\left|\gamma_Y^{-\frac{\alpha}{2}}(\gamma_Y^\alpha)'\gamma_Y^{-\frac{\alpha}{2}}\right|\right|\right| = \left|\left|\left|(Y^*\gamma^\alpha Y)^{-\frac12}(Y^*\gamma^\alpha Y)'(Y^*\gamma^\alpha Y)^{-\frac12}\right|\right|\right| \\
&= \left|\left|\left|(Y^*\gamma^\alpha Y)^{-\frac12}Y^*(\gamma^\alpha)'Y(Y^*\gamma^\alpha Y)^{-\frac12}\right|\right|\right| \\
&= \left|\left|\left|\sqrt{(Y^*\gamma^\alpha Y)^{-\frac12}Y^*(\gamma^\alpha)'Y(Y^*\gamma^\alpha Y)^{-1}Y^*(\gamma^\alpha)'Y(Y^*\gamma^\alpha Y)^{-\frac12}}\right|\right|\right| \\
&= \left|\left|\left|\sqrt{(Y^*\gamma^\alpha Y)^{-\frac12}Y(\gamma^\alpha)'\gamma^{-\alpha}(\gamma^\alpha)'Y^*(Y^*\gamma^\alpha Y)^{-\frac12}}\right|\right|\right| \\
&= \left|\left|\left|\sqrt{\gamma^{-\frac{\alpha}{2}}(\gamma^\alpha)'Y(Y^*\gamma^\alpha Y)^{-1}Y^*(\gamma^\alpha)'\gamma^{-\frac{\alpha}{2}}}\right|\right|\right| \\
&= \left|\left|\left|\sqrt{\gamma^{-\frac{\alpha}{2}}(\gamma^\alpha)'\gamma^{-\alpha}(\gamma^\alpha)'\gamma^{-\frac{\alpha}{2}}}\right|\right|\right| = \left|\left|\left|\gamma^{-\frac{\alpha}{2}}(\gamma^\alpha)'\gamma^{-\frac{\alpha}{2}}\right|\right|\right| = \alpha L_\alpha(\dot{\gamma};\gamma).
\end{aligned}$$

□

To show the theorem, we use the formula for $H(t) = \alpha \log \gamma(t)$;

$$(\gamma(t)^\alpha)' = \frac{d}{dt}e^{H(t)} = \int_0^1 e^{uH(t)}H'(t)e^{(1-u)H(t)}du$$

and the following 'logarithmic-geometric mean inequality' due to Hiai-Kosaki [6]:

Hiai-Kosaki inequality: $$\left|\left|\left|\int_0^1 H^u X K^{1-u}du\right|\right|\right| \geqq |||H^{1/2}XK^{1/2}|||.$$

Proof of Theorem 2. The Hiai-Kosaki inequality implies

$$\begin{aligned}
\alpha L_{[\alpha]}(\dot{\gamma};\gamma) &= \left|\left|\left|\gamma^{-\frac{\alpha}{2}}(\gamma^\alpha)'\gamma^{-\frac{\alpha}{2}}\right|\right|\right| \\
&= \left|\left|\left|e^{-\frac{H(t)}{2}}\left(\int_0^1 e^{uH(t)}H'(t)e^{(1-u)H(t)}du\right)e^{-\frac{H(t)}{2}}\right|\right|\right| \\
&= \left|\left|\left|\int_0^1 e^{uH(t)}e^{-\frac{H(t)}{2}}H'(t)e^{-\frac{H(t)}{2}}e^{(1-u)H(t)}du\right|\right|\right| \\
&\geqq \left|\left|\left|e^{\frac{H(t)}{2}}e^{-\frac{H(t)}{2}}H'(t)e^{-\frac{H(t)}{2}}e^{\frac{H(t)}{2}}\right|\right|\right| = |||H'(t)|||.
\end{aligned}$$

Now suppose a path γ from A to B attains the shortest length. By the above lemma, this condition is equivalent to that the path $\delta(t) \equiv \gamma_{A^{-\alpha/2}}$ from I to $(A^{-\frac{\alpha}{2}}B^\alpha A^{-\frac{\alpha}{2}})^{1/\alpha}$ attains the shortest length $|||\log(A^{-\frac{\alpha}{2}}B^\alpha A^{-\frac{\alpha}{2}})^{1/\alpha}|||$. So we consider H for δ instead of γ. Then the length $\ell(\gamma)$ satisfies

$$\begin{aligned}
|||\log(A^{-\frac{\alpha}{2}}B^\alpha A^{-\frac{\alpha}{2}})^{\frac{1}{\alpha}}||| = \ell(\delta) \equiv \frac{1}{\alpha}\int_0^1 L_{[\alpha]}(\dot{\delta};\delta)dt \geqq \frac{1}{\alpha}\int_0^1 |||H'(t)|||dt \\
\geqq \frac{1}{\alpha}\left|\left|\left|\int_0^1 H'(t)dt\right|\right|\right| = \frac{1}{\alpha}|||H(1)-H(0)||| = |||\log(A^{-\frac{\alpha}{2}}B^\alpha A^{-\frac{\alpha}{2}})^{\frac{1}{\alpha}}|||,
\end{aligned}$$

so that, it must satisfy

$$\int_0^1 |||H'(t)|||dt = \left|\left|\left|\int_0^1 H'(t)dt\right|\right|\right| = |||\log A^{-\frac{\alpha}{2}}B^\alpha A^{-\frac{\alpha}{2}}|||.$$

Therefore, similarly to the preceding proof, we have $\delta(t)$ equals $\left(A^{-\frac{\alpha}{2}}B^{\alpha}A^{-\frac{\alpha}{2}}\right)^{t/\alpha}$, and hence $\gamma(t)$ equals $A\ p_{\alpha,t}B$ as paths. $\square$

Also, we can show that the shortest paths are not unique for Ky Fan norm $\|\ \|_{(k)}$ as in the following example:

Example 2. Let $B = (b_j)$ be a non-scalar positive definite diagonal matrix with $b_j > 1$ for all j which is strictly monotone decreasing. Then, a path

$$\delta(t) = (1 - t + tB^{\alpha})^{1/\alpha},$$

differs from the geodesic $\gamma_{[\alpha]}(t) = Ip_{\alpha,t}B = (B^{t\alpha})^{1/\alpha} = B^t$, and

$$L_{[\alpha],k}(\dot{\delta};\delta) = \frac{1}{\alpha}\|\delta^{-\alpha/2}(\dot{\delta^{\alpha}})\delta^{-\alpha/2}\|_{(k)} = \|(1-t+tB^{\alpha})^{-1}(B^{\alpha}-1)\|_{(k)}.$$

Putting

$$f(x) = \frac{x^{\alpha}-1}{1-t+tx^{\alpha}},$$

we have

$$f'(x) = \frac{\alpha x^{\alpha-1}}{(1-t+tx^{\alpha})^2} > 0$$

and hence f is monotone increasing. Therefore $L_{[\alpha],k}(\dot{\delta};\delta) = \sum_{j=1}^{k} f(b_j)$ and hence

$$\begin{aligned}\ell(\delta) &= \int_0^1 \sum_{j=1}^{k} f(b_j)dt = \sum_{j=1}^{k}\int_0^1 \frac{b_j^{\alpha}-1}{1-t+tb_j^{\alpha}}dt = \frac{1}{\alpha}\sum_{j=1}^{k}\left[\log(1-t+tb_j^{\alpha})\right]_0^1 \\ &= \frac{1}{\alpha}\sum_{j=1}^{k}\log b_j^{\alpha} = \frac{1}{\alpha}\|\log B^{\alpha}\|_{(k)} = \|(\log B^{\alpha})^{\frac{1}{\alpha}}\|_{(k)} = \ell(\gamma_{[\alpha]}(t)),\end{aligned}$$

so that we have δ is one of the shortest paths.

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