A CONSTRUCTION OF HYPERSEMIGROUPS FROM A FAMILY OF PREORDERED SEMIGROUPS

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Abstract. In this article the notion of weak mutual associativity (w.m.a.) and the necessary and sufficient condition for a (L, Γ)-associated hypersemigroup $(H, *)$ derived from some family of $\lesssim$-preordered semigroups to be a hypergroup, are given.

1 Introduction The first step in the history of the development of hyperstructures theory was the 8th congress of scandinavian mathematicians from 1934, when Marty [6] introduced the notion of hypergroup, analyzed its properties and applied them to noncommutative groups, algebraic functions and rational fractions. Nowadays hypergroups are studied from the theoretical point of view and for their applications to many subjects of pure and applied mathematics: geometry, topology, cryptography and coding theory, graphs and hypergraphs, probability theory, binary relations, theory of fuzzy and rough sets and automata theory, hyperalgebras and hyper-coalgebras, etc. (see [3, 4]).

The correspondence between hyperstructures and binary relations is implicitly contained in Nieminen [8] who associated hypergroups to connected simple graphs. Next, Chvalina [1] found a correspondence between partially ordered sets and hypergroups, then a construction of hypergroups from ordered structure has been introduced by D.A. Hort [5]. In this paper, first we introduce the weak mutual associativity of two hyperoperations and then we introduce a construction of hypergroups from a family of weak mutual associative preordered semigroups. In [3], the mutual associativity of two hyperoperations has been introduced by P. Corsini.

In the current section we give the preliminaries which will be used throughout this article.

Definition 1.1. Consider some set P and a binary relation $\lesssim$ on P. Then $\lesssim$ is a preorder, if it is reflexive and transitive, i.e., for all x, y and z in P, we have that: (a) $x \lesssim x$ (reflexivity);

(b) if $x \lesssim y$ and $y \lesssim z$, then $x \lesssim z$ (transitivity).

A set that is equipped with a preorder is called a preordered set.

Definition 1.2. A hypergroupoid is a nonempty set H together with a map $* : H \times H \longrightarrow P^*(H)$ which is called hyperoperation, where $P^*(H)$ denotes the set of all non-empty subsets of H.

Remark 1.3. A hyperoperation $* : H \times H \longrightarrow P^*(H)$ yields an operation $\times : P^*(H) \times P^*(H) \longrightarrow P^*(H)$, defined by $A \times B = \bigcup\limits_{a \in A, b \in B} a * b$. Conversely an operation on $P^*(H)$ yields a hyperoperation on H, defined by $x * y = \{x\} \times \{y\}$.

Definition 1.4. (i) A hypersemigroup is a hypergroupoid $(H, *)$ such that for all a, b and c in H we have $(a * b) * c = a * (b * c)$.

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(ii) A quasihypergroup is a hypergroupoid $(H, *)$ which satisfies the reproductive law, i.e., for all $a \in H$, $H * a = a * H = H$, where $H * a = \bigcup_{h \in H} h * a$ (resp. $a * H$).

(iii) A hypergroup is a hypersemigroup which is also a quasihypergroup.

Definition 1.5. (i) A hypersemigroup $(H, *)$ is called commutative if for all $a, b \in H$ we have $a * b = b * a$.

(ii) A hypersemigroup $(H, *)$ is called complete if for all $(x_1, x_2, ..., x_n) \in H^n$ and $(y_1, y_2, ..., y_m) \in H^m$ where $n, m \geqslant 2$ we have the following implication:

$$\prod_{i=1}^{n} x_i \bigcap \prod_{j=1}^{m} y_j \neq \emptyset \Rightarrow \prod_{i=1}^{n} x_i = \prod_{j=1}^{m} y_j$$

where $\prod\limits_{i=1}^{n} x_i = x_1 * \prod\limits_{i=2}^{n} x_i$.

The notion of complete hypergroup is defined by Migliorato [7] and was studied by many other researchers (see [3]).

Theorem 1.6. A hypersemigroup $(H, *)$ is complete if $H = \bigcup_{s \in S} A_s$, where S and A_s satisfy the conditions: (1) $(S, \cdot)$ is a semigroup;

(2) for all $s, t \in S^2$ where $s \neq t$ we have $A_s \bigcap A_t = \emptyset$;

(3) if $(a, b) \in A_s \times A_t$, then $a * b = A_{s \cdot t}$.

Definition 1.7. Let $(H, *)$ be a hypergroup and $b \in H$. The element b is called a left scalar if for all $x \in H$, $b * x$ is a singleton set and b is called a two-sided scalar or simply scalar if it is a left and a right scalar.

Theorem 1.8. Let $(H, *)$ be a hypergroup, $b \in H$ a scalar element. Then the set of all scalar elements is a group.

Definition 1.9. We say that two binary hyperoperations $\langle *_1 \rangle$, $\langle *_2 \rangle$ on the same set H are mutually associative (m.a.) if for all $(x, y, z) \in H^3$, we have:

$$(x *_1 y) *_2 z = x *_1 (y *_2 z) \text{ and } (x *_2 y) *_1 z = x *_2 (y *_1 z)$$

Mutual associativity for the class of hypergroupoids was introduced by Corsini (see [3]).

Definition 1.9. Suppose $(H, \circ)$ and $(H, \circ')$ are two hypersemigroups. A function $f : H \longrightarrow H'$ is called a homomorphism if $f(x \circ y) \subseteq f(x) \circ' f(y)$ for all x and y in H. We call f a good homomorphism if for all x and y in H $f(x \circ y) = f(x) \circ' f(y)$.

2 A construction of hypersemigroups from some semigroups

In this section we introduce the notion of weak mutual associativity and construct a hyperoperation from the family of weak mutual associative preordered semigroups.

Definition 2.1. Suppose that $*_1$ and $*_2$ are two hyperoperations on H. We say that $*_1$ and $*_2$ are weak mutually associative (for simplicity we say $*_1$ and $*_2$ are w.m.a) if for all x, y and z in H we have:

$$\bigcup_{x,y,z \in H} \{(x *_1 y) *_2 z, (x *_2 y) *_1 z\} = \bigcup_{x,y,z \in H} \{x *_1 (y *_2 z), x *_2 (y *_1 z)\}.$$

We also say that the pair $((H, *_1), (H, *_2))$ is weak mutually associative (or for simplicity w.m.a).

Definition 2.2. We say $(H, \circ, \lesssim)$ is a $\lesssim$-preordered groupoid (or semigroup) where $(H, \circ)$ is a groupoid (or semigroup) and $(H, \lesssim)$ is a preordered set such that for all x, y and z in H the following axioms are fulfilled:
(i) $x \lesssim y$ implies $x \circ z \lesssim y \circ z$;
(ii) $x \lesssim y$ implies $z \circ x \lesssim z \circ y$.

Definition 2.3. Suppose that $(H, \lesssim)$ is a preordered set, for $x \in H$ we define:

$$L(x) :\overset{\text{def}}{=} \{h \in H |\, h \lesssim x\} \text{ and } U(x) :\overset{\text{def}}{=} \{h \in H |\, x \lesssim h\} \text{ and;}$$

$$\emptyset \neq X \subseteq H, L(X) :\overset{\text{def}}{=} \bigcup_{x \in X} L(x) \text{ and } U(X) :\overset{\text{def}}{=} \bigcup_{x \in X} U(x).$$

Lemma 2.4. Suppose that Γ is a family of $\lesssim$-preordered semigroups such that every pair $((H, \circ_1), (H, \circ_2)) \in \Gamma^2$ is weak mutually associative. Then the following equality for all a, b and c in H holds:

$$\bigcup_{(H,\ \circ) \in\ \Gamma} \{L(t \circ c) |\, t \in \bigcup L(a \circ b)\} = \bigcup_{(H,\ \circ) \in\ \Gamma} \{L(a \circ s) |\, s \in \bigcup L(b \circ c)\}.$$

Proof: Suppose that $(H, \circ) \in \Gamma$ and $x \in \bigcup_{t \in \bigcup L(a \circ b)} L(t \circ c)$ are given. So $x \lesssim t_0 \circ c$ for an appropriate $t_0 \lesssim a \circ' b$ where $t_0 \in H$ and $(H, \circ') \in \Gamma$ thus by Definition 2.2, we have $x \lesssim (a \circ' b) \circ c$. Since the pair $((H, \circ), (H, \circ'))$ belongs to Γ^2, then it is w.m.a and we have $(a \circ' b) \circ c = a \circ' (b \circ c)$ or $(a \circ' b) \circ c = a \circ (b \circ' c)$. First suppose $(a \circ' b) \circ c = a \circ' (b \circ c)$. Put $d :\overset{\text{def}}{=} b \circ c$, so $x \lesssim a \circ' d$ and hence $x \in L(a \circ' d) \subseteq \bigcup_{s \in \bigcup b\ \circ\ c} L(a \circ s)$. Now suppose $(a \circ' b) \circ c = a \circ (b \circ' c)$. Put $u :\overset{\text{def}}{=} b \circ' c$, so we have $x \lesssim a \circ u$ and hence $x \in L(a \circ u) \subseteq \bigcup_{s \in \bigcup b\ \circ\ c} L(a \circ s)$. Similarly we have the opposite assertion.

Proposition 2.5. Suppose that Γ is a family of $\lesssim$-preordered semigroups such that every pair $((H, \circ_1), (H, \circ_2)) \in \Gamma^2$ is w.m.a. The hyperoperation $*$ on H where $a * b :\overset{\text{def}}{=} \bigcup_{(H, \circ) \in \Gamma} L(a \circ b)$ gives a hypersemigroup $(H, *)$ and we say $(H, *)$ is a (L, Γ)-associative hypersemigroup.

Proof: It is easy to see that $*$ is a well-defined map so we must show that $*$ is an associative hyperoperation. For all $(H, \circ) \in \Gamma$ and a, b and c in H we have $(a * b) * c = \bigcup_{t \in \bigcup L(a\ \circ\ b)} L(t \circ c)$, so by Lemma 2.4., we have $(a * b) * c = \bigcup_{s \in \bigcup L(b\ \circ\ c)} L(a \circ s) = a * (b * c)$.

Proposition 2.6. Suppose that Γ is a family of $\lesssim$-preordered semigroups and that every pair $((H, \circ_1), (H, \circ_2)) \in \Gamma^2$ is w.m.a. The binary hyperoperation $\star$ on H defined by $a \star b :\overset{\text{def}}{=} \bigcup_{(H, \circ) \in \Gamma} U(a \circ b)$ gives a hypersemigroup $(H, \star)$ and we say $(H, \star)$ is a (U, Γ)-associative hypersemigroup.

Proof: The proof is similar to Proposition 2.5.

Theorem 2.7. Suppose that Γ is a family of $\lesssim$-preordered semigroups such that every pair $((H, \circ_1), (H, \circ_2)) \in \Gamma^2$ is w.m.a. Then the following conditions are equivalent:
(i) for all $(a, b) \in H^2$ there exist $(c, c') \in H^2$ and a pair $((H, \circ_1), (H, \circ_2)) \in \Gamma^2$ such that $a \lesssim b \circ_1 c$ and $a \lesssim c' \circ_2 b$;

(ii) a (L, Γ)-associated hypersemigroup $(H, *)$ satisfies the reproductive law (i.e., $(H, *)$ is a hypergroup).

Proof: (i) $\Rightarrow$ (ii) Suppose that $t \in H$ is given, so the inclusions $t * H \subseteq H$ and $H * t \subseteq H$ are obviously fulfilled. We must prove the opposite inclusion. Let $s \in H$ be an arbitrary element. For any $s, t \in H$ there exist $c, c' \in H$ and a pair $((H, \circ_1), (H, \circ_2)) \in \Gamma^2$ such that $s \lesssim t \circ_1 c$ and $s \lesssim c' \circ_2 t$, so we have:

$$s \in L(t \circ_1 c) \subseteq t * c \subseteq t * H \quad \text{and} \quad s \in L(c' \circ_2 t) \subseteq c' * t \subseteq H * t$$

and consequently $H \subseteq t * H$ and $H \subseteq H * t$.

(ii) $\Rightarrow$ (i) Let $(H, *)$ be a hypergroup and $a, b \in H$ arbitrary elements, so $b * H = H * b = H$ and it follows that $a \in b * H = \bigcup_{t \in H} b * H$ which means $a \in b * c$ for an appropriate element $c \in H$, i.e., $a \lesssim b \circ c$. Similarly $a \in H * b$ which implies $a \lesssim c' \circ b$ for an appropriate element $c' \in H$.

Theorem 2.8. Suppose that Γ is a family of commutative $\lesssim$-preordered semigroups such that every pair $((H, \circ_1), (H, \circ_2)) \in \Gamma^2$ is w.m.a. Then the (L, Γ)-associated hypersemigroup $(H, *)$ (and similarly (U, Γ)-associated hypersemigroup $(H, \star)$) is commutative.

Proof: The proof is straightforward.

Theorem 2.9. Let $\Gamma = \{(H, \circ)\}$ be a singleton $\lesssim$-preordered semigroup and $(H, *)$ be (L, Γ)-associated hypersemigroup. We have the following:

(i) for all $(x, y, z, t) \in H^4$ if $x \circ y \in z * t$, then $x * y \subseteq z * t$;

(ii) for all nonempty subset A of H define $m_A \overset{\text{def}}{:=} Max\{card(a * b) \ : \ a, b \in A\}$ where $card(a * b)$ means the cardinal number of the subsets $a * b$. If $H = \bigcup_{a,b \in A} a * b$ and m_A exists, then for all $(x, y) \in H^2$, $card(x * y) \leqslant m_A$.

Proof: (i) Let $u \in x * y$ so $u \lesssim x \circ y$. Since $x \circ y \in z * t$, then $x \circ y \lesssim z \circ t$. Thus $u \lesssim x \circ y \lesssim z \circ t$, by transitivity it follows $u \lesssim z \circ t$. Therefore $u \in z * t$.

(ii) Since $x \circ y \in x * y \subseteq H$, then there exist a and b in A such that $x \circ y \in a * b$ and by (i) we have $x * y \subseteq a * b$. So $card(x * y) \leqslant card(a * b) \leqslant m_A$.

Corollary 2.10. Let $\Gamma = \{(H, \circ)\}$ be a $\lesssim$-preordered semigroup. If $(H, *)$ is a (L, Γ)-associated hypergroup and $b \in H$ is a left scalar element, then $(H, *)$ is a group.

Theorem 2.11. Every complete hypersemigroup is derived from a $\lesssim$-preordered semigroup.

Proof: Let $(H, *)$ be a complete hypersemigroup. According to the Theorem 1.6., we have $H = \bigcup_{s \in S} A_s$ where S is a semigroup and $\{A_s\}_{s \in S}$ is a nonempty family of sets which are mutually disjoint. Consider the choice function C over the set $\{A_s |\ s \in S\}$, i.e., $C : \{A_s |\ s \in S\} \to \bigcup_{s \in S} A_s = H$, where $C(A_s) \in A_s$. It is easy to see that:

(i) H endowed with the operation "$\circ$" that defined by $a \circ b \overset{\text{def}}{:=} C(A_{st})$ for all $(a, b) \in A_s \times A_t$ and $s, t \in S$, is a semigroup;

(ii) the relation $\lesssim$ over H defined by:

(1) for all $a \in H$, $a \lesssim a$;

(2) for all $s, t \in S$ and $(a, b) \in A_s \times A_t$ and for all $x \in A_{st}$, $a \circ b \lesssim x$

is a preorder relation on H and the (L, Γ)-associated hypersemigroup, where $\Gamma = \{(H, \circ)\}$, is $(H, *)$.

Examples

(1) Suppose that $H = \{e, a, b\}$ and the preorder relation $\lesssim$ on H is discrete relation. Let $\Gamma = \{(H, \circ_1), (H, \circ_2)\}$ be a $\lesssim$-preordered hypersemigroup where $\circ_1$ and $\circ_2$ are hyperoperations on H; see Table 1.

$\circ_1$	e	a	b
e	e	e	e
a	e	e	$\{e,a\}$
b	e	a	b

$\circ_2$	e	a	b
e	e	e	e
a	e	e	a
b	e	$\{e,a\}$	b

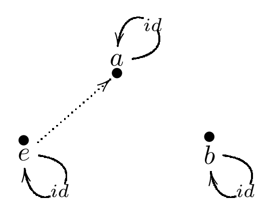
Table 1: The hyperoperations of H

The pair $((H, \circ_1), (H, \circ_2))$ is w.m.a (in fact mutual associative)and non of the hypersemigroups $(H, \circ_1)$ and $(H, \circ_2)$ is commutative. The operation of the (L, Γ)-associative hypersemigroup $(H, *)$ is given by the following table:

$*$	e	a	b
e	e	e	e
a	e	e	$\{e,a\}$
b	e	$\{e,a\}$	b

Table 2: The hyperoperation of H

(2) Suppose that $H = \{e, a, b\}$ and $(H, \lesssim)$ is a partially preorder set as follows:



Let $\Gamma' = \{(H, \circ)\}$ be the set of the following $\lesssim$-preordered semigroup:

$\circ$	e	a	b
e	e	e	e
a	e	e	a
b	e	a	b

Table 3: The operation of H

The (L, Γ')-associative hypersemigroup $(H, *')$ is the same with $(H, *)$ at Example 1.

The following tables show that we cannot produce any associated hypersemigroup with a singleton $\lesssim$-preordered semigroup.

(3) Suppose that $H = \{e, a, b, c\}$ and that the preorder relation $\lesssim$ on H is the discrete relation. Let $\Gamma = \{(H, \circ_1), (H, \circ_2)\}$ be the set of the following $\lesssim$-preordered semigroups:

$\circ_1$	e	a	b	c
e	e	a	b	c
a	a	b	c	e
b	b	c	e	a
c	c	e	a	b

$\circ_2$	e	a	b	c
e	e	a	b	c
a	a	e	c	b
b	b	c	e	a
c	c	b	a	e

Table 4: The operations of H

We can see that the pair $((H,\circ_1),(H,\circ_2))$ is w.m.a which is not mutual associative and $(H,*)$ is (L,Γ)-associative; see Table 5.

$*$	e	a	b	c
e	e	a	b	c
a	a	$\{e,b\}$	c	$\{e,b\}$
b	b	c	e	a
c	c	$\{e,b\}$	a	$\{e,b\}$

Table 5: The hyperoperation of H

By Corollary 2.10, we conclude that we cannot produce $(H,*)$ with a singleton $\lesssim$-preordered semigroup.

3 On morphisms of some associated hypersemigroups

In this section we give a connection between morphisms of some associated hypersemigroups and morphisms of preordered structures.

Proposition 3.1. Let $f : H_1 \longrightarrow H_2$ be a mapping of a preordered set $(H_1,\lesssim_1)$ into another one $(H_2,\lesssim_2)$. The following conditions are equivalent:
(i) f is monotone,
(ii) $f(L(x)) \subseteq L(f(x))$ for all $x \in H_1$,
(iii) $L(f^{-1}(f(x))) \subseteq f^{-1}(L(f(x)))$ for all $x \in H_1$.

Proof: (i) $\Rightarrow$ (ii) Let $x \in H_1$ be an arbitrary element and suppose that $y \in f(L(x))$. Then there exists $z \in L(x)$, i.e., $z \lesssim_1 x$ such that $y = f(z)$. Since f is a monotone, $f(z) \lesssim_2 f(x)$ which implies $y \in L(f(x))$. Thus $f(L(x)) \subseteq L(f(x))$.

(ii) $\Rightarrow$ (iii) Let $x \in H_1$ be an arbitrary element. Suppose that $y \in L(f^{-1}(f(x)))$. Then there exists $z \in f^{-1}(f(x))$, i.e., $f(z) = f(x)$ such that $y \lesssim_1 z$ which means $y \in L(z)$. Therefore $f(y) \in f(L(z)) \subseteq L(f(z)) = L(f(x))$ and hence $y \in f^{-1}(f(y)) \subseteq f^{-1}(L(f(x)))$. Consequently $L(f^{-1}(f(x))) \subseteq f^{-1}(L(f(x)))$.

(iii) $\Rightarrow$ (i) Suppose that $x, y \in H_1$ and $x \lesssim_1 y$ are given. Since $y \in f^{-1}(f(y))$, we have $x \in L(y) \subseteq L(f^{-1}(f(y))) \subseteq f^{-1}(L(f(y\sigma)))$. Thus $f(x) \in L(f(y))$ and hence $f(x) \lesssim_2 f(y)$.

Theorem 3.2. Let $(H_1,\circ_1,\lesssim_1)$ and $(H_2,\circ_2,\lesssim_2)$ be perordered semigroups and furthermore let $f : (H_1,\circ_1) \longrightarrow (H_2,\circ_2)$ be a homomorphism. Then f is a homomorphism of associated hypersemigroups $(H_1,*_1)$ and $(H_2,*_2)$.

Proof: For any $x, y \in H_1$ there exists an unique element $z \in H_1$ such that $x \circ_1 y = z$. With respect to Proposition 3.1 we have:

$$\begin{aligned} f(x *_1 y) &= f(L(x \circ_1 y)) \\ &= f(L(z)) \\ &\subseteq L(f(z)) \\ &= L(f(x \circ_1 y)) \\ &= L(f(x) \circ_2 f(y)) \\ &= f(x) *_2 f(y). \end{aligned}$$

Proposition 3.3. Let $f : H_1 \longrightarrow H_2$ be mapping of an preordered set $(H_1, \lesssim_1)$ into another one $(H_2, \lesssim_2)$. The following conditions are equivalent:
(i) f is a strongly monotone mapping,
(ii) $f(L(x)) = L(f(x))$ for all x in H.

Proof: (i) $\Rightarrow$ (ii) For an strongly monotone mapping f it is enough to prove that set inclusion $L(f(x)) \subseteq f(L(x))$ because of the Proposition 3.1. Suppose $y \in L(f(x))$ is an arbitrary element, then $y \lesssim_2 f(x)$. Since the mapping f is strongly monotone, there exists such $x' \in H_1$ that $x' \lesssim_1 x$ and $f(x') = y$. Therefore $x' \in L(x)$ and hence $y = f(x') \in f(L(x))$ which means $L(f(x)) \subseteq f(L(x))$.

(ii) $\Rightarrow$ (i) Let $x_1 \in H_1$ and $x_2 \in H_2$ be such elements that $x_2 \lesssim_2 f(x_1)$. Since $x_2 \in L(f(x_1)) = f(L(x_1))$, there exists $x_1' \in L(x_1)$, i.e., $x_1' \lesssim_1 x_1$ such that $f(x_1') = x_2$. On the other hand if there exists $x_1' \in H_1$ such that $x_1' \lesssim_1 x_1$ and $f(x_1') = x_2$, then $x_1' \in L(x_1)$ and $f(x_1') \in f(L(x_1)) = L(f(x_1))$ which implies $x_2 \in L(f(x_1))$ and consequently $x_2 \lesssim_2 f(x_1)$.

Theorem 3.4. Let $(H_1, \circ_1, \lesssim_1)$ and $(H_2, \circ_2, \lesssim_2)$ be preordered semigroups and further let $f : (H_1, \circ_1) \longrightarrow (H_2, \circ_2)$ be a homomorphism which is strongly monotone. Then f is a good homomorphism of associated hypersemigroups $(H_1, *_1)$ and $(H_2, *_2)$.

Proof: As in the proof of the Proposition 3.3, let x, y and z in H_1 with $xy = z$ be given. With respect to Proposition 3.1, we have:

$$\begin{aligned} f(x *_1 y) &= f(L(x \circ_1 y)) \\ &= f(L(z)) \\ &= L(f(z)) \\ &= L(f(x \circ_1 y)) \\ &= L(f(x) \circ_2 f(y)) \\ &= f(x) *_2 f(y). \end{aligned}$$

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