PATH TRANSFERABILITY OF PLANAR GRAPHS

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Received July 15, 2009

ABSTRACT. We regard a path as a train moving on a graph. A graph G is called n-transferable if any path of length n can be moved to any other such path by several steps. We will show that every planar graph with minimum degree at least three is at most 10 transferable.

1 Introduction. The graphs discussed here are finite, simple, undirected, and connected. A *path* consists of distinct vertices $v_0, v_1, \ldots, v_n$ and edges $v_0v_1, v_1v_2, \ldots, v_{n-1}v_n$. When the direction of a path P needs to be emphasized, we use the notation $\langle\ \rangle$, such as $P = \langle v_0v_1 \cdots v_n \rangle$. The reverse path of P is denoted by P^{-1}. The number of edges in a path P is called its *length*, and a path of length n is called an *n-path.* The last(resp. first) vertex of a path P in its direction is called the *head* (resp. *tail*) of P and is denoted by $h(P)$(resp. $t(P)$); for $P = \langle v_0v_1 \cdots v_{n-1}v_n \rangle$, we set $h(P) = v_n$ and $t(P) = v_0$. The set of all inner vertices of P, the vertices that are neither the head nor the tail, is denoted by $Inn(P)$.

This paper focuses on the movement of a path along a graph: Let P be an n-path. If $h(P)$ has a neighboring vertex $v \notin Inn(P)$, then we have a new n-path P' by removing the vertex $t(P)$ from P and adding v to P as its new head. We say that P takes a *step* to v, and denote it by $P \xrightarrow{v} P'$ (or briefly $P \to P'$). If there is a sequence $P \to \cdots \to Q$, then we say that P *can transfer* (*or move*) *to* Q, and denote it by $P \dashrightarrow Q$.

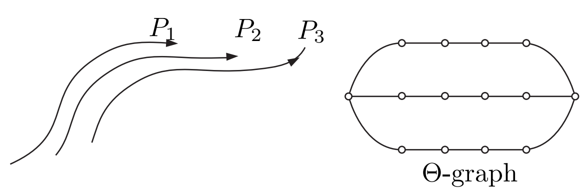

Figure 1: [L] The movement of a path. [R] Θ-graph.

Let $\P_n(G)$ be the set of the all directed n-paths in a graph G. A graph G is called *n-transferable* if $\P_n(G) \neq \emptyset$ and if $P \dashrightarrow Q$ for any pair of directed n-paths $P, Q \in \P_n(G)$.

Theorem [To1]. *Let G be a connected graph. If G is n-transferable, then G is $(n-1)$-transferable.*

2000 *Mathematics Subject Classification.* 05C10 .
Key words and phrases. Path transferability; Graph decomposition .

The maximum number n for which G is n-transferable is called the *transferability* of G, and is denoted by $\tau(G)$. The author further showed the following:

Theorem [To2]. *We assume that a connected graph G has the minimum degree $\delta \geq 2$. If G is neither a complete graph nor a cycle graph, then G is δ-transferable.*

A Θ_n-*graph* consists of three internally-disjoint n-paths with common heads and tails. This graph has the transferability $2n-1$, and hence there are planar graphs which have arbitrary large transferability. In this paper we will show that any planar graph with minimum degree at least three has the transferability at most 10.

2 Local structures of planar graphs. Given a connected plane graph, not necessarily simple, the *degree of a face* f is the length of any facial walk of f. Vertices and faces of degree i are called i-*vertices* and i-*faces*, respectively. A *plane map* is defined to be a connected plane graph with no bridges, and a plane map is called *normal* if degrees of all vertices and faces are not less than three. We notice that loops and multiple edges can appear in a normal plane map.

By Euler polyhedral formula, a simple planar graph has a vertex of degree ≤ 5 or, dually, any plane graph without 1-, 2-vertices has a face of degree ≤ 5. Local structures of planar graphs are studied by Jendrol' and Skupień, and the following lemma is a weak result derived from their Theorem 2 in [JS] (This result was originally proved by H. Lebesgue [L] in a bit weaker version, and was later strengthened by O. V. Borodin [B]).

Lemma 1 ([JS]). *Every normal plane map contains one of the following configurations:*

1. *a 3-face such that if its three vertices have degrees $a \leq b \leq c$ then*
   - *(a) $a = 3 \leq b \leq 10$ or*
   - *(b) $a = 4 \leq b \leq 7$ or*
   - *(c) $a = 5 \leq b \leq 6$;*
2. *a 4-face such that if its four vertices have degrees $a \leq b \leq c \leq d$ then*
   - *$a = 3 \leq b \leq c \leq 5$;*
3. *a 5-face with four 3-vertices.*

The dual graph of any simple plane graph with minimum degree at least three is a normal plane map. Thus we can obtain the following.

Lemma 2. *Every simple plane graph with minimum degree at least three contains one of the following configurations:*

1. *a 3-vertex such that if its three surrounded faces have degrees $a \leq b \leq c$ then*
   - *(a) $a = 3 \leq b \leq 10$ or*
   - *(b) $a = 4 \leq b \leq 7$ or*
   - *(c) $a = 5 \leq b \leq 6$;*
2. *a 4-vertex such that if its four surrounded faces have degrees $a \leq b \leq c \leq d$ then*

- *$a = 3 \leq b \leq c \leq 5$;*

3. a 5-vertex with its four surrounded 3-faces.

Such i-vertices, $i = 3, 4, 5$, are called *light* vertices.

A graph G is 3-connected if and only if $G - v$ is 2-connected for each vertex $v \in V(G)$. It is well known that a plane graph is 2-connected if and only if all its facial walks are cycles. Each vertex v of a plane graph G is contained in exactly one face of $G - v$. Such a face is called the *star neighbor* of v, and its facial boundary walk is called the *link* of v. Using this notation, we can say that a plane graph G is 3-connected if and only if the link of each vertex of G is a cycle.

Lemma 3. *The transferability of a simple 3-connected planar graph is at most 10.*

Proof. Let G be such a graph. By Lemma 2, there is a light vertex in G, say v. We assume that it is a 3-vertex whose faces are of degrees $a \leq b \leq c$, $a = 3, b = 10$. Let $C = u_1u_2u_3 \cdots u_9u_{10} \cdots u_lu_1$ be the cycle that is the link of v (see Figure 2). In this case, we notice that the 11-path $P = \langle u_{11}u_{10}u_9 \cdots u_3u_2u_1v \rangle$ cannot take a step in G.

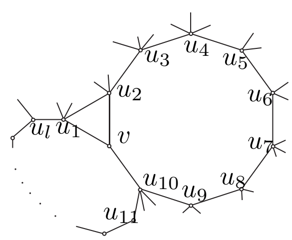

Figure 2:

For the other cases, we can similarly find l-paths, $l \leq 11$, which cannot take a step in G, hence the transferability of G is at most 10. $\square$

This result is best possible, the truncated dodecahedron (see Figure 3) gives an example for this (any two 10-paths can actually transfer from one to another in this graph).

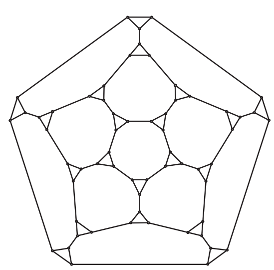

Figure 3: Truncated dodecahedron.

3 Decomposition of a planar graph. In this section we will extend Lemma 3 to 1-, 2-connected graphs. If a graph G is not 3-connected, then the link of each vertex is not always a cycle. However, we will find a light vertex in G whose link is a cycle.

The uniqueness of decompositions of 2-connected graphs has been studied by MacLane [M], Tutte [Tu], Hopcroft and Tarjan [HT]. Cunningham and Edmonds [CE] have proved that a 2-connected graph has a unique minimal decomposition into graphs, each of which is either a 3-connected graph, a bond (i.e., two vertices and multiple edges between them) or a cycle. For the decomposition of 2-connected graphs, Tutte [Tu] use the notation $Blk_3(G)$ as the *tree of* 3-*blocks* of G. The definitions and notations follow from Tutte [Tu].

Lemma 4. *The transferability of a simple* 2*-connected planar graph with minimum degree at least three is at most* 10*.*

Proof. Let G be such a graph. Since the assertion holds for 3-connected graphs, we assume that G is 2-connected but not 3-connected. Let $Blk_3(G)$ be the tree of 3-blocks of G, and J an *extremal* 3-block of G, i.e., the induced subgraph corresponding to a leaf of $Blk_3(G)$. We assume that this block has the *virtual edge* $e = ab$. As far as restricting for simple graphs with minimum degree at least three, $J \cup \{e\}$ will be a simple 3-connected graph. There is a projection of $J \cup \{e\}$ such that the edge e lies on the infinite face of the plane (we use the same notation J for such a projection of J). A projection of G can be obtained as an extension of the projection of J.

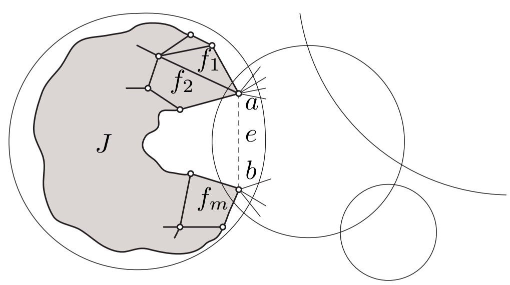

Figure 4: An extremal 3-block in G.

Let $V(J) = \{a, b, v_1, v_2, \ldots, v_n\}$ be the set of the vertices of J, and $F(J) = \{f_1, f_2, \ldots, f_m\} \cup \{f_\infty\}$ the set of the faces of J, each of f_i, $1 \le i \le m$, is a bounded face and f_∞ is the infinite face of J. We will show that there is a light vertex in $V(J) - \{a, b\}$.

We prepare twelve copies of J to make a new graph H: Let $J^{(1)}, J^{(2)}, \ldots, J^{(12)}$ be the copies of J. The graph H is constructed from the cube graph by inserting $J^{(i)}$, $1 \le i \le 12$, between its twelve edges (see Figure 5). We notice that this plane graph H has $12|V(J)| + 8 = 12(n + 2) + 8$ vertices and $12m + 6$ faces, and that all vertices of H have degrees ≥ 3. This graph H has a light vertex by Lemma 2. The eight 3-vertices that do not belong to any copies of J are not light because the new six faces of H have degrees ≥ 12. By the same reason, the copies of the two vertices a or b are also not light. We therefore conclude that $V(J) - \{a, b\}$ contains a light vertex, say v_k.

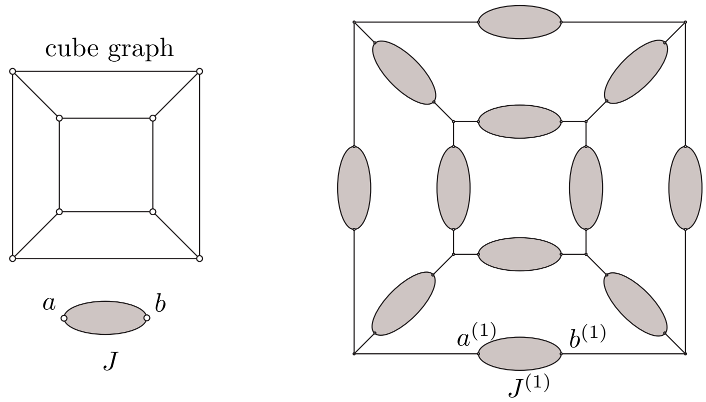

Figure 5: The graph H constructed from twelve copies of J.

Since $V(G) - v_i$ is 2-connected for any $v_i \in V(J) - \{a, b\}$, the link of v_k is a cycle (this cycle may go through another 3-block of G). And then we can find a l-path, $l \leq 11$, which cannot move in G as in Lemma 3, hence the transferability of G is at most 10. □

We can deduce the same proposition for a planar graph which is connected but not 2-connected: Let G be a connected planar graph with minimum degree at least three and J an extremal block of G with cut vertex v. Let H be the graph that constructed from six copies of J with an additional 6-vertex which is adjacent to the six copies of v. Since H is a simple plane graph with minimum degree at least three, there is a light vertex in $V(J) - v$ whose link is a cycle, as in Lemma 4. Therefore we can establish the following theorem.

Theorem 5. *The transferability of a connected planar graph with minimum degree at least three is at most* 10. □

As long as we consider triangle-free graphs, light vertices are only 3-vertices, that correspond to $1(b)$ or $1(c)$ of the configurations in Lemma 2. And therefore we can deduce the following in the same way.

Corollary 6. *The transferability of a simple triangle-free planar graph with minimum degree at least three is at most* 8. □

This is best possible; the truncated icosahedron, the frame of a "soccer ball", has the transferability 8.

Acknowledgements.
I would like to express my sincere thanks to Professor Sin'ichi Suzuki and Professor Kouki Taniyama for their encouragement.

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