# $CI$-ALGEBRAS

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ABSTRACT. In this paper we introduce the notion of $CI$ -algebras as a generalization of $BE$-algebras and dual $BCK/BCI/BCH$-algebras, we investigate its elementary properties. Relations of $CI$-algebras and $BE$-algebras are discussed. Finally we prove that in transitive $BE$-algebras, the notion of ideals is equivalent to one of filters.

## 1. Introduction

The study of $BCK/BCI$-algebras was initiated by K.Iséki in 1966 as a generalization of propositional logic (see[4, 5, 6]). There exist several generalizations of $BCK/BCI$-algebras, as such $BCH$-algebras[3], dual $BCK$-algebras[10], $d$-algebras[9], etc. Especially, H.S.Kim and Y.H.Kim[7] introduced the notion of $BE$-algebras which was deeply studied by S.S.Ahn and Y.H.Kim in [1], S.S.Ahn and K.S.So in [2],H.S.Kim and K.J.Lee in [8], A. Walendziak in [11]. In this paper we will introduce the notion of $CI$-algebras as a generalization of $BE$-algebras and $BCK/BCI/BCH$-algebras, and study its important properties and relations with $BE$-algebras. We prove that in transitive $BE$-algebras, the notion of ideals is equivalent to one of filters. In the sequel, let $\mathbb{N}$ denote the set of all positive integers.

## 2. Preliminaries

**Definition 2.1**[7]. An algebra $(X; *, 1)$ of type (2,0) is said to be a $BE$-algebra if it satisfies the following:
(BE1) $x * x = 1$,
(BE2) $x * 1 = 1$,
(BE3) $1 * x = x$,
(BE4) $x * (y * z) = y * (x * z)$ .

**Definition 2.2**[10]. A dual $BCK$-algebra is an algebra $(X; *, 1)$ of type (2,0) satisfying (BE1), (BE2), and the following axioms:
(dBCK1) $x * y = y * x = 1$ implies $x = y$,
(dBCK2) $(x * y) * ((y * z) * (x * z)) = 1$,
(dBCK3) $x * ((x * y) * y) = 1$.

**Lemma 2.3**[10]. Let $(X; *, 1)$ be a dual $BCK$-algebra. Then (BE3) and (BE4) hold in $X$.

Similarly we can define dual $BCI$-algebras as follows.

**Definition 2.4.** A dual $BCI$-algebra is an algebra $(X; *, 1)$ of type (2,0) satisfying the axioms (BE1), (dBCK1), (dBCK2) and (dBCK3).

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**Proposition 2.5.** Let $(X; *, 1)$ be a dual $BCI$-algebra and $x \in X$. If $1 * x = 1$ then $x = 1$.

*Proof.* Let $x \in X$ be such that $1 * x = 1$. By (BE1) and (dBCK3) we have

$$x * 1 = x * (1 * x) = x * [(x * x) * x] = 1.$$

It follows from (dBCK1) that $x = 1$.

By using very similar arguments as in $BCI$-algebras one can prove the following.

**Proposition 2.6.** Let $(X; *, 1)$ be a dual $BCI$-algebra and $x, y, z \in X$. Then the followings hold:
(1) if $x * y = 1$, then $(y * z) * (x * z) = 1$,
(2) if $x * y = 1$ and $y * z = 1$, then $x * z = 1$,
(3) $y * (z * x) = z * (y * x)$, i.e., (BE4) holds in dual $BCI$-algebra $X$,
(4) $1 * x = x$, i.e., (BE3) holds in dual $BCI$-algebra $X$.

**Lemma 2.7**[11]. Any dual $BCK$-algebra is a $BE$-algebra.

The definition of dual $BCH$-algebras can similarly be complete.

**Definition 2.8.** An algebra of type (2,0) satisfying (BE1), (dBCK1) and (BE4) is said to be a dual $BCH$-algebra.

We easily prove the following.

**Proposition 2.9.** (BE3) holds in any dual $BCH$-algebra.

## 3. The notion and elementary properties of $CI$-algebras

In this section we first introduce the notion of $CI$-algebras and next study some of elementary properties.

**Definition 3.1.** A $CI$-algebra is an algebra $(X; *, 1)$ of type (2,0) satisfying the following axioms:
(CI1) $x * x = 1$,
(CI2) $1 * x = x$,
(CI3) $x * (y * z) = y * (x * z)$.

Obviously, every dual $BCK/BCI/BCH$-algebra is a $CI$-algebra. The axioms (CI1), (CI2) and (CI3) are (BE1), (BE3) and (BE4), respectively. For any $CI$-algebra $X$, denote $B(X) = \{x \in X \mid x * 1 = 1\}$. Hence a $CI$-algebra is a $BE$-algebra if and only if $X = B(X)$.

**Proposition 3.2.** Any $CI$-algebra $X$ satisfies for any $x, y \in X$,

$$y * ((y * x) * x) = 1.$$

*Proof.* By (CI3) and (CI1) we have

$$y * ((y * x) * x) = (y * x) * (y * x) = 1,$$

completing the proof.

**Proposition 3.3.** Any $CI$-algebra of degree 2 is either a dual $BCI$-algebra or a dual $BCK$-algebra.

*Proof.* Let $X := \{1, a\}$ be a $CI$-algebra where $a \neq 1$. Then its Cayley table is the following form:

| * | 1 | a |
|---|---|---|
| 1 | 1 | a |
| a | $x$ | 1 |

where $x$ is either $a$ or 1. When $x = a$, $X$ is a dual $BCI$-algebra; When $x = 1$, $X$ is a dual $BCK$-algebra. This completes the proof.

**Example 3.4.** Let $X := \{1, a, b\}$ be a set with the following Cayley table:

| * | 1 | a | b |
|---|---|---|---|
| 1 | 1 | a | b |
| a | 1 | 1 | 1 |
| b | 1 | 1 | 1 |

We can check that $(X; *, 1)$ is a $CI$-algebra. It is worth to note that in this algebra, $a * b = b * a = 1$, but $a \neq b$. Thus this is not a dual $BCK/BCI/BCH$ -algebra, hence the class of dual $BCK/BCI/BCH$ -algebras is a proper subclass of the class of $CI$-algebras

**Example 3.5.** Let $X := \{1, 2, 3, 4, 5, 6\}$ be a set with the following Cayley table:

| * | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|
| 1 | 1 | 2 | 3 | 4 | 5 | 6 |
| 2 | 1 | 1 | 1 | 4 | 4 | 4 |
| 3 | 1 | 1 | 1 | 4 | 4 | 4 |
| 4 | 4 | 5 | 1 | 1 | 2 | 3 |
| 5 | 4 | 4 | 4 | 1 | 1 | 1 |
| 6 | 4 | 4 | 4 | 1 | 1 | 1 |

Then $(X; *, 1)$ is a $CI$-algebra. But it is not a $BE$-algebra because $4*1 = 5*1 = 6*1 = 4 \neq 1$. Therefore the class of $BE$-algebras is a proper subclass of the class of $CI$-algebras.

**Example 3.6.** Let $X := \{1, a, b, c, d\}$ be a set with the following Cayley table:

| * | 1 | a | b | c | d |
|---|---|---|---|---|---|
| 1 | 1 | a | b | c | d |
| a | 1 | 1 | b | b | d |
| b | 1 | a | 1 | a | d |
| c | 1 | 1 | 1 | 1 | d |
| d | d | d | d | d | 1 |

We can check that $(X; *, 1)$ is a $CI$-algebra.

**Proposition 3.7.** Any $CI$-algebra $X$ satisfies for any $x, y \in X$,

$$(x * 1) * (y * 1) = (x * y) * 1.$$

*Proof.* By (CI3) and (CI1) we have

$$\begin{array}{rcl}(x*1)*(y*1) & = & (x*1)*\{y*[(x*y)*(x*y)]\} \\ & = & (x*1)*\{(x*y)*[x*(y*y)]\} \\ & = & (x*1)*[(x*y)*(x*1)] \\ & = & (x*y)*[(x*1)*(x*1)] \\ & = & (x*y)*1,\end{array}$$

ending the proof.

By induction we easily obtain

**Corollary 3.8.** Let $X$ be a $CI$-algebra and $n \in \mathbb{N}$. Then for any $y, x_1, \cdots, x_n \in X$ we have

$$(x_n * 1) * (\cdots * ((x_1 * 1) * (y * 1)) \cdots) = (x_n * (\cdots * (x_1 * y) \cdots)) * 1.$$

## 4. Relations with $BE$-algebras

It is easy to see from definitions of $BE$-algebras and $CI$-algebras the following

**Proposition 4.1.** *$BE$-algebras must be $CI$-algebras; A $CI$-algebra $X$ is a $BE$ -algebra if and only if $X = B(X)$.*

**Proposition 4.2.** *Let $(X; *', 1)$ be a $BE$-algebra and $a \notin X$. We define for $x, y$, the $x * y$ on $X \cup a$ as follows*

$$x * y = \begin{cases} x *' y & \text{if } x, y \in X, \\ a & \text{if } x = a \text{ and } y \neq a, \\ a & \text{if } x \neq a \text{ and } y = a, \\ 1 & \text{if } x = y = a \end{cases}$$

*then $(X \cup \{a\};, *, 1)$ is a $CI$-algebra.*

*Proof.* It is sufficient to verify (CI3). Let $x, y \in X$. Because
$a * (x * y) = a = x * a = x * (a * x)$,
$x * (y * a) = x * a = a = y * a = y * (x * a)$,
$a * (x * a) = a * a = 1 = x * 1 = x * (a * a)$,
hence (CI3) holds for $X$. Verification of others are trivial. So $(X \cup \{a\}; *, 1)$ is a $CI$-algebra. This completes the proof.

H.S.Kim and Y.H.Kim[7] introduced a self distributivity of $BE$-algebras. A $CI$-algebra $X$ is said to be *self distributive* if for any $x, y, z \in X$,

$$x * (y * z) = (x * y) * (x * z).$$

**Proposition 4.3.** *Any self distributive $CI$-algebra $X$ is $BE$-algebras.*

*Proof.* For any $x, y \in X$,

$$x * 1 = x * (y * y) = (x * y) * (x * y) = 1,$$

hence $X$ is a $BE$-algebra, ending the proof.

**Proposition 4.4.** *A $CI$-algebra $X$ satisfying the condition*

(P) for any $x, y \in X$, $x * (x * y) = x * y$
is a $BE$-algebra.

*Proof.* Let $x = y$ in (P). Then by (CI1) we have $x * 1 = x * (x * x) = x * x = 1$. Hence $x * 1 = 1$ for any $x \in X$, and so $X$ is a $BE$ -algebra.

**Proposition 4.5.** A $CI$-algebra $X$ satisfying the condition
(I) for any $x, y \in X$, $(x * y) * x = x$
is a $BE$-algebra.

*Proof.* Let $x = 1$ in (I). Then by (CI2) we have $y * 1 = (1 * y) * 1 = y$. Hence $y * 1 = 1$ for any $y \in X$, hence $X$ is a $BE$-algebra.

**Definition 4.6.** A $CI$-algebra $X$ is said to be commutative if it satisfies
(C) for any $x, y \in X$, $(x * y) * y = (y * x) * x$.

**Proposition 4.7.** Any commutative $CI$-algebra $X$ is a dual $BCK$-algebra.

*Proof.* Let a $CI$-algebra $X$ be commutative. Now we prove that $X = B(X)$. If $X \neq B(X)$, take any $a \in X - B(X)$. By (C)

$$(a * 1) * 1 = (1 * a) * a = a * a = 1,$$

and so $a * 1 = a * [(a * 1) * 1] = 1$, this shows $a \in B(X)$, a contradiction. Therefore $X = B(X)$, i.e., $X$ is a $BE$-algebra.

For any $x, y \in X$, if $x * y = y * x = 1$, then by (CI2) and (CI3) we have

$$x = 1 * x = (y * x) * x = (x * y) * y = 1 * y = y.$$

Hence (dBCK1) holds.

Applying (CI3), (C) we have

$$\begin{aligned}
(x * y) * [(y * z) * (x * z)] &= (x * y) * \{x * [(y * z) * z]\} \\
&= (x * y) * \{x * [(z * y) * y]\} \\
&= (x * y) * [(z * y) * (x * y)] \\
&= (z * y) * [(x * y) * (x * y)] \\
&= (x * y) * 1 = 1,
\end{aligned}$$

and so (dBCK2) holds.

Proposition 3.2 shows that (dBCK3) holds in $X$. Therefore $X$ is a dual $BCK$-algebra.

By the above proof we have

**Corollary 4.8.** Any commutative $CI$-algebra is a $BE$-algebra.

## 5. Ideals and filters in $CI$-algebras

S.S.Ahn and Y.H.So[1] and H.S.Kim and Y.H.Kim[7] introduced concepts of ideals and filters in $BE$-algebras, respectively. In this section we discuss these concepts in $CI$-algebras.

**Definition 5.1.** Let $X$ be a $CI$-algebra and $I$ a nonempty subset of $X$. $I$ is said to be an ideal of $X$ if it satisfies: $\forall x,\ a,\ b \in X$
(I1) $a \in I$ implies $x * a \in I$, i.e., $X * I \subseteq I$;
(I2) $a \in I$ and $b \in I$ imply $(a * (b * x)) * x \in I$.

**Definition 5.2.** Let $X$ be a $CI$-algebra and $I$ a nonempty subset of $X$. $I$ is said to be a subalgebra of $X$ if $x \in I$ and $y \in I$ imply $x * y \in I$.

For any $CI$-algebra $X$, $\{1\}$ and $X$ are trivial ideals(resp. subalgebras) of $X$. Obviously every ideal in a $CI$-algebra is a subalgebra.

**Definition 5.3.** A non-empty subset $F$ of a $BE$-algebra $X$ is said to be a filter of $X$ if it satisfies:
(F1) $1 \in F$,
(F2) $x * y \in F$ and $x \in F$ imply $y \in F$.

**Definition 5.4.** A $CI$-algebra $X$ is said to be *transitive* if for all $x, y, z \in X$,

$$(y * z) * [(x * y) * (x * z)] = 1.$$

In what follows we will prove that in a transitive $BE$-algebra, the notion of ideals coincides with one of filters.

**Proposition 5.5.** Let $X$ be a transitive $BE$-algebra and $A$ a nonempty subset of $X$. Then $A$ is an ideal of $X$ if and only if $A$ is a filter of $X$.

*Proof.* Suppose $A$ is an ideal of $X$. Take any $a \in I$, then $1 = a * a \in A$. Hence (F1) holds. Let $x, y \in X$ be such that $x * y \in A$ and $x \in A$. Since $(x * y) * y = [1 * (x * y)] * y$, it follows from (I2) that $(x * y) * y \in A$. Denote $\alpha = (x * y) * y$ and $\beta = x * y$, by (I2) we have

$$y = 1 * y = \{[(x * y) * y] * [(x * y) * y]\} * y = [\alpha * (\beta * y)] * y \in A.$$

(F2) is true. Therefore $A$ is a filter of $X$.

Conversely let $A$ be a filter of $X$. Assume that $x \in X$ and $a \in A$. Since $a * (x * a) = x * (a * a) = x * 1 = 1 \in A$ by (F1), it follows from (F2) that $x * a \in A$. Hence (I1) is true. Let $a, b \in A$ and $x \in X$. Because $a * [(a * x) * x] = (a * x) * (a * x) = 1 \in A$ by (F1), and so $(a * x) * x \in A$ by (F2). Using transitivity of $X$ we have

$$[(a * x) * x] * \{[b * (a * x)] * (b * x)\} = 1 \in A,$$

and so $[b * (a * x)] * (b * x) \in A$ by (F2). Hence $b * \{[b * (a * x)] * x\} \in A$. By $b \in A$ and (F2) we obtain $[b * (a * x)] * x \in A$, i.e. (I2) holds. Therefore $A$ is an ideal of $X$.

**Proposition 5.6.** Let $X$ be a transitive $BE$-algebra and $A$ a nonempty subset of $X$. Then $A$ is a filter of $X$ if and only if $A$ satisfies: for any $a, b \in A$ and $x \in X$, $a * (b * x) = 1$ implies $x \in A$.

*Proof.* ($\Leftarrow$) Assume $a \in A$. Since $a * (a * 1) = 1$, it follows that $1 \in A$. (F1) holds for $A$. Suppose $a * x \in A$ and $a \in A$. Because $a * [(a * x) * x] = 1$, and so $x \in A$. (F2) is true. Therefore $A$ is a filter of $X$.

($\Rightarrow$) Let $A$ be a filter of $X$. Assume $a, b \in A$ and $x \in X$ such that $a * (b * x) = 1$. By (F1) we have $a * (b * x) \in A$. Then applying (F2) twice we obtain $x \in A$. This completes the Proof.

By induction we easily obtain

**Corolary 5.7.** Let $X$ be a transitive $BE$-algebra and $A$ a nonempty subset of $X$. Then $A$ is a filter of $X$ if and only if $A$ satisfies: for any $a_i \in A$ $(i \in \mathbb{N})$ and $x \in X$, $a_n * (\cdots * (a_1 * x) \cdots) = 1$ implies $x \in A$.

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