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ESTIMATION OF THE ORDERS ON HYPERGROUP EXTENSIONS

Ryo Ichihara and Satoshi Kawakami

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Abstract. We establish that the ranges of the orders of hypergroup extensions are described in the form of orders with respect to their subhypergroup and quotient hypergroup. We obtain two families of hypergroup extensions which include the extensions of all orders in the range of our estimation. Remarkable fact is that there exist hypergroup extensions which have an order higher than one of the direct product of a subhypergroup and a quotient hypergroup.

1 Introduction Let $\mathcal{K}$ be a finite commutative hypergroup. If $\mathcal{K}$ has a subhypergroup $\mathcal{H}$, then the quotient $\mathcal{K}/\mathcal{H} = \mathcal{L}$ is a hypergroup. In this case, $\mathcal{K}$ is called a *hypergroup extension* of $\mathcal{L}$ by $\mathcal{H}$. Fix commutative hypergroups $\mathcal{L}$ and $\mathcal{H}$. The problem of extensions in the category of commutative hypergroups is to determine all commutative hypergroup extensions $\mathcal{K}$ of $\mathcal{L}$ by $\mathcal{H}$ and to analyze them.

We have discussed the extension problem of hypergroups in [HKKK], [HK1] and [K] in general situation for splitting extensions. In the papers [HK2], [KST], we have succeeded to determine all extensions in the case that $\mathcal{H}$ is a group. Moreover we have analyzed the almost all classes of hypergroup extensions in the case that $\mathcal{L}$ is a group [IKS].

Wildberger[W2] determined all strong hypergroups of order three using the harmonic analysis of hypergroups. In the paper [IK], we have considered the extension problem in the case of order four that both of $\mathcal{H}$ and $\mathcal{L}$ are hypergroups of order two. This is the model case that $\mathcal{H}$ and $\mathcal{L}$ are not necessarily groups. We have determined all hypergroups of order four and characterized strong hypergroups among them. The present paper is devoted to giving explicit answers to the question whether there exists a hypergroup extension of order five or much more. Theorem 9 and Theorem 11 show the existence of a hypergroup of order five in some cases.

The maximal order of hypergroup extension is calculated in Proposition 4. But the hypergroup extension $\mathcal{K}$ of the maximal order does not always exist for arbitrary hypergroups $\mathcal{H}$ and $\mathcal{L}$ [IK][IKS]. Furthermore Corollary 7 shows that there exist many extension problems which the maximal order of Proposition 4 does not appear in the case of group $\mathcal{L}$.

Applying the result in Preliminaries, we finally give two series of signed hypergroup models which are described by character tables and have arbitrary orders, as in Proposition 8 and Proposition 10. These models give hypergroup examples which attain all orders in the range which is estimated in Proposition 4. Furthermore we can know that there exist many extension problems such that hypergroup extensions have orders in the range of Proposition 4.

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2 Preliminaries We recall some notions and facts on finite commutative hypergroups from Wildberger's paper [W1] and Bloom-Heyer's book [BH]. A pair $(\mathcal{K}, A(\mathcal{K}))$ is called a finite commutative *signed hypergroup* of order $n+1$ if the following conditions (a1)-(a6) are satisfied.

(a1) *Unit*: $A(\mathcal{K})$ is a $*$-algebra over $\mathbb{C}$ with unit c_0,

(a2) *Basis*: $\mathcal{K} = \{c_0, c_1, \dots, c_n\}$ is a $\mathbb{C}$-linear basis of $A(\mathcal{K})$,

(a3) *Involution*: $\mathcal{K}^* = \mathcal{K}$,

(a4) *Structure constants*: $c_i c_j = \sum_{k=0}^n n_{ij}^k c_k$, where $n_{ij}^k \in \mathbb{R}$ such that

(i) $c_i^* {=} c_j \iff n_{ij}^0 > 0$, (ii) $c_i^* \neq c_j \iff n_{ij}^0 = 0$,

(a5) *Stochastic*: $\sum_{k=0}^n n_{ij}^k = 1$ for any i, j,

(a6) *Commutativity*: $c_i c_j = c_j c_i$ for any i, j.

We simply write $\mathcal{K} = (\mathcal{K}, A)$ and the order $|\mathcal{K}| = n+1$. In the case that $n_{ij}^k \geq 0$ for any i, j, k, we call $\mathcal{K}$ a finite commutative *hypergroup*. If $c_i^* = c_i$ for all $i = 1, 2, \dots, n$, then $\mathcal{K}$ is called a *hermitian* hypergroup. We note that finite commutative groups are hypergroups. If a finite hypergroup is indexed by numbers $0, \dots, n$, then we choose the element indexed by zero for the unit.

Let $\mathrm{L}(q) = \{\ell_0, \ell_1 \ s.t.\ \ell_1^2 = q\ell_0 + (1-q)\ell_1, \ell_0 \text{ unit}\}$ be the smallest non trivial hypergroup of order 2 for $0 < q \leq 1$. Specially L(1) equals $\mathbb{Z}_2$.

The *weight* of an element $c_i \in \mathcal{K}$ is defined by $w(c_i) := (n_{ij}^0)^{-1}$ where $c_j = c_i^*$. We define the *weight* of a subset $S \subset \mathcal{K}$:

$$w(S) = \sum_{s \in S} w(s).$$

and the *total weight* $w(\mathcal{K}) := \sum_{i=0}^n w(c_i)$. Then $w(\mathrm{L}(q)) = 1 + 1/q$.

For a finite commutative signed hypergroup $\mathcal{K}$, a complex valued function χ on $\mathcal{K}$ is called a *character* if

$$(1)\ \chi(c_0) = 1, \quad (2)\ \chi(c_i^*) = \chi(c_i)^-, \quad (3)\ \chi(c_i)\chi(c_j) = \sum_{k=0}^n n_{ij}^k \chi(c_k).$$

It is well known that the conjugate function of a character also is a character. The set $\mathcal{K}^\wedge$ of all characters of $\mathcal{K}$ forms a finite commutative signed hypergroup of the same order $|\mathcal{K}|$ where the involution of $\chi \in \mathcal{K}^\wedge$ is the conjugate function $\chi^- \in \mathcal{K}^\wedge$. The duality $\mathcal{K}^{\wedge\wedge} \cong \mathcal{K}$ as signed hypergroups holds in the sense of isomorphisms between signed hypergroups[W1][Z]. A finite hypergroup $\mathcal{K}$ is called a *strong* hypergroup if the dual $\mathcal{K}^\wedge$ is a hypergroup. If $\mathcal{K}^\wedge = \{\chi_0, \dots, \chi_n\}$, then the matrix $\left(\chi_i(c_j)\right)_{ij} \in M_{n+1}(\mathbb{C})$ for $\chi_i \in \mathcal{K}^\wedge$, $c_j \in \mathcal{K}$ is said to be a *character table*.

Definition 1. Let $U \in M_{n+1}(\mathbb{C})$ be a square matrix of order $n+1$. If the conjugate of any column vector of U is some column vector and the conjugate of any line vector of U is some line vector, then U is called *real constants type*.

Lemma 1. *Let a matrix $\left(\chi_i(c_j)\right)_{ij}$ with positive values $\{w(c_i)\}_i$, $\{w(\chi_i)\}_i$ be given in a form :*

	c_0	c_1	$\cdots$	c_n	
χ_0	1	1	$\cdots$	1	$w(\chi_0)=1$
χ_1	1	$\chi_1(c_1)$	$\cdots$	$\chi_1(c_n)$	$w(\chi_1)$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
χ_n	1	$\chi_n(c_1)$	$\cdots$	$\chi_n(c_n)$	$w(\chi_n)$
	$w(c_0)=1$	$w(c_1)$	$\cdots$	$w(c_n)$	W

where $W=\sum_i w(c_i)=\sum_i w(\chi_i)$. *If the matrix* $\big(\chi_i(c_j)w(\chi_i)^{1/2}w(c_j)^{1/2}W^{-1/2}\big)_{ij}$ *is a real constants type unitary, then there exists a signed hypergroup whose characters and weights table coincide with the above form.*

Proof. Consider a column vector of the above table $c_i={}^T\big(\chi_0(c_i),\ldots,\chi_n(c_i)\big)$ in $*$-algebra $\mathbb{C}^{n+1}$ and $\mathcal{K}=\{c_0,\ldots,c_n\}$, where ${}^TX$ is a transposed matrix of X. Then products and involution are given as:

$$c_ic_j={}^T\big(\chi_0(c_i)\chi_0(c_j),\ldots,\chi_n(c_i)\chi_n(c_j)\big),\ c_i^*={}^T\big(\chi_0(c_i)^-,\ldots,\chi_n(c_i)^-\big).$$

It is obvious that (a1)(a6) in the axiom of signed hypergroups hold. Now (a2) is shown as matrix $\big(\chi_i(c_j)\big)_{ij}$ is regular and (a3) $c_i^*\in\mathcal{K}$ is from the assumption of "real constant type". By the assumption of "unitary" we have

$$\delta_{ij}=w(\mathcal{K})^{-1}\sum_k\chi_k(c_i)\chi_k(c_j)^-w(\chi_k)w(c_i)$$

and

$$\delta_{ij}=w(\mathcal{K})^{-1}\sum_k\chi_i(c_k)\chi_j(c_k)^-w(c_k)w(\chi_i).$$

Hence we have that $\langle c_i,c_j\rangle=w(c_i)^{-1}\delta_{ij}$ and $w(c_i^*)=w(c_i)$ where the standard inner product $\langle c_i,c_j\rangle=w(\mathcal{K})^{-1}\sum_k\chi_k(c_i)\chi_k(c_j)^-w(\chi_k)$ [W2]. Similarly $w(\chi_i^-)=w(\chi_i)$. Noting that $\{c_0,\ldots,c_n\}$ is an orthogonal system, the structure constants are given by $n_{ij}^k=w(c_k)\langle c_ic_j,c_k\rangle$ for all i,j,k, where $c_ic_j=\sum_k n_{ij}^kc_k$ is expanded. Since the sum of two summands χ,χ^-:

$$\chi(c_ic_j)\chi(c_k^*)w(\chi)+\chi(c_ic_j)^-\chi(c_k^*)^-w(\chi^-)=2w(\chi)\Re\big(\chi(c_ic_j)\chi(c_k^*)\big)$$

is real, it is obtained that all n_{ij}^k are real. Thus (a4) is shown. Applying the trivial character χ_0 to $c_ic_j=\sum_k n_{ij}^kc_k$, we have (a5).

Therefore it follows that $\mathcal{K}$ is a signed hypergroup and its character table is a given one. □

3 Main Results Let $\mathcal{K}$ be a finite commutative hypergroup and $\mathcal{H}\subset\mathcal{K}$ be a subhypergroup. It is well known that the quotient $\mathcal{K}/\mathcal{H}$ is also a hypergroup [SW]. In order to describe this situation, we often use the form of short exact sequence $1\to\mathcal{H}\to\mathcal{K}\xrightarrow{\varphi}\mathcal{L}\to 1$ where $\mathcal{L}=\mathcal{K}/\mathcal{H}$ and φ is the quotient homomorphism. $\mathcal{K}$ is said to be a hypergroup extension of $\mathcal{L}$ by $\mathcal{H}$.

The extension problem is to determine all extensions $\mathcal{K}$ and parametrize some useful families of them when $\mathcal{H}$ and $\mathcal{L}$ are given.

Now fix a finite commutative hypergroup $\mathcal{K}$ and a subhypergroup $\mathcal{H}$ of $\mathcal{K}$.

$$\mathcal{H}(s) = \{h \in \mathcal{H} \ s.t. \ hs = s\}$$

is called a *stabilizer* of $s \in \mathcal{K}$ under $\mathcal{H}$-multiplication.

Lemma 2. *Let $1 \to \mathcal{H} \to \mathcal{K}$ be an exact sequence of finite commutative hypergroups. An element $h \in \mathcal{H}(s)$ is characterized by the following property $(*)$:*

$(*)$ $\qquad \chi(h) = 1$ *for* $\chi \in \mathcal{K}^\wedge$ *such that* $\chi(s) \neq 0$.

Furthermore $\mathcal{H}(s)$ is a subhypergroup.

Proof. For $\chi \in \mathcal{K}^\wedge$ such that $\chi(s) \neq 0$, we have that $\chi(hs) = \chi(h)\chi(s) = \chi(s)$, so that $\chi(h) = 1$. Conversely it is shown that $\chi(hs) = \chi(s)$ for all $\chi \in \mathcal{K}$. Thus $hs = s$ and $h \in \mathcal{H}(s)$. To start with showing that $\mathcal{H}(s)$ is a subhypergroup, we notice that $h^* \in \mathcal{H}(s)$ if $h \in \mathcal{H}(s)$ from the property $(*)$. Therefore $\mathcal{H}(s)^* = \mathcal{H}(s)$. For $h, h' \in \mathcal{H}(s)$, we write $hh' = \sum_i n_i c_i$ where $\sum_i n_i = 1$, $c_i \in \mathcal{K}$. Assume that $\chi \in K^\wedge$ such that $\chi(s) \neq 0$. It is always satisfied $|\chi(c_i)| \leq 1$ for a hypergroup $\mathcal{K}$ [W2], then $\Re(\chi(c_i)) \leq 1$. We then get $1 = \chi(h)\chi(h') = \chi(hh') = \sum_i n_i\chi(c_i) = \sum_i n_i \cdot \Re(\chi(c_i)) \leq \sum_i n_i = 1$ because the sum of imaginary part is vanishing.

We note that $\Re(\chi(c_i)) = 1$ if and only if $\chi(c_i) = 1$. It is clear to see that $\chi(c_i) = 1$ for i such that $n_i \neq 0$. Thus hh' belongs to a real linear span of elements in $\mathcal{H}(s)$. □

Provided that $\mathcal{K}$ is a group, it is well known that the above subhypergroup $\mathcal{H}(s) = \{c_0\}$. We can find an example which has the maximal stabilizer, i.e. $\mathcal{H} \vee \mathcal{L}$ has $\mathcal{H}(s) = \mathcal{H}$ for an element $s \notin \mathcal{H}$.

We give some relations among weights of hypergroups in a short exact sequence.

Lemma 3. *Let $1 \to \mathcal{H} \to \mathcal{K} \xrightarrow{\varphi} \mathcal{L} \to 1$ be an exact sequence of finite commutative hypergroups. For any $\ell \in \mathcal{L}$, the following conditions* (1)-(3) *are satisfied*:

(1) $w(\mathcal{K}) = w(\mathcal{H})w(\mathcal{L})$,

(2) $w(\varphi^{-1}(\ell)) = w(\mathcal{H})w(\ell)$,

(3) $w(\ell) \leq w(s) \leq w(\mathcal{H})w(\ell)$ *for* $s \in \varphi^{-1}(\ell)$.

Proof. Let $e_0^{\mathcal{K}} = w(\mathcal{K})^{-1}\sum_{c_i \in \mathcal{K}} w(c_i)c_i$ be the normalized Haar measure of $\mathcal{K}$. The image $\varphi(e_0^{\mathcal{K}})$ equals the normalized Haar measure $e_0^{\mathcal{L}}$ of $\mathcal{L}$. Let ℓ_0 be the unit of $\mathcal{L}$, and $\varphi^{-1}(\ell_0) = \mathcal{H}$. From the expression

$$\begin{aligned} \varphi(e_0^{\mathcal{K}}) &= w(\mathcal{K})^{-1}\big(\sum_{c \in \mathcal{H}} w(c)\ell_0 + \cdots + \sum_{d \in \varphi^{-1}(\ell)} w(d)\ell + \cdots\big) \\ &= w(\mathcal{K})^{-1}\big(\ w(\mathcal{H})\ell_0 \ + \cdots + \ w(\varphi^{-1}(\ell))\ell \ + \cdots\big), \end{aligned}$$

it is obtained that $w(\mathcal{K})^{-1}w(\mathcal{H}) = w(\mathcal{L})^{-1}$ and $w(\mathcal{K})^{-1}w(\varphi^{-1}(\ell)) = w(\mathcal{L})^{-1}w(\ell)$. This shows (1),(2) and the second inequality of (3).

The first one of (3) is shown by the fact $w(s)^{-1} \leq w(\ell)^{-1}$ since

$$\varphi(ss^*) = \varphi(w(s)^{-1}c_0 + \cdots) = (w(s)^{-1} + \cdots)\ell_0 + \cdots = w(\ell)^{-1}\ell_0 + \cdots .$$

□

In the category of signed hypergroup extensions, namely if $\mathcal{K}$ is a signed hypergroup, the conditions (1),(2) and (3) $w(s) \le w(\mathcal{H})w(\ell)$ in Lemma 3 also hold but (3) $w(\ell) \le w(s)$ does not always hold. It turns that

(3′) $0 < w(s) \le w(\mathcal{H})w(\ell)$.

Using Lemma 3 on the relations about weights, we give a general condition about orders of hypergroups in a short exact sequence.

Proposition 4. *Let* $1 \to \mathcal{H} \to \mathcal{K} \xrightarrow{\varphi} \mathcal{L} \to 1$ *be an exact sequence of finite commutative hypergroups. Then the inequalities*:

$$|\mathcal{H}| + |\mathcal{L}| - 1 \ \le\ |\mathcal{K}| \ \le\ |\mathcal{H}| + \lfloor w(\mathcal{H}) \rfloor (|\mathcal{L}| - 1)$$

hold, where the floor function $\lfloor x \rfloor = \max\{n \in \mathbb{Z} \mid n \le x\}$ *for a real number* x.

Proof. The first inequality is obvious, equality occurs if and only if $\mathcal{K} = \mathcal{H} \vee \mathcal{L}$, i.e. a join hypergroup of $\mathcal{H}$ with $\mathcal{L}$. From the above Lemma 3 (2)(3), $w(\mathcal{H})w(\ell) = \sum_{s\in\varphi^{-1}(\ell)} w(s) \ge \sum_{s\in\varphi^{-1}(\ell)} w(\ell) = |\varphi^{-1}(\ell)|w(\ell)$. Hence $w(\mathcal{H}) \ge |\varphi^{-1}(\ell)|$. The second inequality holds from $\mathcal{K} = \bigcup_{\ell\in\mathcal{L}} \varphi^{-1}(\ell)$. □

This estimation is strict since we will show that there exist hypergroup extensions of several order in the inequalities. It is trivial that the direct product hypergroup $\mathcal{K} = \mathcal{H} \times \mathcal{L}$ is a hypergroup extension of order $|\mathcal{H}| \cdot |\mathcal{L}|$.

We first show some cases in which the maximal order of hypergroup extension $\mathcal{K}$ of $\mathcal{L}$ by $\mathcal{H}$ equals $|\mathcal{H}| \cdot |\mathcal{L}|$.

When $\mathcal{H}$ is a group, we have $w(\mathcal{H}) = \lfloor w(\mathcal{H}) \rfloor = |\mathcal{H}|$. The condition of Proposition 4 is in the form $|\mathcal{H}| + |\mathcal{L}| - 1 \le |\mathcal{K}| \le |\mathcal{H}| \cdot |\mathcal{L}|$ which is established in [K]. In the case of $|\mathcal{H}| \le w(\mathcal{H}) < |\mathcal{H}| + 1$, the above condition also holds.

We define for subsets M, N of $\mathcal{K}$,

$$MN = \{c_i c_j \in A(\mathcal{K}) \ s.t.\ c_i \in M, c_j \in N\}.$$

We note that the set MN is not usually enclosed in $\mathcal{K}$.

Proposition 5. *Let* $1 \to \mathcal{H} \to \mathcal{K} \xrightarrow{\varphi} \mathcal{L} \to 1$ *be an exact sequence of finite commutative hypergroups and* $\mathcal{H}$ *be a group. Fix* $\ell \in \mathcal{L}$. *The following statements* (1)-(4) *are satisfied*:

(1) $hs \in \varphi^{-1}(\ell)$ *for* $s \in \varphi^{-1}(\ell)$ *and* $h \in \mathcal{H}$,

(2) $\mathcal{H}\{s\} = \varphi^{-1}(\ell)$ *for* $s \in \varphi^{-1}(\ell)$,

(3) $\mathcal{H}(s) = \mathcal{H}(s')$ *for* $s, s' \in \varphi^{-1}(\ell)$, *and* $|\varphi^{-1}(\ell)| = |\mathcal{H}/\mathcal{H}(s)|$,

(4) $w(s) = w(\ell) \cdot |\mathcal{H}|/|\varphi^{-1}(\ell)|$ *for* $s \in \varphi^{-1}(\ell)$.

Proof. The statements (1),(2) and (3) were proved in the section 3 of [K]. The values of character $|\chi(h)| = 1$ $\forall \chi \in \mathcal{K}^\wedge$ as $\mathcal{H}$ is a group. For $s' = hs$, $h \in \mathcal{H}$, writing $\mathcal{K} = \{\chi_0, \dots, \chi_n\}$, we have

$$\begin{aligned} w(s')^{-1} = \langle s', s' \rangle &= w(\mathcal{K})^{-1} \sum_{k=0}^{n} |\chi_k(s')|^2 w(\chi_k) \\ &= w(\mathcal{K})^{-1} \sum_{k=0}^{n} |\chi_k(h)\chi_k(s)|^2 w(\chi_k) \\ &= w(\mathcal{K})^{-1} \sum_{k=0}^{n} |\chi_k(s)|^2 w(\chi_k) = w(s)^{-1}. \end{aligned}$$

Finally Lemma 3 (3) and the above show the last (4). □

Proposition 5 gives many examples of hypergroups whose stabilizer have any subgroup of $\mathcal{H}$.

In the paper [IKS], it was shown that if $\mathcal{H} = \mathrm{L}(q)$ and $\mathcal{L}$ is a group, then $|\mathcal{H}|+|\mathcal{L}|-1 \le |\mathcal{K}| \le |\mathcal{H}|\cdot|\mathcal{L}|$ holds, more precisely $|\mathcal{L}|+1 \le |\mathcal{K}| \le 2\cdot|\mathcal{L}|$.

It is obvious from Proposition 3 that the maximal order $\mathcal{K}$ is $|\mathcal{H}|\cdot|\mathcal{L}|$ provided that $|\mathcal{H}| \le w(\mathcal{H}) < |\mathcal{H}|+1$. For example, $\mathcal{H} = \mathrm{L}(q)$ for $1/2 < q \le 1$.

What happens in the case of a general finite commutative hypergroup $\mathcal{H}$ and of any finite commutative group $\mathcal{L}$? The following is an answer to this question.

Proposition 6. *Let $1 \to \mathcal{H} \to \mathcal{K} \xrightarrow{\varphi} \mathcal{L} \to 1$ be an exact sequence of finite commutative hypergroups. If $\mathcal{L}$ is a group, then $|\varphi^{-1}(\ell)| \le |\mathcal{H}|$ for $\ell \in \mathcal{L}$ and $w(\chi) = w(\chi')$ for $\chi, \chi' \in \mathcal{K}^\wedge$ such that their restrictions $\chi|_{\mathcal{H}} = \chi'|_{\mathcal{H}} \in \mathcal{H}^\wedge$.*

Proof. We first obtain that the matrix $[\chi(s_i)]_{\chi\in\mathcal{K}^\wedge,\, s_i\in\varphi^{-1}(\ell)}$ of the characters matrix has rank $m+1$ since the column vectors are linearly independent from Axiom of hypergroups. Now we will calculate this rank in another way.

Fix a non-trivial character $\chi^{\mathcal{H}} \in \mathcal{H}^\wedge$. We define the subset $\Sigma(\chi^{\mathcal{H}}) = \{\chi \in \mathcal{K}^\wedge \text{ s.t. } \chi|_{\mathcal{H}} = \chi^{\mathcal{H}}\}$ of $\mathcal{K}^\wedge$. Write $\varphi^{-1}(\ell) = \{s_0, \ldots, s_m\} \subset \mathcal{K}$.

Assume that $\chi(s_0) \ne 0$ for some character $\chi \in \Sigma(\chi^{\mathcal{H}})$. Since $\mathcal{L}$ is a group, we have $(s_0)^* s_i \in A(\mathcal{H})$, so that the value $\chi((s_0)^* s_i)$ is independent of the choice of $\chi \in \Sigma(\chi^{\mathcal{H}})$. Then $V(\chi^{\mathcal{H}}, 0, i) := \chi((s_0)^* s_i)$. It is obvious that $|\chi(s_0)|^2 = V(\chi^{\mathcal{H}}, 0, 0) \ne 0$. This shows that the absolute value of $\chi(s_0)$ is independent of χ. It is also obtained that $\chi(s_i) = V(\chi^{\mathcal{H}}, 0, i)\chi(s_0^*)^{-1}$. Hence the vector $(\chi(s_0), \ldots, \chi(s_m))$ is parallel to a vector $(V(\chi^{\mathcal{H}}, 0, 0), \ldots, V(\chi^{\mathcal{H}}, 0, m))$. The sub-matrix $[\chi_j(s_i)]$ for $\chi_j \in \Sigma(\chi^{\mathcal{H}})$ and $s_i \in \varphi^{-1}(\ell)$ has rank one. If $\chi_j(s_i) = 0$ for all χ_j, $s_i \in \varphi^{-1}(\ell)$, then the sub-matrix $[\chi_j(s_i)]$ for $\chi_j \in \Sigma(\chi^{\mathcal{H}})$ turns to be zero, i.e. the rank is zero. Hence the sub-matrix $[\chi_j(s_i)]$ for $\chi_j \in \Sigma(\chi^{\mathcal{H}})$ and $s_i \in \varphi^{-1}(\ell)$ has at most rank one. Therefore the rank of the matrix $[\chi(s_i)]_{\chi\in\mathcal{K}^\wedge,\, s_i\in\varphi^{-1}(\ell)}$ is at most $|\mathcal{H}^\wedge| = |\mathcal{H}|$. This shows that $|\varphi^{-1}(\ell)| = m+1 \le |\mathcal{H}|$.

It is already shown that the values of $\chi(s_i)$ have the same absolute value for all $\chi \in \Sigma(\chi^{\mathcal{H}})$ which depends only on $s_i \in \varphi^{-1}(\ell)$. If $c \in \mathcal{K}$ is fixed then $|\chi(c)|$ has the same value for $\chi \in \Sigma(\chi^{\mathcal{H}})$. From the formula for weights of characters, we have that for $\chi, \chi' \in \Sigma(\chi^{\mathcal{H}})$,

$$\begin{aligned} w(\chi)^{-1} &= w(\mathcal{K})^{-1}\sum_{c\in\mathcal{K}} |\chi(c)|^2 w(c) \\ &= w(\mathcal{K})^{-1}\sum_{c\in\mathcal{K}} |\chi'(c)|^2 w(c) = w(\chi')^{-1}. \end{aligned}$$

This implies that $w(\chi) = w(\chi')$. □

We have the next corollary on the order of hypergroup extensions.

Corollary 7. *Let $1 \to \mathcal{H} \to \mathcal{K} \xrightarrow{\varphi} \mathcal{L} \to 1$ be an exact sequence of finite commutative hypergroups. If $\mathcal{L}$ is a group, then*

$$|\mathcal{H}| + |\mathcal{L}| - 1 \le |\mathcal{K}| \le |\mathcal{H}|\cdot|\mathcal{L}|.$$

Proof. It is a trivial consequence of $|\varphi^{-1}(\ell)| \le |\mathcal{H}|$ together with the arguments in the proof of Proposition 4. □

We note that if a hypergroup $\mathcal{K}$ is strong, Corollary 7 is immediately consequence of the duality theory and S. Kawakami's Lemma [K]. The maximal order $|\mathcal{H}| \cdot |\mathcal{L}|$ is achieved when $\mathcal{K}$ is a direct product $\mathcal{H} \times \mathcal{L}$, but there does not always exist hypergroup extensions with order of any integer between $|\mathcal{H}|+|\mathcal{L}|-1$ and $|\mathcal{H}|\cdot|\mathcal{L}|$ in Corollary 7, which was shown in [IKS].

Does there exist an extension $\mathcal{K}$ with order higher than $|\mathcal{H}| \cdot |\mathcal{L}|$?

Now we give the models of hermitian signed hypergroups for the answer to this question.

Proposition 8. *Let n be an integer and $0 < q \leq 1$. The following character table $\chi_i(c_j)$ with weights $w(\chi_i), w(c_j)$ determines a signed hypergroup $\mathcal{K}(n,q) = \{c_0, \ldots, c_{n+1}\}$ of order $n+2$ and the total weight $w(\mathcal{K}) = (1+q)^2 q^{-2}$.*

	c_0	c_1	c_2	c_3	$\cdots$	c_{n+1}	$w(\chi_i)$
χ_0	1	1	1	1	$\cdots$	1	1
χ_1	1	1	$-q$	$-q$	$\cdots$	$-q$	$1/q$
χ_2	1	$-q$	$(n-1)q$	$-q$	$\cdots$	$-q$	$\frac{1+q}{nq^2}$
χ_3	1	$-q$	$-q$	$(n-1)q$	$\cdots$	$-q$	$\frac{1+q}{nq^2}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
χ_{n+1}	1	$-q$	$-q$	$-q$	$\cdots$	$(n-1)q$	$\frac{1+q}{nq^2}$
$w(c_i)$	1	$1/q$	$\frac{1+q}{nq^2}$	$\frac{1+q}{nq^2}$	$\cdots$	$\frac{1+q}{nq^2}$	$\frac{(1+q)^2}{q^2}$

Moreover, $\mathcal{K}(n,q)^\wedge = \{\chi_0, \ldots, \chi_{n+1}\} \cong \mathcal{K}(n,q)$.

Proof. Since the values of characters and weights of c_i are real, it is clear that the matrix $\left(\chi_i(c_j) w(\chi_i)^{1/2} w(c_j)^{1/2} w(\mathcal{K})^{-1/2}\right)_{ij}$ is a real constant type orthogonal matrices. Thus the condition of Lemma 1 is satisfied. So that $\mathcal{K}(n,q)$ is a signed hypergroup from Lemma 1. The weight of c_i equals $(1+q)n^{-1}q^{-2}$ for all $i \geq 2$, which is calculated by using Wildberger's harmonic analysis [W2].

Symmetry of the table leads $\mathcal{K}(n,q)^\wedge \cong \mathcal{K}(n,q)$. □

Now we calculate structure constants of $\mathcal{K}(n,q)$ to determine which model $\mathcal{K}(n,q)$ is a hypergroup or not.

Theorem 9. *For the models $\mathcal{K}(n,q) = \{c_0, \ldots, c_{n+1}\}$ for an integer n and $0 < q \leq 1$ in Proposition 8, the following statements hold:*

(t1) *$\mathcal{K}(n,q)$ is a hypergroup if and only if $1 \geq q(n-1)$,*

(t2) *$\{c_0, c_1\} \subset \mathcal{K}(n,q)$ is a subhypergroup isomorphic to $\mathrm{L}(q)$,*

(t3) *the quotient $\mathcal{K}(n,q)/\{c_0, c_1\}$ is isomorphic to $\mathrm{L}(q)$.*

Proof. According to Wildberger's formula about structure constants, values of characters and weights [W2], the multiplications have the following expression:

(1) $c_1^2 = qc_0 + (1-q)c_1$,

(2) $c_1c_i = (-q)c_i + \dfrac{1+q}{n}(c_2 + \cdots + c_{n+1}),$

(3) $c_i^2 = \dfrac{nq^2}{1+q}c_0 + \dfrac{1-q(n-1)}{1+q}qc_1 + (n-2)qc_i + \dfrac{1-q(n-1)}{n}(c_2 + \cdots + c_{n+1}),$

(4) $c_ic_j = qc_1 + (-q)(c_i + c_j) + \dfrac{1+q}{n}(c_2 + \cdots + c_{n+1})$

for $2 \le i,\ 2 \le j$ and $i \ne j$.

(t2) is obvious from (1). Hence non negativity of coefficients in the above (2)-(4) implies (t1). From (2), we have that $\frac{1}{1+q}(qc_0 + c_1)c_i = \frac{1}{n}(c_2 + \cdots + c_{n+1})$, so that

$$\frac{1}{1+q}(qc_0 + c_1)\frac{1}{n}(c_2 + \cdots + c_{n+1}) = \frac{1}{n}(c_2 + \cdots + c_{n+1}).$$

Summarizing (3) and (4), and writing $\ell_1 := \frac{1}{n}(c_2 + \cdots + c_{n+1})$, we have

$$\begin{aligned} \ell_1 c_i &= \frac{1}{n}\Big[\frac{nq^2}{1+q}c_0 + \frac{1-q(n-1)}{1+q}qc_1 + (n-1)qc_1 + (n-2)qc_i + (-q)(n-1)c_i \\ &\qquad +(-q)n\ell_1 + qc_i + (1-q(n-1) + (1+q)(n-1))\ell_1\Big] \\ &= q\left(\frac{q}{1+q}c_0 + \frac{1}{1+q}c_1\right) + (1-q)\ell_1. \end{aligned}$$

Write $\ell_0 := (1+q)^{-1}(qc_0 + c_1)$. Therefore,

$$\begin{aligned} \ell_1^2 = \frac{1}{n}\sum_{i=2}^{n+1}\ell_1 c_i &= q\left(\frac{q}{1+q}c_0 + \frac{1}{1+q}c_1\right) + (1-q)\ell_1 \\ &= q\ell_0 + (1-q)\ell_1. \end{aligned}$$

Thus the quotient $\mathcal{K}(n,q)/\{c_0, c_1\} = \{\ell_0, \ell_1\} \cong \mathrm{L}(q)$, which shows (t3). □

The statements in Theorem 9 mean that $1 \to \mathrm{L}(q) \to \mathcal{K}(n,q) \to \mathrm{L}(q) \to 1$ is exact. $\mathcal{K}(n,q)$ is a self-dual hypergroup, so that it is a strong hypergroup.

Remark 1. We comment here about the condition between n and q in which $\mathcal{K}(n,q)$ is a hypergroup. It is easy to see that $\mathcal{K}(1,q) = \mathrm{L}(q) \vee \mathrm{L}(q)$, a join hypergroup which has order three. The condition (t1) means $n \le w(\mathcal{H}) = 1 + 1/q$ which is proved in Proof of Proposition 4. If an integer $n > 1 + 1/q$, then $\mathcal{K}(n,q)$ is a signed hypergroup but not a hypergroup. There exists a signed hypergroup extension with any order higher than three. It is well known that hypergroup extensions of a group by another group are also groups. As $\mathrm{L}(1) = \mathbb{Z}_2$, there exist well known group extension $\mathbb{Z}_2 \times \mathbb{Z}_2 = \mathcal{K}(2,1)$ and $\mathbb{Z}_4$, only two group extensions, but there exist infinite many signed hypergroup extensions $\mathcal{K}(n,1)$ for $n \ge 3$ which have subhypergroup $\mathbb{Z}_2$ and the quotient $\mathbb{Z}_2$. Although non splitting group extension $\mathbb{Z}_4$ does not appear in our models. When the maximal integer n such that $1 + 1/q \ge n$, namely $n = 1 + \lfloor 1/q \rfloor$, this model $\mathcal{K}(n,q)$ is a self dual hypergroup with the maximal order mentioned in Proposition 4. Moreover there exits a hypergroup extension $\mathcal{K}(n,q)$ whose order is any integer between 3 and $3 + \lfloor 1/q \rfloor$.

When $1/2 < q \le 1$, we have $w(\mathrm{L}(q)) < 3$, so that $\lfloor w(\mathrm{L}(q)) \rfloor = 2$. According to estimation in Proposition 4, the hypergroup extensions of $\mathrm{L}(q)$ by $\mathrm{L}(q)$ must have an order less than or equal to four. Thus there exists no extension of order higher than four.

When $0 < q \leq 1/2$, there exist many extensions of orders higher than four. If $q = 1/2$, then it is easily shown that hermitian extension of order five is isomorphic to $\mathcal{K}(3, 1/2)$, i.e. uniquely determined hermitian hypergroup extension provided that the character table is symmetric. In this case it is easily seen from the table that $\mathcal{K}(3, 1/2)$ has four subhypergroups $\{c_0, c_1\} \cong \{c_0, c_2\} \cong \{c_0, c_3\} \cong \{c_0, c_4\} \cong \mathrm{L}(1/2)$ and their quotients are isomorphic to $\mathrm{L}(1/2)$. The automorphism group of $\mathcal{K}(3, 1/2)$ is a permutation group of degree four on the set $\{c_1, c_2, c_3, c_4\}$. Thus there exist three cross sections ε_i for $i = 2, 3, 4$ in the following exact sequence:

$$1 \to \mathrm{L}(1/2) \to \mathcal{K}(3, 1/2) \leftrightarrows \mathrm{L}(1/2) \to 1$$

where hypergroup homomorphism ε_i maps $\ell_1 \in \mathrm{L}(1/2)$ to c_i. However $\mathcal{K}(3, 1/2)$ is not isomorphic to a direct product $\mathrm{L}(1/2) \times \mathrm{L}(1/2)$.

Furthermore $\mathcal{K}(n+1, 1/n)$ for an integer $n \geq 3$ has the cross sections in the short exact sequences.

When $1/3 < q < 1/2$, there exists many hermitian extensions which are not isomorphic to $\mathcal{K}(3, q)$. After suitable calculations, the set of all hypergroup extensions is parametrized by two real dimension with respect to two weights, but we omit them. Therefore the entire list of hypergroup extensions of $\mathrm{L}(q)$ by $\mathrm{L}(q)$ for a fixed positive number $0 < q < 1/2$ and $q \neq 1/n$ $(n = 2, 3, \dots)$ is intricate for us to express their all structure constants completely. When the number q is near to 0, it is very intricate for us to done it completely even if in the hermitian case.

We can see the above models as signed hypergroup extensions $\mathcal{K}(n, q)$ of a signed hypergroup $\mathrm{L}(q)$ by another one when $q > 1$. But these analysis in the category of signed hypergroups is too complex to be completed.

Remark 2. The action π of $\{c_0, c_1\} \cong \mathrm{L}(q)$ on a set $\{c_2, \dots, c_{n+1}\}$ is irreducible and has the matrix form T_{ij} appeared in [SW]. For an example, $\pi(e_0^{\mathcal{H}}) = (1/n)J_n$ where J_n is $n \times n$-matrix all components 1. The matrix in the character table in Proposition 8 equals $(1+q)^{-1}nq\big(\pi(c_0) - \pi(c_1)\big) = nq\big(\pi(c_0) - \pi(e_0^{\mathcal{H}})\big)$. Moreover the action π is a irreducible $*$-action in Example 1 in [SW].

Since the value of characters in $\mathcal{K}(n, q)$ does not vanish if $n \neq 1$, so that the stabilizer $\mathrm{L}(q)(c_i) = \{c_0\} \subset \mathrm{L}(q)$ for any $c_i \in \mathcal{K}(n, q)$. Hypergroup $\mathcal{K}(n, q)$ for $n > 2$ and $q \leq 1/2$ is considered as an example such that the cardinal number of the section $\varphi^{-1}(\ell)$ for ℓ does not equal to two when ℓ is not a unit. However that of the quotient $\mathrm{L}(q)/\mathrm{L}(q)(c_i) = \mathrm{L}(q)$ equals two. Therefore the method of stabilizers can not go well in general as same as group theory. In a general extension problem of exact sequence $1 \to \mathcal{H} \to \mathcal{K} \xrightarrow{\varphi} \mathcal{L} \to 1$, the cardinal number of $\mathcal{H}/\mathcal{H}(c)$ for $c \in \varphi^{-1}(\ell)$ does not always equal the cardinal number of $\varphi^{-1}(\ell)$.

Finally we illustrate more general models for hermitian signed hypergroup extensions having three parameters.

Proposition 10. *The following character table with weights determines a signed hypergroup* $\mathcal{K}(n, p, q) = \{c_0, \dots, c_{n+1}\}$ *for an integer* n *and* $0 < p \leq 1$, $0 < q \leq 1$. *In addition* $\mathcal{K}(n, p, q)$ *has order* $n + 2$.

	c_0	c_1	c_2	c_3	$\cdots$	c_{n+1}	$w(\chi_i)$
χ_0	1	1	1	1	$\cdots$	1	1
χ_1	1	1	$-q$	$-q$	$\cdots$	$-q$	$1/q$
χ_2	1	$-p$	$(n-1)\sqrt{pq}$	$-\sqrt{pq}$	$\cdots$	$-\sqrt{pq}$	$\frac{1+q}{npq}$
χ_3	1	$-p$	$-\sqrt{pq}$	$(n-1)\sqrt{pq}$	$\cdots$	$-\sqrt{pq}$	$\frac{1+q}{npq}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
χ_{n+1}	1	$-p$	$-\sqrt{pq}$	$-\sqrt{pq}$	$\cdots$	$(n-1)\sqrt{pq}$	$\frac{1+q}{npq}$
$w(c_i)$	1	$1/p$	$\frac{1+p}{npq}$	$\frac{1+p}{npq}$	$\cdots$	$\frac{1+p}{npq}$	$\frac{(1+p)(1+q)}{pq}$

Proof. In a similar way to Proposition 9, it is checked that $\mathcal{K}(n,p,q)$ is a signed hypergroup. □

Here the structure of $\mathcal{K}(n,p,q)$ is calculated as:

(1) $c_1^2 = pc_0 + (1-p)c_1$,

(2) $c_1 c_i = (-p)c_i + \frac{1+p}{n}(c_2 + \cdots + c_{n+1})$,

(3) $$c_i^2 = \frac{npq}{1+p}c_0 + \frac{1+p-np}{1+p}qc_1 + (n-2)\sqrt{pq}c_i + \frac{1-q-\sqrt{pq}(n-2)}{n}(c_2 + \cdots + c_{n+1}),$$

(4) $c_i c_j = qc_1 + (-\sqrt{pq})(c_i + c_j) + \frac{1-q+2\sqrt{pq}}{n}(c_2 + \cdots + c_{n+1})$

for $2 \le i,\ 2 \le j$ and $i \ne j$.

The above structure constants are non-negative if the following conditions (T1)-(T2) hold:

(T1) $1 + p - pn \ge 0$,

(T2) $1 - q - \sqrt{pq}(n-2) \ge 0$.

Hence $\mathcal{K}(n,p,q)$ is a hypergroup under the above condition. Thus the next theorem follows.

Theorem 11. *$\mathcal{K}(n,p,q)$ is a hypergroup if (T1)-(T2) are satisfied. Moreover $\mathcal{K}(n,p,q)$ has subhypergroup $\{c_0, c_1\} \cong \mathrm{L}(p)$ such that $\mathcal{K}(n,p,q)/\{c_0,c_1\} \cong \mathrm{L}(q)$ and is a hypergroup extension of $\mathrm{L}(q)$ by $\mathrm{L}(p)$.*

Proof. The first statement is already proved. The statement about the quotient hypergroup is clear in a similar way to the proof of Theorem 9. □

Obviously we see that $\mathcal{K}(n,p,p) = \mathcal{K}(n,p)$. From Theorem 11 we have similarly that $1 \to \mathrm{L}(p) \to \mathcal{K}(n,p,q) \to \mathrm{L}(q) \to 1$ is exact. This sequence gives the models of hypergroup extensions to the problem: an exact sequence $1 \to \mathrm{L}(p) \to \mathcal{K} \to \mathrm{L}(q) \to 1$ in the hermitian case of orders higher than four. These problems of hypergroup extensions in the case of order four are described in [IK], and are completely analyzed.

Remark 3. When $p = 1$, the condition (T1) turns out $n \leq 2$, which is shown in [K]. When $q = 1$, the condition (T2) also turns out $n \leq 2$, which is shown in [IKS].

We possess many hypergroup extensions of $\mathrm{L}(q)$ by $\mathrm{L}(p)$ with order higher than four in some case. Furthermore given $0 < q < 1$ and an integer $n \geq 3$, there exists a suitable $0 < p < 1$ such that $\mathcal{K}(n-2, p, q)$ is a hypergroup with the order n. This is shown by checking that (T1)-(T2) hold as $p \to +0$.

If $\mathcal{L}$ of order two is not a group, then we can choose a suitable hypergroup $\mathcal{H}$ of order two such that there exists a hypergroup extension of any order higher than four. Fixed n larger than two, it is shown that the region of all point (p, q) for $0 < p < 1$ and $0 < q < 1$ such that there exists a hypergroup extension of order $n+2$ is not symmetric with respect to two parameters p, q. Thus $\mathcal{K}(n, p, q)$ is not always strong even if $\mathcal{K}(n, p, q)$ is a hypergroup.

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Ryo Ichihara : Nara National College of Technology
Yatacho Yamato-Koriyama city
Nara, 639-1080 Japan
e-mail : ichihara@libe.nara-k.ac.jp

Satoshi Kawakami : Nara University of Education
Deparment of Mathematics
Takabatakecho
Nara, 630-8528 Japan
e-mail : kawakami@nara-edu.ac.jp