KANTOROVICH TYPE INEQUALITIES CHARACTERIZE THE CHAOTIC ORDER FOR POSITIVE OPERATORS

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Abstract. In our previous note, we showed the difference version of Kantorovich type inequality with two positive parameters under the usual order. As a continuation of our preceding note, we shall show the difference type Kantorovich inequalities with two positive parameters under the chaotic order in terms of a two parameters version of the Mond-Shisha difference: Let A and B be positive and invertible operators on a Hilbert space H such that $MI \geq B \geq mI$ for some scalars $0 < m < M$ and $h = \frac{M}{m}$. Then the following assertions are mutually equivalent:

(1) $\log A \geq \log B$,

(2) $A^q + \frac{(M^{p+q}-m^{p+q})^2 - 4m^q M^q (M^p - m^p)(M^q - m^q)}{4m^q (M^q - m^q)^2} I \geq B^p$ for all $p, q > 0$,

(3) $A^q + \frac{p}{q}\frac{M^p - m^p}{\log M^p - \log m^p} \log \left(m^{p-q} \frac{(h^p-1)h^{\frac{q}{h^p-1}}}{eq \log h} \right) I \geq B^p$ for all $p, q > 0$.

1 Introduction. Throughout this paper, we consider bounded linear operators on a complex Hilbert space H. An operator T is said to be positive (denoted by $T \geq 0$) if $(Tx, x) \geq 0$ for all $x \in H$. The positivity defines the usual order $A \geq B$ for selfadjoint operators A and B. For the sake of convenience, $T > 0$ means T is positive and invertible. The Löwner-Heinz inequality asserts that $A \geq B \geq 0$ ensures $A^\alpha \geq B^\alpha$ for all $0 \leq \alpha \leq 1$. However $A \geq B \geq 0$ does not ensure $A^\alpha \geq B^\alpha$ for $\alpha > 1$ in general. For positive invertible operators A and B, the chaotic order is defined by $\log A \geq \log B$, which is weaker than the usual order $A \geq B$.

Yamazaki [11] showed the following characterizations of the chaotic order in terms of difference type Kantorovich inequalities:

Theorem A. *Let A and B be positive operators on H such that $MI \geq B \geq mI$ for some scalars $M > m > 0$. Then following assertions are mutually equivalent:*

(1) $\log A \geq \log B$,

(2) $A^p + \frac{(M^p - m^p)^2}{4m^p} I \geq B^p$ *for all* $p > 0$,

(3) $A^p + L(m^p, M^p) \log S_h(p) I \geq B^p$ *for all* $p > 0$

where the logarithmic mean $L(m, M) = \frac{M-m}{\log M - \log m}$, *the generalized Specht ratio* $S_h(p) = \frac{(h^p-1)h^{\frac{p}{h^p-1}}}{pe \log h}$ *and* $h = \frac{M}{m}$.

Mond and Shisha[9, 10] made an estimate of the difference between the arithmetic mean and the geometric one: For positive numbers $x_1, ..., x_n \in [m, M]$ with $M > m > 0$ and $h = \frac{M}{m}$,

$$\sqrt[n]{x_1 x_2 ... x_n} + D(m, M) \geq \frac{x_1 + x_2 + ... + x_n}{n}$$

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where

$$D(m,M) = \theta M + (1-\theta)m - M^{\theta}m^{1-\theta} \quad \text{and } \theta = \log\left(\frac{h-1}{\log h}\right)\frac{1}{\log h}$$

which we call the *Mond-Shisha difference.* It is known in [5] that

$$D(m^p, M^p) = L(m^p, M^p)\log S_h(p) \quad \text{for } M > m > 0 \quad \text{and } h = \frac{M}{m} > 1.$$

In this paper, we will show difference type Kantorovich inequalities with two positive parameters under the chaotic order which is an extension of Theorem A. Among others, if A and B are positive operators such that $MI \geq B \geq mI$ for some scalars $M > m > 0$ and $h = \frac{M}{m}$, then the chaotic order $\log A \geq \log B$ is equivalent to

$$A^q + \frac{p}{q}L(m^p, M^p)\log\left(m^{p-q}\frac{(h^p-1)h^{\frac{q}{h^p-1}}}{eq\log h}\right)I \geq B^p \qquad \text{for all } p,\ q > 0.$$

2 Difference type Kantorovich inequalities. First of all, we cite the following main tool obtained in [7].

Theorem B. *Let A and B be positive operators on H such that $MI \geq B \geq mI$ for some scalars $M > m > 0$. If $A \geq B$, then*

$$A^q + C(m,M,p,q)I \geq B^p$$

for all $p,\ q > 1$.

Here *the Kantorovich constant for the difference* $C(m,M,p,q)$ for $p,q > 1$ is defined by

$$C(m,M,p,q) = \begin{cases} \frac{Mm^p - mM^p}{M-m} + (q-1)\left(\frac{M^p-m^p}{q(M-m)}\right)^{\frac{q}{q-1}} & \text{if } m \leq \left(\frac{M^p-m^p}{q(M-m)}\right)^{\frac{1}{q-1}} \leq M, \\ m^p - m^q & \text{if } \left(\frac{M^p-m^p}{q(M-m)}\right)^{\frac{1}{q-1}} < m, \\ M^p - M^q & \text{if } M < \left(\frac{M^p-m^p}{q(M-m)}\right)^{\frac{1}{q-1}}. \end{cases}$$

In this section, we propose difference type Kantorovich inequalities under the chaotic order. The following theorem is a two parameters version of (2) in Theorem A.

Theorem 1. *Let A and B be positive operators on H such that $MI \geq B \geq mI$ for some scalars $0 < m < M$. Then following assertions are equivalent:*

(1) $\log A \geq \log B$,

(2) $A^q + \frac{(M^{p+q}-m^{p+q})^2 - 4m^qM^q(M^p-m^p)(M^q-m^q)}{4m^q(M^q-m^q)^2}I \geq B^p$ *for all $p,\ q > 0$.*

Proof. (1) $\Rightarrow$ (2). Put $q = 2$ in Theorem B. Then we have the following inequality: If $A_1 \geq B_1$ and $M_1 I \geq B_1 \geq m_1 I$ for $M_1 > m_1 > 0$, then

$$A_1{}^2 + \left\{ \frac{M_1 m_1{}^{p_1} - m_1 M_1{}^{p_1}}{M_1 - m_1} + \frac{(M_1{}^{p_1} - m_1{}^{p_1})^2}{4(M_1 - m_1)^2} \right\} I \geq B_1{}^{p_1} \tag{2.1}$$

for all $p_1 > 1$ by definition of $C(m, M, p, q)$. And $\log A \geq \log B$ ensures that

$$(B^{\frac{q}{2}} A^q B^{\frac{q}{2}})^{\frac{1}{2}} \geq B^q$$

for all $q > 0$, see [1], also [2] and [3]. Put $A_1 = (B^{\frac{q}{2}} A^q B^{\frac{q}{2}})^{\frac{1}{2}}, B_1 = B^q, M_1 = M^q, m_1 = m^q$ and $p_1 = \frac{p+q}{q} (> 1)$ for $p > 0$ in (2.1). Then we have

$$B^{\frac{q}{2}} A^q B^{\frac{q}{2}} + \left\{ \frac{M^q m^{p+q} - m^q M^{p+q}}{M^q - m^q} + \frac{(M^{p+q} - m^{p+q})^2}{4(M^q - m^q)^2} \right\} I \geq B^{p+q}.$$

And, we have

$$A^q + \left\{ \frac{M^q m^q (m^p - M^p)}{M^q - m^q} + \frac{(M^{p+q} - m^{p+q})^2}{4(M^q - m^q)^2} \right\} B^{-q} \geq B^p,$$

then

$$A^q + \left\{ \frac{M^q m^q (m^p - M^p)}{M^q - m^q} + \frac{(M^{p+q} - m^{p+q})^2}{4(M^q - m^q)^2} \right\} m^{-q} I \geq B^p \qquad \text{since} \;\; m^{-q} I \geq B^{-q} > 0.$$

Therefore, we have

$$A^q + \left\{ \frac{(M^{p+q} - m^{p+q})^2 - 4M^q m^q (M^p - m^p)(M^q - m^q)}{4m^q (M^q - m^q)^2} \right\} I \geq B^p$$

for all $p, \ q > 0$.
(2) $\Rightarrow$ (1). Put $q = p$ in (2), then we have

$$A^p + \frac{(M^p - m^p)^2}{4m^p} I \geq B^p$$

for all $p > 0$. Then, it follows from Theorem A that $A^p + \frac{(M^p - m^p)^2}{4m^p} I \geq B^p$ for all $p > 0$ implies $\log A \geq \log B$. Therefore we have the desired result. □

The following theorem is a characterization of the chaotic order in terms of the Kantorovich constant for the difference.

Theorem 2. *Let A and B be positive operators on H such that $MI \geq B \geq mI$ for some scalars $0 < m < M$. Then the following assertions are equivalent:*

(1) $\log A \geq \log B$,

(2) $A^q + \frac{1}{m^r} C(m^r, M^r, \frac{p+r}{r}, \frac{q+r}{r}) I \geq B^p$ *for all* $p, \ q, \ r > 0$.

Proof. (1) $\Rightarrow$ (2). Since $\log A \geq \log B$, it follows from the chaotic Furuta inequality [2, 3] that

$$(B^{\frac{r}{2}} A^q B^{\frac{r}{2}})^{\frac{r}{q+r}} \geq B^r \quad \text{for all } q,\ r > 0.$$

Put $A_1 = (B^{\frac{r}{2}} A^q B^{\frac{r}{2}})^{\frac{r}{q+r}}, B_1 = B^r, M_1 = M^r, m_1 = m^r, q_1 = \frac{q+r}{r}\ (>1)$ and $p_1 = \frac{p+r}{r}\ (>1)$ for all $p,\ q,\ r > 0$ in Theorem B. Then we have

$$B^{\frac{r}{2}} A^q B^{\frac{r}{2}} + C\left(m^r, M^r, \frac{p+r}{r}, \frac{q+r}{r}\right) I \geq B^{p+r},$$

and

$$A^q + C\left(m^r, M^r, \frac{p+r}{r}, \frac{q+r}{r}\right) B^{-r} \geq B^p.$$

Since $m^{-r} I \geq B^{-r} > 0$, we have

$$A^q + \frac{1}{m^r} C\left(m^r, M^r, \frac{p+r}{r}, \frac{q+r}{r}\right) I \geq B^p \quad \text{for all } p,\ q,\ r > 0.$$

(2) $\Rightarrow$ (1). If we put $p = q = r$ in (2), then we have

$$A^p + \frac{1}{m^p} C(m^p, M^p, 2, 2) I \geq B^p \quad \text{for all } p > 0$$

and $\frac{1}{m^p} C(m^p, M^p, 2, 2) = \frac{(M^p - m^p)^2}{4m^p}$. Therefore we have the implication of (2) $\Rightarrow$ (1) by using Theorem A. □

The following corollary is a chaotic order version of Theorem B in some sense.

Corollary 3. *Let A and B be positive operators on H such that $MI \geq B \geq mI$ for some scalars $0 < m < M$. If $\log A \geq \log B$, then*

$$A^q + \frac{1}{m} C(m, M, p+1, q+1) I \geq B^p \quad \textit{for all } \ p,\ q > 0.$$

Proof. Put $r = 1$ in (2) of Theorem 2.

□

To obtain the following corollary, we need an estimate of $C(m, M, p, q)$ in [6, Lemma 2.3]:

Lemma 4. *Let $M > m > 0$, then*

$$M(M^{p-1} - m^{p-1}) \frac{M^{\frac{p-1}{q-1}} - m}{M - m} \geq C(m, M, p, q)$$

for all $p,\ q > 1$ such that

$$m \leq \left(\frac{M^p - m^p}{q(M - m)} \right)^{\frac{1}{p-1}} \leq M.$$

Corollary 5. *Let A and B be positive operators on H such that $MI \geq B \geq mI$ for some scalrs $M > m > 0$. If $\log A \geq \log B$, then*

$$A^q + \frac{(M^p - m^p)M^r\left(m^{-r}M^{\frac{rp}{q}} - 1\right)}{M^r - m^r}I \geq A^q + \frac{1}{m^r}C\left(m^r, M^r, \frac{r+p}{r}, \frac{r+q}{r}\right)I \geq B^p$$

for all $p,\ q,\ r > 0$ such that $m \leq \left(\frac{r(M^{p+r}-m^{p+r})}{(q+r)(M^r-m^r)}\right)^{\frac{1}{p}} \leq M$.

Proof. Put $M_1 = M^r, m_1 = m^r, p_1 = \frac{p+r}{r}$ and $q_1 = \frac{q+r}{r}$ in Lemma 4. Then we have the result directly from the following inequality.

$$\frac{1}{m^r}C\left(m^r, M^r, \frac{r+p}{r}, \frac{r+q}{r}\right) \leq \frac{1}{m^r}\frac{(M^p - m^p)M^r\left(M^{\frac{rp}{q}} - m^r\right)}{M^r - m^r}$$

if $m^r \leq \left(\frac{(M^r)^{\frac{p+r}{r}} - (m^r)^{\frac{p+r}{r}}}{\frac{q+r}{r}(M^r - m^r)}\right)^{\frac{1}{\frac{p+r}{r}-1}} \leq M^r$. □

3 Mond-Shisha difference.

We shall show the following characterization of the chaotic order in terms of a two parameters version of the Mond-Shisha difference.

Theorem 6. *Let A and B be positive operators on H such that $MI \geq B \geq mI$ for some scalars $0 < m < M$ and $h = \frac{M}{m}$. Then the following assertions are equivalent:*

(1) $\log A \geq \log B$,

(2) $A^q + \frac{p}{q}L(m^p, M^p)\log\left(m^{p-q}\frac{(h^p-1)h^{\frac{q}{h^p-1}}}{eq\log h}\right)I \geq B^p$ *for all $p,\ q > 0$,*

where $L(m, M)$ is the logarithmic mean.

Proof. (1) ⇒ (2). By Theorem 2, it follows that

$$A^q + \frac{1}{m^r}C(m^r, M^r, \frac{p+r}{r}, \frac{q+r}{r})I \geq B^p \quad \text{for all } p,\ q,\ r > 0. \tag{3.1}$$

Noting that

$$C(m, M, p, q) \leq \frac{Mm^p - mM^p}{M - m} + (q-1)\left(\frac{M^p - m^p}{q(M-m)}\right)^{\frac{q}{q-1}} \qquad \text{for all } p, q > 1$$

by definition of $C(m, M, p, q)$, we estimate the constant in (3.1):

$$\begin{aligned}
&\frac{1}{m^r}C(m^r, M^r, \frac{p+r}{r}, \frac{q+r}{r}) \\
&\leq \frac{1}{m^r}\left(\frac{M^r m^{p+r} - m^r M^{p+r}}{M^r - m^r} + \frac{q}{r}\left(\frac{r(M^{p+r} - m^{p+r})}{(q+r)(M^r - m^r)}\right)^{\frac{q+r}{q}}\right) \\
&= \frac{r}{q}\frac{M^r(M^p - m^p)}{M^r - m^r}\frac{q}{r}\left(-1 + \frac{q}{q+r}\frac{M^{p+r} - m^{p+r}}{m^r M^r(M^p - m^p)}\left(\frac{r(M^{p+r} - m^{p+r})}{(q+r)(M^r - m^r)}\right)^{\frac{r}{q}}\right) \\
&= \frac{r}{q}\frac{M^r(M^p - m^p)}{M^r - m^r}\frac{q}{r}\left(-1 + \left(\left(\frac{q}{q+r}\frac{M^{p+r} - m^{p+r}}{m^r M^r(M^p - m^p)}\right)^{\frac{q}{r}}\frac{r(M^{p+r} - m^{p+r})}{(q+r)(M^r - m^r)}\right)^{\frac{r}{q}}\right).
\end{aligned}$$

Put

$$F(r)=\left(\frac{q}{q+r}\frac{M^{p+r}-m^{p+r}}{m^rM^r(M^p-m^p)}\right)^{\frac{q}{r}}\frac{r(M^{p+r}-m^{p+r})}{(q+r)(M^r-m^r)}$$

and

$$f(x)=\log(h^{p+x}-1).$$

Since $\frac{M^{p+r}-m^{p+r}}{m^rM^r(M^p-m^p)}=\frac{h^{p+r}-1}{M^r(h^p-1)}$, we have

$$\lim_{r\to 0}\log\left(\frac{h^{p+r}-1}{M^r(h^p-1)}\right)^{\frac{1}{r}}=f'(0)-\log M=\frac{h^p\log h}{h^p-1}-\log mh$$
$$=\log\frac{1}{m}h^{\frac{1}{h^p-1}}.$$

Hence

$$\lim_{r\to 0}F(r)=\frac{1}{e}\frac{1}{m^q}h^{\frac{q}{h^p-1}}\frac{m^p(h^p-1)}{q\log h}=m^{p-q}\frac{(h^p-1)h^{\frac{q}{h^p-1}}}{eq\log h}$$

and this implies

$$\lim_{r\to 0}\frac{F(r)^{\frac{r}{q}}-1}{\frac{r}{q}}=\log\left(m^{p-q}\frac{(h^p-1)h^{\frac{q}{h^p-1}}}{eq\log h}\right).$$

Therefore, we have

$$\lim_{r\to 0}\frac{r}{q}\frac{M^r(M^p-m^p)}{M^r-m^r}\frac{q}{r}\left(-1+\left(\left(\frac{q}{q+r}\frac{M^{p+r}-m^{p+r}}{m^rM^r(M^p-m^p)}\right)^{\frac{q}{r}}\frac{r(M^{p+r}-m^{p+r})}{(q+r)(M^r-m^r)}\right)^{\frac{r}{q}}\right)$$
$$=\lim_{r\to 0}\frac{r}{q}\frac{M^r(M^p-m^p)}{M^r-m^r}\frac{F(r)^{\frac{r}{q}}-1}{\frac{r}{q}}$$
$$=\frac{1}{q}\frac{M^p-m^p}{\log M-\log m}\log\left(m^{p-q}\frac{(h^p-1)h^{\frac{q}{h^p-1}}}{eq\log h}\right).$$

By (3.1), we have the desired inequality

$$A^q+\frac{1}{q}\frac{M^p-m^p}{\log M-\log m}\log\left(m^{p-q}\frac{(h^p-1)h^{\frac{q}{h^p-1}}}{eq\log h}\right)I\geq B^p$$

for all $p,q>0$.

(2) $\Rightarrow$ (1). If $p=q$ in (2), then $m^{p-q}\frac{(h^p-1)h^{\frac{q}{h^p-1}}}{eq\log h}$ just coinsides with the generalized Specht ratio $S_h(p)$ and hence we have

$$A^p+L(m^p,M^p)\log S_h(p)I\geq B^p\quad\text{ for all }\ p>0.$$

Thus it follows from Theorem A. □

Remark 1. We recall the generaized Specht ratio with two parameters in [4]:

$$S_h(p,q)=\begin{cases}m^{p-q}\frac{(h^p-1)h^{\frac{q}{h^p-1}}}{eq\log h} & \text{if } q\leq\frac{h^p-1}{\log h}\leq qh^p,\\ m^{p-q} & \text{if } \frac{h^p-1}{\log h}<q,\\ M^{p-q} & \text{if } qh^p<\frac{h^p-1}{\log h}.\end{cases}$$

Therefore, Theorem 6 is regarded as a characterization of the chaotic order due to a two parametres version of the Mond-Shisha difference.

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