

EVOLUTIONARILY STABLE STRATEGIES BASED ON BAYESIAN GAMES

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Received November 2, 2010

ABSTRACT. Evolutionarily stable strategies(ESSs) of asymmetric animal contests are formulated by using Bayesian games. A criterion which is a generalization of that of Abakukus([1]) and a strengthened Selten's theorem([9]) are given. As an example, a Hawk-Dove game with roles, in which the animals having the same role may possibly meet each other, is investigated and all possible Nash equilibria and ESSs are obtained for given parameters.

1 Introduction

In his famous book(1982, [7]), Maynard-Smith has discussed asymmetric games and their evolutionarily stable strategies between animal contests as normal form games, where the payoff matrix between strategies is obtained from the situation explained intuitively. In fact, his payoff matrix, Table 13(p.101) about the Hawk-Dove-Bourgeois game involves some miscalculations, and they are corrected in the Japanese book translated by Teramoto-Kakehasi published in 1985.

Weibull(1995, [14]) has also discussed the same situations as role conditioned behaviors using extensive form games and gives a simple proof of Selten's theorem(1980, [9]).

In this paper, we will characterize ESS of asymmetric animal contests by using Bayesian games due to Harsanyi(1967-68, [3]) and generalize Abakukus's criterion(1980, [1]) of ESS for normal form games to that for Bayesian games and strengthen Selten's theorem. Finally, we shall discuss a Hawk-Dove game with roles based on Bayesian games and its ESS under more general setting than that of Maynard-Smith, where Selten's theorem is not applicable in general and give all possible Nash equilibria and ESSs.

In addition, we remark the following:

1. In his book(pp.25–31), Vega-Redondo(1996, [13]) has also given an equivalent formulation with ours, but he discusses only the case where animals with the same role never meet each other. In case of 'owner' and 'intruder', they may naturally never meet themselves but in case of 'old' and 'young', 'small' and 'large', or 'male' and 'female', they may possibly meet themselves.

Taylor(1979, [12]) has also discussed ESS with two types of player, but his formulation is a little bit different from ours.

2. Selten(1983, [10], pp.278–280) has also discussed a Hawk-Dove game with incomplete information. However his framework is a little bit different from ours. In fact, his 'possessor' knows the value 'good' or 'bad' of the territory, but the 'possessor' or 'intruder' never meet themselves. Moreover, his symmetrization of asymmetric animal conflicts using extensive form games seems to be unnatural.

3. Abakukus(1980, [1]) corrected the original theorem on a criterion for ESS due to Haigh(1975, [2]) whose statement was inadequate and also his proof is valid only in the

2000 *Mathematics Subject Classification.* 91A22, 91A05, 91A10.

Key words and phrases. Hawk-dove game, Games with incomplete information. Asymmetric animal conflicts .

restricted case. Hines's necessary and sufficient condition (1980, [4], p.336) is assumed to be mixed ESSs with full support. In this case Haigh's proof is validated.

2 Formulation

In this section, we reexamine two-player Bayesian games and their symmetric versions for our purpose. First, let us list up some notation:

T (Type space) : a finite set of player's types, say, male or female, large size or small size, owner or intruder and so on, but not necessarily two types. To avoid a trivial case, we assume $|T| \geq 2$, where $|T|$ means the cardinal number of the set T .

S (Strategy space) : a finite set of strategies for all players. We also assume $|S| \geq 2$.

$u_n(s_1, s_2; t_1, t_2)$ (Payoff for player n , $n=1,2$) : a real valued function defined on $S \times S \times T \times T$.

$\mathcal{P}(M)$ denotes the set of all probability distributions on a finite set M .

We assume that at the first stage nature gives a probability distribution $\{p(t_1, t_2)\} \in \mathcal{P}(T \times T)$. Here, we always assume that for all $t \in T$, $\sum_{t_1 \in T} p(t_1, t)$ and $\sum_{t_2 \in T} p(t, t_2)$ are not zero.

We call a family of probability distributions of $\mathcal{P}(S)$: $X := (\{x(s/t)\}_{s \in S}; t \in T)$ a behavior strategy¹ of players and the set of all behavior strategies is denoted by \mathcal{B} . That is, each player chooses an element from \mathcal{B} at the same time after the first stage.

The expected payoff $\bar{u}_n(X_1, X_2)$ for player n when player n adopts a strategy X_n is defined by

$$\bar{u}_n(X_1, X_2) = \sum_{t_1 \in T} \sum_{t_2 \in T} \sum_{s_1 \in S} \sum_{s_2 \in S} u_n(s_1, s_2; t_1, t_2) p(t_1, t_2) x_1(s_1/t_1) x_2(s_2/t_2).$$

When the above framework are given, we call $\Gamma = (T, S, u_n, p; n = 1, 2)$ a (two player) Bayesian game.

Definition 1. A behavior strategy profile $(X_1^*, X_2^*) \in \mathcal{B} \times \mathcal{B}$ is called a Bayesian Nash equilibrium if and only if

$$(i) \forall X_1 \in \mathcal{B}, \bar{u}_1(X_1^*, X_2^*) \geq \bar{u}_1(X_1, X_2^*),$$

and

$$(ii) \forall X_2 \in \mathcal{B}, \bar{u}_2(X_1^*, X_2^*) \geq \bar{u}_2(X_1^*, X_2),$$

hold.

Now in order to define ESS under our framework, we introduce a symmetric Bayesian game.

Definition 2. A Bayesian game $\Gamma = (T, S, u_n, p; n = 1, 2)$ is called symmetric if and only if

$$(i) \forall (s_n, t_n) \in S \times T, n = 1, 2, u_2(s_1, s_2; t_1, t_2) = u_1(s_2, s_1; t_2, t_1),$$

and

$$(ii) \forall (t_1, t_2) \in T \times T, p(t_1, t_2) = p(t_2, t_1)$$

hold.

We can easily verify that for a symmetric Bayesian game, $\bar{u}_2(X_1, X_2) = \bar{u}_1(X_2, X_1)$ holds for all $(X_1, X_2) \in \mathcal{B} \times \mathcal{B}$.

¹A behavior strategy can be represented by one random variable. See Kōno([5], [6]).

Now we can define ESS for a symmetric Bayesian game $\Gamma = (T, S, u_n, p; n = 1, 2)$ as follows:

Definition 3. A behavior strategy $X^* \in \mathcal{B}$ is said to be an ESS if and only if

(i) $\forall X \in \mathcal{B}, \bar{u}_1(X^*, X^*) \geq \bar{u}_1(X, X^*),$

and

(ii) if $\bar{u}_1(X^*, X^*) = \bar{u}_1(X, X^*)$ and $X \neq X^*$, then $\bar{u}_1(X^*, X) > \bar{u}_1(X, X)$

hold.

We observe that if a behavior strategy $X^* \in \mathcal{B}$ is an ESS, then condition (i) of Definition 3 implies that the behavior strategy profile (X^*, X^*) is a Bayesian Nash equilibrium. In this case, we call a behavior strategy X^* a symmetric Bayesian Nash equilibrium or simply a Bayesian Nash equilibrium if no confusion arises.

In the following discussions, we need a refinement of the notion of a mixed strategy, which is a little bit different from the usual one.

We denote the support of a probability distribution $\{x(s)\}_{s \in S} \in \mathcal{P}(S)$ by $Supp(\{x(s)\})$, that is $Supp(\{x(s)\}) := \{s \in S; x(s) > 0\}$.

Definition 4. A behavior strategy $X = (\{x(s/t)\}_{s \in S}; t \in T)$ is called a **t-pure behavior strategy** if $|Supp(\{x(s/t)\})| = 1$, a **t-mixed behavior strategy** if $|Supp(\{x(s/t)\})| \geq 2$, a **totally pure behavior strategy** if it is a t-pure behavior strategy for all t. We observe that our t-mixed behavior strategy is different from a mixed strategy in an extensive form game but our totally pure behavior strategy coincides with a pure strategy in an extensive form game.

Now, we have the following lemma equivalent to Definition 1.

Lemma 1. A behavior strategy profile $(X_1^* = (\{x_1^*(s/t)\}_{s \in S}; t \in T), X_2^* = (\{x_2^*(s/t)\}_{s \in S}; t \in T))$ is a Bayesian Nash equilibrium if and only if

(i) $\forall t \in T, \forall \{x_1(s)\}_{s \in S} \in \mathcal{P}(S) \text{ C}$

$$\sum_{s_1 \in S} \sum_{s_2 \in S} \sum_{t_2 \in T} u_1(s_1, s_2; t, t_2)(x_1^*(s_1/t) - x_1(s_1))x_2^*(s_2/t_2)p(t, t_2) \geq 0,$$

and

(ii) $\forall t \in T, \forall \{x_2(s)\}_{s \in S} \in \mathcal{P}(S) \text{ C}$

$$\sum_{s_1 \in S} \sum_{s_2 \in S} \sum_{t_1 \in T} u_2(s_1, s_2; t_1, t)x_1^*(s_1/t_1)(x_2^*(s_2/t) - x_2(s_2))p(t_1, t) \geq 0$$

hold.

The following lemma is well known in case of normal form games.

For a behavior strategy $X = (\{x(s/t)\}_{s \in S}; t \in T)$, set

$$\begin{aligned} S_1(X/t) &:= \{s \in S; \max_{i \in S} \sum_{s_2 \in S} \sum_{t_2 \in T} u_1(i, s_2; t, t_2)x(s_2/t_2)p(t, t_2) \\ &= \sum_{s_2 \in S} \sum_{t_2 \in T} u_1(s, s_2; t, t_2)x(s_2/t_2)p(t, t_2)\}, \end{aligned}$$

and

$$\begin{aligned} S_2(X/t) &:= \{s \in S; \max_{j \in S} \sum_{s_1 \in S} \sum_{t_1 \in T} u_2(s_1, j; t_1, t) x(s_1/t_1) p(t_1, t) \\ &= \sum_{s_1 \in S} \sum_{t_1 \in T} u_2(s_1, s; t_1, t) x(s_1/t_1) p(t_1, t)\}. \end{aligned}$$

Lemma 2. A behavior strategy profile $(X_1^* = (\{x_1^*(s/t)\}_{s \in S}; t \in T), X_2^* = (\{x_2^*(s/t)\}_{s \in S}; t \in T))$ is a Bayesian Nash equilibrium if and only if

$$(i) \quad \forall t \in T, \text{Supp}(\{x_1^*(s/t)\}) \subset S_1(X_2^*/t),$$

and

$$(ii) \quad \forall t \in T, \text{Supp}(\{x_2^*(s/t)\}) \subset S_2(X_1^*/t)$$

hold.

Proof. First, we assume that (i) of Lemma 2 holds. Then, by the definition of $S_1(X_2^*/t)$, for all $j \in S$ and $t \in T$, we have

$$\begin{aligned} \max_{i \in S} \sum_{s_2 \in S} \sum_{t_2 \in T} u_1(i, s_2; t, t_2) x_2^*(s_2/t_2) p(t, t_2) \\ = \sum_{s_1 \in S} \sum_{s_2 \in S} \sum_{t_2 \in T} u_1(s_1, s_2; t, t_2) x_1^*(s_1/t) x_2^*(s_2/t_2) p(t, t_2) \\ \geq \sum_{s_2 \in S} \sum_{t_2 \in T} u_1(j, s_2; t, t_2) x_2^*(s_2/t_2) p(t, t_2). \end{aligned}$$

Here, multiply the both sides of the above inequality by $x_1(j)$ of any probability distribution $\{x_1(j)\}_{j \in S} \in \mathcal{P}(S)$ and sum with respect to j . Then, (i) of Lemma 1 holds by virtue of $\sum_{j \in S} x_1(j) = 1$.

Conversely, if there exists $t_1 \in T$ such that $s_{10} \notin S_1(X_2^*/t_1)$ and $x_1^*(s_{10}/t_1) > 0$, then by the definition of $S_1(X_2^*/t_1)$, we have

$$\max_{i \in S} \sum_{s_2 \in S} \sum_{t_2 \in T} u_1(i, s_2; t_1, t_2) x_2^*(s_2/t_2) p(t_1, t_2) > \sum_{s_2 \in S} \sum_{t_2 \in T} u_1(s_{10}, s_2; t_1, t_2) x_2^*(s_2/t_2) p(t_1, t_2).$$

Therefore, for any $\{x_1(s)\}_{s \in S} \in \mathcal{P}(S)$ such that $\text{Supp}(\{x_1(s)\}) \subset S_1(X_2^*/t_1)$, we have the following estimations:

$$\begin{aligned} & \sum_{s_1 \in S} \sum_{s_2 \in S} \sum_{t_2 \in T} u_1(s_1, s_2; t_1, t_2) x_1^*(s_1/t_1) x_2^*(s_2/t_2) p(t_1, t_2) \\ = & \sum_{s_1 \in S, s_1 \neq s_{10}} \sum_{s_2 \in S} \sum_{t_2 \in T} u_1(s_1, s_2; t_1, t_2) x_1^*(s_1/t_1) x_2^*(s_2/t_2) p(t_1, t_2) \\ & + \sum_{s_2 \in S} \sum_{t_2 \in T} u_1(s_{10}, s_2; t_1, t_2) x_1^*(s_{10}/t_1) x_2^*(s_2/t_2) p(t_1, t_2) \\ & < \max_{i \in S} \sum_{s_2 \in S} \sum_{t_2 \in T} u_1(i, s_2; t_1, t_2) x_2^*(s_2/t_2) p(t_1, t_2) \\ = & \sum_{s_1 \in S} \sum_{s_2 \in S} \sum_{t_2 \in T} u_1(s_1, s_2; t_1, t_2) x_1(s_1) x_2^*(s_2/t_2) p(t_1, t_2). \end{aligned}$$

But, this inequality contradicts condition (i) of Lemma 1. In a similar manner, we have the equivalence between (ii) of Lemma 1 and (ii) of Lemma 2. (Q.E.D.)

By rewriting Definition 3, we have the following lemma.

Lemma 3. A behavior strategy $X^* = (\{x^*(s/t)\}_{s \in S}; t \in T)$ is an ESS if and only if
 (i) the behavior strategy X^* is a Bayesian Nash equilibrium, and
 (ii) for all behavior strategy $(\{x(s/t)\}_{s \in S}; t \in T) \neq (\{x^*(s/t)\}_{s \in S}; t \in T)$ such that

$$(1) \quad \forall t_1 \in T, \sum_{s_1 \in S} \sum_{s_2 \in S} \sum_{t_2 \in T} u_1(s_1, s_2; t_1, t_2)(x^*(s_1/t_1) - x(s_1/t_1))x^*(s_2/t_2)p(t_1, t_2) = 0,$$

the following strict inequality holds:

$$(2) \quad \sum_{s_1 \in S} \sum_{s_2 \in S} \sum_{t_1 \in T} \sum_{t_2 \in T} u_1(s_1, s_2; t_1, t_2)(x^*(s_1/t_1) - x(s_1/t_1))x(s_2/t_2)p(t_1, t_2) > 0.$$

Definition 5. In a symmetric Bayesian game, a Bayesian Nash equilibrium $X^* = (\{x^*(s/t)\}_{s \in S}; t \in T)$ is called a **t-strong Bayesian Nash equilibrium** if and only if for $\forall \{x(s)\}_{s \in S} \in \mathcal{P}(S) \neq \{x^*(s/t)\}_{s \in S}$,

$$(3) \quad \sum_{s_1 \in S} \sum_{s_2 \in S} \sum_{t_2 \in T} u_1(s_1, s_2; t, t_2)(x^*(s_1/t) - x(s_1))x^*(s_2/t_2)p(t, t_2) > 0$$

holds. If it is a t-strong Bayesian Nash equilibrium for all $t \in T$, then it is simply called a **strong Bayesian Nash equilibrium**.

Remark. (i) By virtue of Lemma 2, a Bayesian Nash equilibrium $X^* = (\{x^*(s/t)\}_{s \in S}, t \in T)$ is t-strong if and only if $|S_1(X^*/t)| = 1$ holds.

(ii) A strong Bayesian Nash equilibrium is a totally pure behavior strategy and an ESS, since (ii) of Lemma 3 holds trivially.

(iii) A Bayesian Nash equilibrium which is a t-pure behavior strategy is not necessarily a t-strong Bayesian Nash equilibrium.

3 Theorems

The following theorem can be obtained immediately from the definition of ESS and Lemma 2.

Theorem 1. If a behavior strategy $X^* = (\{x^*(s/t)\}_{s \in S}; t \in T)$ is an ESS, then any other behavior strategy $X = (\{x(s/t)\}_{s \in S}; t \in T) \neq X^*$ such that $Supp(\{x(s/t)\}) \subset S_1(X^*/t)$ hold for all $t \in T$ is not a Bayesian Nash equilibrium.

Corollary. If a behavior strategy $X^* = (\{x^*(s/t)\}_{s \in S}; t \in T)$ such that $Supp(\{x^*(s/t)\}) = S$ hold for all $t \in T$ is an ESS, then this behavior strategy is only one symmetric Bayesian Nash equilibrium in this game.

Now, in order to obtain an criterion whether a Bayesian Nash equilibrium $X^* = (\{x^*(s/t)\}_{s \in S}; t \in T)$ is an ESS or not, let us continue an analysis of Lemma 3. By taking account of Lemma 2, the equation (1) holds if and only if $\forall t \in T, Supp(\{x(s/t)\}) \subset S_1(X^*/t)$, and $\{x(s/t)\}$ can be different from $\{x^*(s/t)\}$ if and only if $|S_1(X^*/t)| \geq 2$. Therefore, setting $T_0 := \{t; |S_1(X^*/t)| \geq 2\}$, and subtracting (2) from (1), we have

$$(4) \quad \sum_{s_1 \in S} \sum_{s_2 \in S} \sum_{t_1 \in T_0} \sum_{t_2 \in T_0} u_1(s_1, s_2; t_1, t_2) \times \\ (x^*(s_1/t_1) - x(s_1/t_1))(x^*(s_2/t_2) - x(s_2/t_2))p(t_1, t_2) < 0.$$

Now, choose an element $s_t^{(0)}$ from $Supp(\{x^*(s/t)\})$ and set

$$S_0(X^*/t) := S_1(X^*/t) - \{s_t^{(0)}\}.$$

By using the relations

$$x^*(s_t^{(0)}/t) = 1 - \sum_{s \in S_0(X^*/t)} x^*(s/t), \text{ and } x(s_t^{(0)}/t) = 1 - \sum_{s \in S_0(X^*/t)} x(s/t),$$

formula (4) turns out to be the following:

$$\sum_{t_1 \in T_0} \sum_{t_2 \in T_0} \sum_{s_1 \in S_0(X^*/t_1)} \sum_{s_2 \in S_0(X^*/t_2)} (u_1(s_1, s_2; t_1, t_2) - u_1(s_{t_1}^{(0)}, s_2; t_1, t_2) \\ - u_1(s_1, s_{t_2}^{(0)}; t_1, t_2) + u_1(s_{t_1}^{(0)}, s_{t_2}^{(0)}; t_1, t_2))p(t_1, t_2) \times \\ (5) \quad (x^*(s_1/t_1) - x(s_1/t_1))(x^*(s_2/t_2) - x(s_2/t_2)) < 0.$$

Here, we introduce a square matrix $U(X^*) = (u((s_1, t_1), (s_2, t_2)))$ of order $d := \sum_{t \in T_0} |S_0(X^*/t)|$, where the entry $u((s_1, t_1), (s_2, t_2)); t_n \in T_0, s_n \in S_0(X^*/t_n), n = 1, 2$ is defined as follows:

$$u((s_1, t_1), (s_2, t_2)) := (-u_1(s_1, s_2; t_1, t_2) + u_1(s_1, s_{t_2}^{(0)}; t_1, t_2) \\ + u_1(s_{t_1}^{(0)}, s_2; t_1, t_2) - u_1(s_{t_1}^{(0)}, s_{t_2}^{(0)}; t_1, t_2))p(t_1, t_2); \\ (6) \quad t_n \in T_0, s_n \in S_0(X^*/t_n), n = 1, 2.$$

We denote also symmetrized matrix of $U(X^*)$ by $U^s(X^*)$, i.e. $U^s(X^*) = (U(X^*) + {}^tU(X^*)/2)$, where tA means transposed matrix of A and let R^d be d -dimensional Euclidean space with the dot product (\cdot, \cdot) as the usual inner product. Then, setting $\vec{y} = \{x^*(s/t) - x(s/t)\} \in R^d$, the formula (5) turns out to be the following:

$$(7) \quad (U(X^*)\vec{y}, \vec{y}) = (U^s(X^*)\vec{y}, \vec{y}) > 0.$$

Then, we have the following theorem on criteria of ESS when a behavior strategy $X^* = (\{x^*(s/t)\}_{s \in S}; t \in T)$ is a Bayesian Nash equilibrium. This theorem is a generalization of that of Abakuks(1980, [1]) to Bayesian games.

Theorem 2.

- (i) If X^* is a strong Bayesian Nash equilibrium, then it is an ESS.
- (ii) If the matrix $U^s(X^*)$ is positive definite, then a Bayesian Nash equilibrium X^* is an ESS.
- (iii) If for all $t \in T_0, |S_1(X^*/t)| = |Supp(\{x^*(s/t)\})|$ holds with at most one point exception t_0 , where $|S_1(X^*/t_0)| = |Supp(\{x^*(s/t_0)\})| + 1$ holds and the behavior strategy X^* is an ESS, then the matrix $U^s(X^*)$ is positive definite².

²The assumption can not be relaxed due to Abakuks([1])'s counter example.

Proof. (i) If a behavior strategy $X^* = (\{x^*(s/t)\}_{s \in S}; t \in T)$ is a strong Bayesian Nash equilibrium, then by Lemma 2, there does not exist such behavior strategies $(\{x(s/t)\}_{s \in S} \in \mathcal{P}(S); t \in T) \neq (\{x^*(s/t)\}_{s \in S}, t \in T)$ that equation (1) holds. Therefore, it is an ESS by definition.

(ii) By the definition of positive definiteness of the matrix $U^s(X^*)$, formula (7) holds for any non-zero vector $\vec{y} \in R^d$.

(iii) Take any non-zero vector $\vec{y} = \{y(s/t)\}_{s \in S_0(X^*/t), t \in T_0} \in R^d$ and set

$$\epsilon = \min_{s \in \text{Supp}(\{x^*(s/t)\}), t \in T_0} \{x^*(s/t)\}.$$

We observe that $\epsilon > 0$. If the exceptional point t_0 exists, then there exists only one point $s_{t_0}^{(1)} = S_1(X^*/t_0) - \text{Supp}(\{x^*(s/t_0)\})$ by the assumption and set

$$(8) \quad \text{sig}\{\vec{y}\} := \begin{cases} 1 & \text{if } y(s_{t_0}^{(1)}/t_0) \geq 0 \text{ or there does not exist such } t_0, \\ -1 & \text{if } y(s_{t_0}^{(1)}/t_0) < 0. \end{cases}$$

Finally, set

$$M := \max_{t \in T_0} \sum_{s \in S_0(X^*/t)} |y(s/t)|,$$

and

$$x(s/t) := x^*(s/t) - (\text{sig}\{\vec{y}\}\epsilon/M)y(s/t); t \in T_0, s \in S_0(X^*/t).$$

By our setting of M , it follows that $x(s/t) \geq 0$ holds for any $t \in T_0, s \in S_0(X^*/t)$ and

$$x(s_t^{(0)}) := 1 - \sum_{s \in S_0(X^*/t)} x(s/t) \geq 1 - \sum_{s \in S_0(X^*/t)} x^*(s/t) - \epsilon = x^*(s_t^{(0)}/t) - \epsilon \geq 0.$$

Therefore, substituting $x^*(s/t) - x(s/t)$ to the formula (7), we have

$$(\text{sig}\{\vec{y}\}\epsilon/M)^2(U^s(X^*)\vec{y}, \vec{y}) > 0.$$

This inequality shows that the matrix $U^s(X^*)$ is positive definite. (Q.E.D.)

The following theorem strengthens that of Selten(1980, [9]).

Theorem 3. If $p(t_0, t_0) = 0$, then ESS implies t_0 -strong Bayesian Nash equilibrium.

Corollary.(Selten [9]) If $p(t,t) = 0$ holds for all $t \in T$, then ESS implies a strong Bayesian Nash equilibrium³.

Proof. Suppose that $|S_1(X^*/t_0)| \geq 2$ holds, that is, X^* is not t_0 -strong Bayesian Nash equilibrium but a Bayesian Nash equilibrium. Then, from $|S_1(X^*/t_0)| \geq 2$ and Lemma 2, there exists a behavior strategy $\{x(s/t_0)\}_{s \in S_1(X^*/t_0)} \in \mathcal{P}(S_1(X^*/t_0))$ such that $\{x(s/t_0)\}_{s \in S_1(X^*/t_0)} \neq \{x^*(s/t_0)\}_{s \in S_1(X^*/t_0)}$, and

$$\sum_{s_1 \in S} \sum_{s_2 \in S} \sum_{t_2 \in T} u_1(s_1, s_2; t_0, t_2)x^*(s_1/t_0)x^*(s_2/t_2)p(t_0, t_2) =$$

³As for Selten's statement, it is not clear the difference between a strong Bayesian Nash equilibrium and a totally pure behavior strategy which is a Bayesian Nash equilibrium.

$$(9) \quad \sum_{s_1 \in S} \sum_{s_2 \in S} \sum_{t_2 \in T} u_1(s_1, s_2; t_0, t_2) x(s_1/t_0) x^*(s_2/t_2) p(t_0, t_2)$$

holds.

Now, we define a new behavior strategy $Y = (\{y(s/t)\}_{s \in S}; t \in T)$ as follows:

$$y(s/t) = \begin{cases} x^*(s/t) & \text{if } t \neq t_0, \\ x(s/t_0) & \text{if } t = t_0 \end{cases}.$$

Then, taking account of $p(t_0, t_0) = 0$, we have

$$\begin{aligned} \bar{u}_1(Y, Y) &= \sum_{t_1 \in T} \sum_{t_2 \in T} \sum_{s_1 \in S} \sum_{s_2 \in S} u_1(s_1, s_2; t_1, t_2) y(s_1/t_1) y(s_2/t_2) p(t_1, t_2) \\ &= \sum_{s_1 \in S} \sum_{s_2 \in S} \left(\sum_{t_1 \neq t_0} \sum_{t_2 \neq t_0} u_1(s_1, s_2; t_1, t_2) x^*(s_1/t_1) y(s_2/t_2) p(t_1, t_2) \right) \\ &\quad + \sum_{t_2 \neq t_0} u_1(s_1, s_2; t_0, t_2) x(s_1/t_0) x^*(s_2/t_2) p(t_0, t_2) \\ &\quad + \sum_{t_1 \neq t_0} u_1(s_1, s_2; t_1, t_0) x^*(s_1/t_1) y(s_2/t_0) p(t_1, t_0) \\ &\quad \text{(by virtue of (9))} \\ &= \sum_{s_1 \in S} \sum_{s_2 \in S} \left(\sum_{t_1 \neq t_0} \sum_{t_2 \neq t_0} u_1(s_1, s_2; t_1, t_2) x^*(s_1/t_1) y(s_2/t_2) p(t_1, t_2) \right) \\ &\quad + \sum_{t_2 \neq t_0} u_1(s_1, s_2; t_0, t_2) x^*(s_1/t_0) y(s_2/t_2) p(t_0, t_2) \\ &\quad + \sum_{t_1 \neq t_0} u_1(s_1, s_0; t_1, t_0) x^*(s_1/t_1) y(s_2/t_0) p(t_1, t_0) \\ &= \bar{u}_1(X^*, Y), \end{aligned}$$

which implies that the Bayesian Nash equilibrium $X^* = (\{x^*(s/t)\}_{s \in S}, t \in T)$ is not an ESS. (Q.E.D.)

4 Bayesian Hawk-Dove Game

As an example of our Bayesian game described above, let us reformulate Maynard-Smith's Hawk-Dove game with roles discussed in his book ([7], §8).

Type Space : $T = \{1, 2\}$. {owner, intruder} or {larger, smaller} for examples. $\{p(k, m)\}_{(k, m) \in T \times T}$: a probability distribution on $T \times T$ given in advance by nature. We assume the following two conditions:

- (i) $p(1, 2) > 0$ (non-triviality), and
- (ii) $p(1, 2) = p(2, 1)$ (symmetricity).

Since any probability distribution satisfying the above conditions can be characterized by two parameters, we introduce two parameters q_1 and q_2 defined as follows:

$$q_1 = \frac{p(1, 1)}{p(1, 1) + p(1, 2)}, \quad q_2 = \frac{p(2, 1)}{p(2, 1) + p(2, 2)}.$$

We remark that non-triviality and symmetricity of $\{p(k, m)\}$ imply $0 \leq q_1 < 1$ and $0 < q_2 \leq 1$.

Strategy Space : $S = \{1, 2\} = \{H(\text{awk}), D(\text{ove})\}$

Payoff Matrices : Payoff matrices $(u_1(i, j; k, m), u_2(i, j; k, m))_{(i,j) \in S \times S}$ for the type $(k, m) \in T \times T$ are defined as follows: Here, Player 1 picks the row, while Player 2 selects the column. The element on the left side in the parenthesis represents Player 1's payoff and that on the right side denotes Player 2's payoff. Here, we assume⁴ $V, v, C, c > 0$.

		H	D
¶ Type (1, 1)F	H	$((V - C)/2, (V - C)/2)$	$(V, 0)$
	D	$(0, V)$	$(V/2, V/2)$
		H	D
¶ Type (1, 2)F	H	$((V - C)/2, (v - c)/2)$	$(V, 0)$
	D	$(0, v)$	$(V/2, v/2)$
		H	D
¶ Type (2, 1)F	H	$((v - c)/2, (V - C)/2)$	$(v, 0)$
	D	$(0, V)$	$(v/2, V/2)$
		H	D
¶ Type (2, 2)F	H	$((v - c)/2, (v - c)/2)$	$(v, 0)$
	D	$(0, v)$	$(v/2, v/2)$

Table 1. Payoff matrices of Bayesian Hawk-Dove game

Since any behavior strategy $(\{x(s/t)\}_{s \in S}; t \in T)$ can be determined by $x(1/1)$ and $x(1/2)$, we represent this behavior strategy by $(x(1/1), x(1/2))$. Then, Maynard-Smith's "Bourgeois" strategy, i.e. if owner, play Hawk; if intruder, play Dove is represented by $(x(1/1), x(1/2)) = (1, 0)$. We denote Bourgeois strategy by "B" strategy. Likewise, we introduce "Anti-Bourgeois" strategy, i.e. if intruder, play Hawk; if owner, play Dove. This strategy is represented by $(x(1/1), x(1/2)) = (0, 1)$. We denote Anti-Bourgeois strategy by "A" strategy. If he/she plays always Hawk, it is represented by $(x(1/1), x(1/2)) = (1, 1)$ and we denote it by "H" strategy. Finally, If he/she plays always Dove, it is represented by $(x(1/1), x(1/2)) = (0, 0)$ and we denote it by "D" strategy. We remark that the utility table $u_1(X, Y)$ calculated by our formulation, where $X, Y = "H", "D", "B", "A"$ under the assumption $p(1, 2) = p(2, 1) = 1/2$ coincides with that of Table 13 in page 110 of the Japanese translation of Maynard-Smith's book. But this result is slightly different from his original table 13 in page 101 of his book.

As is shown below, the mathematical structure is completely determined by five parameters, $0 \leq q_1 < 1, 0 < q_2 \leq 1, r_1 := V/C > 0, r_2 := v/c > 0$ and $\gamma := C/c > 0$. Bayesian Nash equilibria are determined by the first four parameters q_1, q_2, r_1 and r_2 , but to determine whether they are ESS or not, the parameter γ is required. We will give a complete classification using these parameters when a behavior strategy $(x^*(1/1), x^*(1/2))$ is a Bayesian Nash equilibrium or an ESS.

The first proposition is about when a Bayesian Nash equilibrium $X^* = (x^*(1/1), x^*(1/2))$ is an ESS or not.

⁴Maynard-Smith has assumed that $V > v > 0$ and $C = c > 0$, but we investigate them under more general setting.

Proposition 1.

(i) Assume $|S_1(X^*/1)| = |S_1(X^*/2)| = 1$. Then, the Bayesian Nash equilibrium X^* is an ESS

(ii) Assume $|S_1(X^*/1)| = 1$ and $|S_1(X^*/2)| = 2$. Then, the Bayesian Nash equilibrium X^* is an ESS if and only if $p(2, 2) > 0$ ($\Leftrightarrow (1 - q_2) > 0$).

(iii) Assume $|S_1(X^*/1)| = 2$ and $|S_1(X^*/2)| = 1$. Then, the Bayesian Nash equilibrium X^* is an ESS if and only if $p(1, 1) > 0$ ($\Leftrightarrow q_1 > 0$).

(iv) Assume $|S_1(X^*/1)| = |S_1(X^*/2)| = 2$, and $|Supp(\{x^*(s/1)\})| = 2$ or $|Supp(\{x^*(s/2)\})| = 2$. Then, the Bayesian Nash equilibrium $X^* = (x^*(1/1), x^*(1/2))$ is an ESS if and only $0 < q_1, q_2 < 1$ and $q_2 < g(q_1)$, where

$$g(q_1) = \frac{4\gamma q_1}{4\gamma + (\gamma - 1)^2(1 - q_1)}.$$

(v) Assume $|S_1(X^*/1)| = |S_1(X^*/2)| = 2$ and $|Supp(\{x^*(s/1)\})| = |Supp(\{x^*(s/2)\})| = 1$. Then, the Bayesian Nash equilibrium is possibly one of four totally pure strategies. In case of “H” or “D” strategy, it is an ESS if and only if $0 < q_1, q_2 < 1$. On the other hand in case of “B” or “A” strategy, it is an ESS if and only if $0 < q_1, q_2 < 1$ and $q_2 < g(q_1)$.

Proof. Theorem 2 is applicable to the cases (i),(ii),(iii) and (iv), but case (v) is obtained directly from the estimation of (5).

(i) In this case, X^* is a strong Bayesian Nash equilibrium. Therefore, it is an ESS by (i) of Theorem 2.

(ii) In this case, $T_0 = \{2\}$ and we can choose $s_2^{(0)} = 1$ or $s_2^{(0)} = 2$. In any case, from the definition of $U(X^*)$, or (6), it is enough to check $d = 1$ dimensional matrix $U(X^*) = u((2, 2), (2, 2))$ or $U(X^*) = u((1, 2), (1, 2))$, but both values coincide and we have

$$\begin{aligned} u((1, 2), (1, 2)) &= u((2, 2), (2, 2)) \\ &= (-u_1(1, 1; 2, 2) + u_1(1, 2; 2, 2) + u_1(2, 1; 2, 2) - u_1(2, 2; 2, 2))p(2, 2). \\ &= cp(2, 2)/2. \end{aligned}$$

Therefore, X^* is an ESS if and only if $p(2, 2) > 0$. We can prove (iii) in a similar manner.

(iv) In this case, $T_0 = \{1, 2\}$ and we can choose $s_1^{(0)} = s_2^{(0)} = 1$ or $s_1^{(0)} = s_2^{(0)} = 2$. In any case, we have the same matrix $U(x^*) = (u(k, m))_{k,m=1,2}$:

$$\begin{aligned} u(k, m) &= (-u_1(1, 1; k, m) + u_1(1, 2; k, m) \\ &\quad + u_1(2, 1; k, m) - u_1(2, 2; k, m))p(k, m), \quad k, m = 1, 2. \end{aligned}$$

By calculation, we get

$$U^s(X^*) = \frac{1}{4} \begin{pmatrix} 2Cp(1, 1) & (C + c)p(1, 2) \\ (C + c)p(2, 1) & 2cp(2, 2) \end{pmatrix}.$$

Therefore, $U^s(X^*)$ is positive definite if and only if $p(1, 1) > 0, p(2, 2) > 0$ and $4det(U^s(X^*)) = 4Ccp(1, 1)p(2, 2) - (C + c)^2(p(1, 2))^2 > 0$. By using q_1, q_2 and γ , we have the equivalent conditions $0 < q_1, q_2 < 1$ and $q_2 < g(q_1)$.

(v) In this case, we can not apply Theorem 2. From the assumption $|Supp(\{x^*(s/1)\})| = |Supp(\{x^*(s/2)\})| = 1$, there are four possibilities, that is, X^* is “H”, “D”, “B” or “A” strategy. We shall check each case.

Case 1. X^* ="H" strategy, i.e. $x^*(1/1) = 1, x^*(1/2) = 1$. In (5), $T_0 = T, s_1^{(0)} = 1, s_2^{(0)} = 1$ and so $S_0(X^*/1) = S_0(X^*/2) = \{2\}$. Therefore, formula (6) turns out to be the following:

$$u((2, k), (2, m)) = (-u_1(2, 2; k, m) + u_1(2, 1; k, m) + u_1(1, 2; k, m) - u_1(1, 1; k, m))p(k, m).$$

Since $\bar{y}(2/k) = x^*(2/k) - x(2/k) = -x(2/k); k = 1, 2$, we have

$$\begin{aligned} (U(X^*)\bar{y}, \bar{y}) &= \\ C p(1, 1)(x(2/1))^2/2 + C p(1, 2)x(2/1)x(2/2)/2 + c p(2, 1)x(2/2)x(2/1)/2 + c p(2, 2)(x(2/2))^2/2 \\ &= C(p(1, 1) + p(1, 2))x(2/1)(q_1x(2/1) + (1 - q_1)x(2/2))/2 \\ (10) \quad &+ c(p(2, 1) + p(2, 2))x(2/2)(q_2x(2/1) + (1 - q_2)x(2/2))/2. \end{aligned}$$

Therefore, for $(x^*(2/1), x^*(2/2)) = (0, 0) \neq (x(2/1), x(2/2))$, if $0 < q_1, q_2 < 1$, (10) is positive, whereas if $q_1 = 0$, then (10) is zero for $x(2/1) \neq 0, x(2/2) = 0$, and if $q_2 = 1$, then (10) is zero for $x(2/1) = 0, x(2/2) \neq 0$.

Case 2. X^* ="D" strategy, i.e. $x^*(1/1) = 0, x^*(1/2) = 0$. As is shown in Proposition 2, "D" strategy can not be a Bayesian Nash equilibrium, so we omit this case.

Case 3. X^* ="B" strategy, i.e. $x^*(1/1) = 1, x^*(1/2) = 0$. In (5), $T_0 = T, s_1^{(0)} = 1, s_2^{(0)} = 2$ and so $S_0(X^*/1) = \{2\}, S_0(X^*/2) = \{1\}$. Therefore, in this case we have

$$U(X^*) = \begin{pmatrix} (C/2)p(1, 1) & -(C/2)p(1, 2) \\ -(c/2)p(2, 1) & (c/2)p(2, 2) \end{pmatrix}.$$

Contrary to case 1, (10) must be positive for $x(2/1)$ and $-x(1/2)$ instead of $x(2/2)$. With a little inspection, we conclude that $U^s(X^*)$ must be positive definite. Therefore, we have the same conditions as those of (iv) of Proposition 1. We can prove the case of "A" strategy in a similar manner. (Q.E.D.)

Now we give a complete classification by using four parameters q_1, q_2, r_1 and r_2 what kind of Bayesian Nash equilibria exist and whether they are ESS or not. Here, imagine the first quadrant $\{(r_1, r_2); r_1 > 0, r_2 > 0\}$ in a Cartesian coordinate system (r_1, r_2) .

Proposition 2.

(I) "D" strategy is never Bayesian Nash equilibrium.

(II) (i) $r_1 > 1, r_2 > 1$: "H" strategy is a unique strong Bayesian Nash equilibrium and an ESS in this domain.

(ii) $r_1 = 1, r_2 > 1$: "H" strategy is a unique Bayesian Nash equilibrium, and it is an ESS if and only if $q_1 > 0$.

(iii) $r_1 > 1, r_2 = 1$: "H" strategy is a unique Bayesian Nash equilibrium, and it is an ESS if and only if $1 - q_2 > 0$.

(iv) $r_1 = 1, r_2 = 1$: "H" strategy is a unique Bayesian Nash equilibrium, and it is an ESS if and only if $0 < q_1, q_2 < 1$.

In case of $r_1 < 1$ or $r_2 < 1$, the results depend on q_1, q_2 and γ .

(III) Assume $q_1 = 0, q_2 = 1$ ($\iff p(1, 1) = p(2, 2) = 0, p(1, 2) = p(2, 1) = 1/2$.)

(v) $r_2 < 1 < r_1$: "B" strategy is a unique strong Bayesian Nash equilibrium and an ESS.

(vi) $r_2 < 1 = r_1$: "B" strategy is a strong Bayesian Nash equilibrium and an ESS, and there are other Bayesian Nash equilibria such as $(0 \leq x^*(1/1) \leq r_2, x^*(1/2) = 1)$ but these equilibria are not ESS.

(vii) $r_1 < 1 < r_2$: “A” strategy is a unique strong Bayesian Nash equilibrium and an ESS.

(viii) $r_1 < 1 = r_2$: “A” strategy is a strong Bayesian Nash equilibrium and an ESS, and there are other Bayesian Nash equilibria such as $(x^*(1/1) = 1, 0 \leq x^*(1/2) \leq r_1)$ but these equilibria are not ESS.

(ix) $r_1 < 1, r_2 < 1$: There are three Bayesian Nash equilibria⁵ (a) “B” strategy (b) “A” strategy and (c) $(x^*(1/1), x^*(1/2)) = (r_2, r_1)$. Among these equilibria, (a) and (b) are strong Bayesian Nash equilibria and ESS, but (c) is not an ESS.

We remark that the cases (v) and (ix) are those investigated by Maynard-Smith and his results coincide with ours.

For real numbers a and b , we denote $\max\{a, b\}$ by $a \vee b$ and $\min\{a, b\}$ by $a \wedge b$, respectively.

(IV) Assume $q_1 = 0, 0 < q_2 < 1$ ($\iff p(1, 1) = 0, 0 < p(2, 2) < 1$).

(x) $r_2 < 1 < r_1$: There is a unique Bayesian Nash equilibrium $(x^*(1/1) = 1, x^*(1/2) = ((r_2 - q_2)/(1 - q_2)) \vee 0)$ which is an ESS.

(xi) $r_2 < 1 = r_1$: There is a Bayesian Nash equilibrium $(x^*(1/1) = 1, x^*(1/2) = ((r_2 - q_2)/(1 - q_2)) \vee 0)$ which is an ESS. In addition, if $r_2 \geq 1 - q_2$, then there are such Bayesian Nash equilibria that $(0 \leq x^*(1/1) \leq (r_2 - (1 - q_2))/q_2, x^*(1/2) = 1)$ which are not ESS.

(xii) $r_1 < 1$: If $r_2 < (1 - q_2)r_1 + q_2$, then there is a Bayesian Nash equilibrium $(x^*(1/1) = 1, x^*(1/2) = ((r_2 - q_2)/(1 - q_2)) \vee 0)$ which is an ESS. On the other hand, if $r_2 > (1 - q_2)r_1$, then there is a Bayesian Nash equilibrium $(x^*(1/1) = 0, x^*(1/2) = (r_2/(1 - q_2)) \wedge 1)$ which is an ESS. In addition, if $(1 - q_2)r_1 \leq r_2 \leq (1 - q_2)r_1 + q_2$, then there is a Bayesian Nash equilibrium $(x^*(1/1) = (r_2 - (1 - q_2)r_1)/q_2, x^*(1/2) = r_1)$ which is not an ESS.

(V) Assume $0 < q_1 < 1, q_2 = 1$ ($\iff 0 < p(1, 1) < 1, p(2, 2) = 0$).

(xiii) $r_1 < 1 < r_2$: There is a unique Bayesian Nash equilibrium $(x^*(1/1) = ((r_1 - (1 - q_1))/q_1) \vee 0, x^*(1/2) = 1)$ which is an ESS.

(xiv) $r_1 < 1 = r_2$: There is a Bayesian Nash equilibrium $(x^*(1/1) = ((r_1 - (1 - q_1))/q_1) \vee 0, x^*(1/2) = 1)$ which is an ESS. In addition, if $r_1 \geq q_1$, then there are such Bayesian Nash equilibrium that $(x^*(1/1) = 1, 0 \leq x^*(1/2) \leq (r_1 - q_1)/(1 - q_1))$ which are not ESS.

(xv) $r_2 < 1$: If $r_1 < q_1r_2 + (1 - q_1)$, then there is a Bayesian Nash equilibrium $(x^*(1/1) = ((r_1 - (1 - q_1))/q_1) \vee 0, x^*(1/2) = 1)$ which is an ESS. On the other hand, if $r_1 > q_1r_2$, then there is a Bayesian Nash equilibrium $(x^*(1/1) = (r_1/q_1) \wedge 1, x^*(1/2) = 0)$ which is an ESS. In addition, if $q_1r_2 \leq r_1 \leq q_1r_2 + (1 - q_1)$, then There is a Bayesian Nash equilibrium $(x^*(1/1) = r_2, x^*(1/2) = (r_1 - q_1r_2)/(1 - q_1))$ which is not an ESS.

(VI) In case of $0 < q_1, q_2 < 1$ ($\iff 0 < p(1, 1) < 1, 0 < p(2, 2) < 1$), we divide it in the following three cases. In these cases, the classifications are a little bit complicated. First, we introduce the following four linear functions having two parameters q_1 and q_2 :

$$\begin{aligned} \ell_1(r_1) &:= (1 - q_2)r_1/(1 - q_1), & \ell_2(r_1) &:= q_2r_1/q_1 + (q_1 - q_2)/q_1, \\ \ell_3(r_2) &:= q_1r_2/q_2, & \ell_4(r_2) &:= (1 - q_1)r_2/(1 - q_2) + (q_1 - q_2)/(1 - q_2). \end{aligned}$$

(VI-1) Assume $0 < q_2 < q_1 < 1$: Set

$$R_{11} := \{(r_1, r_2); r_1 < 1, r_2 > \ell_1(r_1) \wedge \ell_2(r_1)\},$$

$$R_{12} := \{(r_1, r_2); r_1 > \ell_3(r_2) \wedge \ell_4(r_2), r_2 < 1\},$$

$$R_{13} := \{(r_1, r_2); r_1 \leq \ell_3(r_2) \wedge \ell_4(r_2), r_2 \leq \ell_1(r_1) \wedge \ell_2(r_1)\} - (1, 1).$$

⁵In this case, there exist two nonsymmetric Bayesian Nash equilibria $(x_1^*(1/1), x_1^*(1/2); x_2^*(1/1), x_2^*(1/2)) = (r_2, 1; 0, r_1)$ or $(r_2, 0; 1, r_1)$.

We remark that the three domains R_{11} , R_{12} and R_{13} make a partition of the domain $\{r_1 < 1\} \cup \{r_2 < 1\}$. As is shown below, there is a unique Bayesian Nash equilibrium in each domain.

(xvi) $(r_1, r_2) \in R_{11}$: There is a unique Bayesian Nash equilibrium $(x^*(1/1) = ((r_1 - (1 - q_1))/q_1) \vee 0, x^*(1/2) = (r_2/(1 - q_2)) \wedge 1)$ which is an ESS.

(xvii) $(r_1, r_2) \in R_{12}$: There is a unique Bayesian Nash equilibrium $(x^*(1/1) = (r_1/q_1) \wedge 1, x^*(1/2) = ((r_2 - q_2)/(1 - q_2)) \vee 0)$ which is an ESS.

(xviii) $(r_1, r_2) \in R_{13}$: There is a unique Bayesian Nash equilibrium $(x^*(1/1) = ((1 - q_2)r_1 - (1 - q_1)r_2)/(q_1 - q_2), x^*(1/2) = (-q_2r_1 + q_1r_2)/(q_1 - q_2))$ which is an ESS if and only if $q_2 < g(q_1)$.

(VI-2) Assume $0 < q_1 < q_2 < 1$: Set

$$R_{21} := \{(r_1, r_2); r_1 < 1, r_2 > \ell_1(r_1) \vee \ell_2(r_1)\},$$

$$R_{22} := \{(r_1, r_2); r_1 > \ell_3(r_2) \vee \ell_4(r_2), r_2 < 1\},$$

$$R_{23} := \{(r_1, r_2); r_1 \geq \ell_3(r_2) \vee \ell_4(r_2), r_2 \geq \ell_1(r_1) \vee \ell_2(r_1)\} - (1, 1).$$

We remark that contrary to case (VI-1), $\{0 < r_1 < 1\} \cup \{0 < r_2 < 1\} = R_{21} \cup R_{22}$ and $R_{23} \supset R_{21} \cap R_{22}$.

(xix) $(r_1, r_2) \in R_{21}$: There is a Bayesian Nash equilibrium $(x^*(1/1) = ((r_1 - (1 - q_1))/q_1) \vee 0, x^*(1/2) = (r_2/(1 - q_2)) \wedge 1)$ which is an ESS.

(xx) $(r_1, r_2) \in R_{22}$: There is a Bayesian Nash equilibrium $(x^*(1/1) = (r_1/q_1) \wedge 1, x^*(1/2) = ((r_2 - q_2)/(1 - q_2)) \vee 0)$ which is an ESS.

(xxi) $(r_1, r_2) \in R_{23}$: There is a Bayesian Nash equilibrium $(x^*(1/1) = ((1 - q_2)r_1 - (1 - q_1)r_2)/(q_1 - q_2), x^*(1/2) = (-q_2r_1 + q_1r_2)/(q_1 - q_2))$ which is not an ESS.

(VI-3) Assume $0 < q_1 = q_2 =: q < 1$ ($\iff p(1, 1) = q^2$). In this case, there are Bayesian Nash equilibria which are ESS obtained by putting $q_1 = q_2 = q$ in cases (xvi), (xvii) of (VI-1) or (xix), (xx) of (VI-2). In addition, if $r_1 = r_2 = r < 1$, then there are Bayesian Nash equilibria $0 \leq x^*(1/1), x^*(1/2) \leq 1$ which satisfy the equation $r - qx^*(1/1) - (1 - q)x^*(1/2) = 0$, but they are not ESS.

Proof. We will prove in detail for readers who are not mathematicians.

Since any probability distribution $\{x(s)\}_{s \in S}$ in our case can be represented by one parameter $x(1) = x; 0 \leq x \leq 1$, from Lemma 1, a behavior strategy $(x^*(1/1), x^*(1/2))$ is a Bayesian Nash equilibrium if and only if

$$(i-1) \quad 0 \leq \forall x \leq 1,$$

$$(11) \quad (x^*(1/1) - x)((V - Cx^*(1/1))p(1, 1) + (V - Cx^*(1/2))p(1, 2)) \geq 0,$$

and

$$(i-2) \quad 0 \leq \forall x \leq 1,$$

$$(12) \quad (x^*(1/2) - x)((v - cx^*(1/1))p(2, 1) + (v - cx^*(1/2))p(2, 2)) \geq 0$$

hold.

By dividing formula (11) or (12) by $C(p(1, 1) + p(1, 2))$ or $c(p(2, 1) + p(2, 2))$ respectively, they turn out to be the following:

$$(i-1) \quad 0 \leq \forall x \leq 1,$$

$$(13) \quad (x^*(1/1) - x)(r_1 - q_1x^*(1/1) - (1 - q_1)x^*(1/2)) \geq 0,$$

and

(i-2) $0 \leq \forall x \leq 1,$

(14) $(x^*(1/2) - x)(r_2 - q_2x^*(1/1) - (1 - q_2)x^*(1/2)) \geq 0$

hold.

Now we start proving each case.

(I) Since $r_1, r_2 > 0$, obviously none of (13) or (14) holds for $x^*(1/1) = x^*(1/2) = 0$ (“D” strategy).

(II) (i) $r_1 > 1, r_2 > 1$: Since $r_1, r_2 > x^*(1/1), x^*(1/2)$, obviously (13) and (14) hold if and only if $x^*(1/1) = x^*(1/2) = 1$ (“H” strategy) and $|S_1(X^*/1)| = |S_1(X^*/2)| = 1$ holds, which means a strong Bayesian Nash equilibrium and an ESS by (i) of Theorem 2.

(ii) $r_1 = 1, r_2 > 1$: Since $1 - q_1x^*(1/1) - (1 - q_1)x^*(1/2) \geq 1$ and $r_2 - q_2x^*(1/1) - (1 - q_2)x^*(1/2) > 0$, obviously (13) and (14) hold if and only if $x^*(1/1) = x^*(1/2) = 1$ (“H” strategy), but in this case we have $|S_1(X^*/1)| = 2, |S_1(X^*/2)| = 1$. Applying (iii) of Proposition 1, this equilibrium is an ESS if and only if $q_1 > 0$.

(iii) $r_1 > 1 = r_2$: Similarly to (ii), “H” strategy is a unique Bayesian Nash equilibrium, but in this case we have $|S_1(X^*/1)| = 1, |S_1(X^*/2)| = 2$. Applying (ii) of Proposition 1, this equilibrium is an ESS if and only if $1 - q_2 > 0$.

(iv) $r_1 = r_2 = 1$: Similarly to (ii), “H” strategy is a unique Bayesian Nash equilibrium, but in this case we have $|S_1(X^*/1)| = |S_1(X^*/2)| = 2$ and $|Supp(\{x^*(s/1)\})| = |Supp(\{x^*(s/2)\})| = 1$. Applying (v) of Proposition 1, this equilibrium is an ESS if and only if $0 < q_1, q_2 < 1$.

(III) Assume $q_1 = 0, q_2 = 1$: (13) and (14) turn out to be the following:

(i-1)

(15) $0 \leq \forall x \leq 1, (x^*(1/1) - x)(r_1 - x^*(1/2)) \geq 0,$

and

(i-2)

(16) $0 \leq \forall x \leq 1, (x^*(1/2) - x)(r_2 - x^*(1/1)) \geq 0$

hold.

(v) $r_2 < 1 < r_1$: Since $r_1 - x^*(1/2) > 0$ and $r_2 - 1 < 0$, we have $x^*(1/1) = 1$ by (15) and $x^*(1/2) = 0$ by (16). Therefore, “B” strategy is a strong Bayesian Nash equilibrium and an ESS.

(vi) $r_2 < 1 = r_1$: If $x^*(1/2) = 1$, then (15) holds for any $x^*(1/1)$. On the other hand, (16) implies $r_2 - x^*(1/1) \geq 0$. In this case, if $x^*(1/1) = r_2$, then we have $|S_1(X^*/1)| = |S_1(X^*/2)| = 2$ and $|Supp(\{x^*(s/1)\})| = 2$. Applying (iv) of Proposition 1, this equilibrium is not an ESS. If $0 \leq x^*(1/1) < r_2$, we have $|S_1(X^*/2)| = 1$, and from (iii) of Proposition 1 it is not an ESS. If $x^*(1/2) < 1$, then (15) holds if and only if $x^*(1/1) = 1$. On the other hand, (16) implies $x^*(1/2) = 0$. Therefore, $(x^*(1/1) = 1, x^*(1/2) = 0)$ (“B” strategy) is a strong Bayesian Nash equilibrium and an ESS.

(vii) $r_1 < 1 < r_2$: In this case, we can obtain the result by interchanging r_1 and r_2 in csse (v).

(viii) $r_1 < 1 = r_2$: In this case, we can obtain the results by interchanging r_1 and r_2 in csse (vi).

(ix) $r_1 < 1$ and $r_2 < 1$: This case is a little bit complicated. If $r_1 - x^*(1/2) = 0$, then (16) holds if and only if $r_2 - x^*(1/1) = 0$. Therefore, $(x^*(1/1), x^*(1/2)) = (r_2, r_1)$ is a Bayesian Nash equilibrium and applying (iv) of Proposition 1, it is not an ESS. If

$r_1 - x^*(1/2) > 0$, (15) holds if and only if $x^*(1/1) = 1$. Then (16) holds if and only if $x^*(1/2) = 0$. That is, $(x^*(1/1), x^*(1/2)) = (1, 0)$ (“B” strategy) is a Bayesian Nash equilibrium. It is easily seen that this equilibrium is a strong Bayesian Nash equilibrium and an ESS. Finally, if $r_1 - x^*(1/2) < 0$, (15) holds if and only if $x^*(1/1) = 0$. Then (16) holds if and only if $x^*(1/2) = 1$. That is, $(x^*(1/1), x^*(1/2)) = (0, 1)$ (“A” strategy) is a Bayesian Nash equilibrium. It is easily seen that this equilibrium is a strong Bayesian Nash equilibrium and an ESS.

(IV) Assume $q_1 = 0, 0 < q_2 < 1$. (13) and (14) turn out to be the following:

(i-1)
 (17) $0 \leq \forall x \leq 1, (x^*(1/1) - x)(r_1 - x^*(1/2)) \geq 0,$

and

(i-2) $0 \leq \forall x \leq 1,$
 (18) $(x^*(1/2) - x)(r_2 - q_2x^*(1/1) - (1 - q_2)x^*(1/2)) \geq 0$

hold.

(x) $r_2 < 1 < r_1$: (17) implies that $x^*(1/1) = 1$, therefore substituting this to (18), we have

(19) $0 \leq \forall x \leq 1, (x^*(1/2) - x)(r_2 - q_2 - (1 - q_2)x^*(1/2)) \geq 0.$

Since $r_2 - q_2 - (1 - q_2) < 0$, $x^*(1/2) = 1$ does not satisfy (19). If $r_2 < q_2$, then (19) holds if and only if $x^*(1/2) = 0$. On the other hand, if $q_2 \leq r_2 < 1$, then (19) holds if and only if $x^*(1/2) = (r_2 - q_2)/(1 - q_2)$. Therefore, $(x^*(1/1) = 1, x^*(1/2) = ((r_2 - q_2)/(1 - q_2)) \vee 0)$ is a Bayesian Nash equilibrium. Since $|S_1(X^*/1)| = 1$, it is an ESS by (i) or (ii) of Proposition 1.

(xi) $r_2 < 1 = r_1$: If $x^*(1/2) = 1$, then (17) holds for any $x^*(1/1)$, but by (18) we have $(r_2 - q_2x^*(1/1) - (1 - q_2)) \geq 0$. Solving this inequality with respect to $x^*(1/1)$, we have $x^*(1/1) \leq (r_2 - (1 - q_2))/q_2$. Since $x^*(1/1) \geq 0$, $r_2 \geq 1 - q_2$ must hold and we have Bayesian Nash equilibria $(0 \leq x^*(1/1) \leq (r_2 - (1 - q_2))/q_2, x^*(1/2) = 1)$ when $r_2 \geq 1 - q_2$, which are not ESS by (iii) or (iv) of Proposition 1. On the other hand, if $0 < x^*(1/2) < 1$, then (17) implies $x^*(1/1) = 1$ and (18) implies $r_2 - q_2 - (1 - q_2)x^*(1/2) = 0$, which means $x^*(1/2) = (r_2 - q_2)/(1 - q_2)$ if $r_2 \geq q_2$. On the contrary, if $r_2 \leq q_2$, (18) still holds for $x^*(1/2) = 0$. Therefore, $(x^*(1/1) = 1, x^*(1/2) = ((r_2 - q_2)/(1 - q_2)) \vee 0)$ is a Bayesian Nash equilibrium and an ESS by (ii) of Proposition 1.

(xii) $r_1 < 1$: We shall investigate three cases of $x^*(1/2)$ for which (17) holds. First, if $x^*(1/2) = r_1$, then from (18), we have $r_2 - q_2x^*(1/1) - (1 - q_2)r_1 = 0$, that is, $x^*(1/1) = (r_2 - (1 - q_2)r_1)/q_2$. Since $0 \leq x^*(1/1) \leq 1$, $(1 - q_2)r_1 \leq r_2 \leq (1 - q_2)r_1 + q_2$ must be satisfied. In this case, by (iv) of Proposition 1, this Bayesian Nash equilibrium $(x^*(1/1) = (r_2 - r_1(1 - q_2))/q_2, x^*(1/2) = r_1)$ is not an ESS. Second, if $x^*(1/2) < r_1$, then by (17) and (18) we have $x^*(1/1) = 1$ and $(r_2 - q_2) - (1 - q_2)x^*(1/2) \leq 0$. More precisely, if $0 < x^*(1/2) < r_1$, it must be $(r_2 - q_2) - (1 - q_2)x^*(1/2) = 0$. Combining all together, if $(r_2 - q_2)/(1 - q_2) < r_1$ we have a Bayesian Nash equilibrium $(x^*(1/1) = 1, x^*(1/2) = ((r_2 - q_2)/(1 - q_2)) \vee 0)$ which is an ESS by (ii) of Proposition 1. Third, if $x^*(1/2) > r_1$, then by (17) and (18) we have $x^*(1/1) = 0$ and $r_2 - (1 - q_2)x^*(1/2) \geq 0$. More precisely, if $1 > x^*(1/2) > r_1$, it must be $r_2 - (1 - q_2)x^*(1/2) = 0$. Combining all together, if $r_2 > r_1(1 - q_2)$ we have a Bayesian Nash equilibrium $(x^*(1/1) = 0, x^*(1/2) = (r_2/(1 - q_2)) \wedge 1)$ which is an ESS by (ii) of Proposition 1.

We can prove case (V) in a similar manner to that of (IV), so we omit it.

(VI) Assume $0 < q_1, q_2 < 1$. (13) and (14) turn out to be the following:

(i-1)
 (20) $0 \leq \forall x \leq 1, (x^*(1/1) - x)f_1(x^*(1/1), x^*(1/2)) \geq 0,$

and

(i-2)
 (21) $0 \leq \forall x \leq 1, (x^*(1/2) - x)f_2(x^*(1/1), x^*(1/2)) \geq 0,$

hold, where

$$f_1(x, y) := r_1 - q_1x - (1 - q_1)y, \quad f_2(x, y) := r_2 - q_2x - (1 - q_2)y$$

We shall investigate the conditions for r_1, r_2, q_1 and q_2 under the supposed Bayesian Nash equilibrium $(x^*(1/1), x^*(1/2))$.

(a) Assume that $(x^*(1/1), x^*(1/2)) = (1, 1)$ (“H” strategy) is a Bayesian Nash equilibrium: Then, (20) and (21) imply $f_1(1, 1) \geq 0, f_2(1, 1) \geq 0$, that is, $f_1(1, 1) = r_1 - 1 \geq 0$ and $f_2(1, 1) = r_2 - 1 \geq 0$. In this case, we have already obtained the results in (II) above.

(b) $(x^*(1/1), x^*(1/2)) = (1, 0)$ (“B” strategy) : $f_1(1, 0) \geq 0, f_2(1, 0) \leq 0$, that is, $f_1(1, 0) = r_1 - q_1 \geq 0$ and $f_2(1, 0) = r_2 - q_2 \leq 0$. More precisely, if $r_1 > q_1, |S_1(X^*/1)| = 1$ and if $r_1 = q_1, |S_1(X^*/1)| = 2$. Similarly, if $r_2 < q_2, |S_1(X^*/2)| = 1$ and if $r_2 = q_2, |S_1(X^*/2)| = 2$. Therefore, in case of $(r_1 > q_1, r_2 < q_2)$, “B” strategy is a strong Bayesian Nash equilibrium and an ESS, in case of $(r_1 = q_1, r_2 < q_2)$ or $(r_1 > q_1, r_2 = q_2)$, applying (ii) or (iii) of Proposition 1, it is also an ESS, but in case of $(r_1 = q_1, r_2 = q_2)$, applying (v) of Proposition 1, it is an ESS if and only if $q_2 < g(q_1)$.

(c) $(x^*(1/1), x^*(1/2)) = (0, 1)$ (“A” strategy) : $f_1(0, 1) \leq 0, f_2(0, 1) \geq 0$, that is, $f_1(0, 1) = r_1 - (1 - q_1) \leq 0, f_2(0, 1) = r_2 - (1 - q_2) \geq 0$. More precisely, if $r_1 < 1 - q_1, |S_1(X^*/1)| = 1$ and if $r_1 = 1 - q_1, |S_1(X^*/1)| = 2$. Similarly, if $r_2 > 1 - q_2, |S_1(X^*/2)| = 1$ and if $r_2 = 1 - q_2, |S_1(X^*/2)| = 2$. Therefore, in case of $(r_1 < 1 - q_1, r_2 > 1 - q_2)$, “A” strategy is a strong Bayesian Nash equilibrium and an ESS, in case of $(r_1 = 1 - q_1, r_2 > 1 - q_2)$ or $(r_1 < 1 - q_1, r_2 = 1 - q_2)$, applying (ii) or (iii) of Proposition 1, it is also an ESS, but in case of $(r_1 = 1 - q_1, r_2 = 1 - q_2)$, applying (v) of Proposition 1, it is an ESS if and only if $q_2 < g(q_1)$.

(d) $(x^*(1/1) = 1, 0 < x^*(1/2) < 1)$: (20) and (21) imply

$$f_1(1, x^*(1/2)) = r_1 - q_1 - (1 - q_1)x^*(1/2) \geq 0 \dots (*) \text{ and}$$

$$f_2(1, x^*(1/2)) = r_2 - q_2 - (1 - q_2)x^*(1/2) = 0 \dots (**).$$

If $f_1(1, x^*(1/2)) > 0$, we have $|S_1(X^*/1)| = 1$ and applying (ii) of Proposition 1, this Bayesian Nash equilibrium is an ESS. If $f_1(1, x^*(1/2)) = 0$, we have $|S_1(X^*/1)| = 2$ and applying (iv) of Proposition 1, this Bayesian Nash equilibrium is an ESS if and only if $q_2 < g(q_1)$. Now solving (**), we have $x^*(1/2) = (r_2 - q_2)/(1 - q_2)$ and substituting this solution to (*), we have $r_1 \geq \ell_4(r_2)$, and $q_2 < r_2 < 1$ from $0 < x^*(1/2) < 1$. Therefore, we have a Bayesian Nash equilibrium $(x^*(1/1) = 1, x^*(1/2) = (r_2 - q_2)/(1 - q_2))$ which is an ESS if $(r_1 > \ell_4(r_2), q_2 < r_2 < 1)$. In case of $(r_1 = \ell_4(r_2), q_2 < r_2 < 1)$, it is an ESS if and only if $q_2 < g(q_1)$.

(e) $(0 < x^*(1/1) < 1, x^*(1/2) = 1)$: (20) and (21) imply

$$f_1(x^*(1/1), 1) = r_1 - q_1x^*(1/1) - (1 - q_1) = 0 \dots (*) \text{ and}$$

$$f_2(x^*(1/1), 1) = r_2 - q_2x^*(1/1) - (1 - q_2) \geq 0 \dots (**).$$

If $f_2(x^*(1/1), 1) > 0$, we have $|S_1(X^*/2)| = 1$ and applying (ii) of Proposition 1, this Bayesian Nash equilibrium is an ESS. If $f_2(x^*(1/1), 1) = 0$, we have $|S_1(X^*/2)| = 2$ and applying (iv) of Proposition 1, this Bayesian Nash equilibrium is an ESS if and only if $q_2 < g(q_1)$. Now solving (*), we have $x^*(1/1) = (r_1 - (1 - q_1))/q_1$ and substituting this solution to (**), we have $r_2 \geq \ell_2(r_1)$, and $1 - q_1 < r_1 < 1$ from $0 < x^*(1/1) < 1$. Therefore,

we have a Bayesian Nash equilibrium $(x^*(1/1) = (r_1 - (1 - q_1))/q_1, x^*(1/2) = 1)$ which is an ESS if $(1 - q_1 < r_1 < 1, r_2 > \ell_2(r_1))$. In case of $(1 - q_1 < r_1 < 1, r_2 = \ell_2(r_1))$, it is an ESS if and only if $q_2 < g(q_1)$.

(f) $(x^*(1/1) = 0, 0 < x^*(1/2)) < 1)$: (20) and (21) imply

$$f_1(0, x^*(1/2)) = r_1 - (1 - q_1)x^*(1/2) \leq 0 \dots (*)$$

$$f_2(0, x^*(1/2)) = r_2 - (1 - q_2)x^*(1/2) = 0 \dots (**).$$

If $f_1(0, x^*(1/2)) < 0$, we have $|S_1(X^*/1)| = 1$ and applying (iii) of Proposition 1, this Bayesian Nash equilibrium is an ESS. If $f_1(0, x^*(1/2)) = 0$, we have $|S_1(X^*/1)| = 2$ and applying (iv) of Proposition 1, this Bayesian Nash equilibrium is an ESS if and only if $q_2 < g(q_1)$. Now solving (**), we have $x^*(1/2) = r_2/(1 - q_2)$ and substituting this solution to (*), we have $r_2 \geq \ell_1(r_1)$, and $0 < r_2 < 1 - q_2$ from $0 < x^*(1/2) < 1$. Therefore, we have a Bayesian Nash equilibrium $(x^*(1/1) = 0, x^*(1/2) = r_2/(1 - q_2))$ which is an ESS if $0 < r_2 < 1 - q_2$ and $r_2 > \ell_1(r_1)$. In case of $0 < r_2 < 1 - q_2$ and $r_2 = \ell_1(r_1)$, it is an ESS if and only if $q_2 < g(q_1)$.

(g) $(0 < x^*(1/1) < 1, x^*(1/2) = 0)$: (20) and (21) imply

$$f_1(x^*(1/1), 0) = r_1 - q_1x^*(1/1) = 0 \dots (*),$$

$$f_2(x^*(1/1), 0) = r_2 - q_2x^*(1/1) \leq 0 \dots (**).$$

If $f_2(x^*(1/1), 0) < 0$, we have $|S_1(X^*/2)| = 1$ and applying (iii) of Proposition 1, this Bayesian Nash equilibrium is an ESS. If $f_2(x^*(1/1), 0) = 0$, we have $|S_1(X^*/1)| = 2$ and applying (iv) of Proposition 1, this Bayesian Nash equilibrium is an ESS if and only if $q_2 < g(q_1)$. Now solving (*), we have $x^*(1/1) = r_1/q_1$ and substituting this solution to (**), we have $r_1 \geq \ell_3(r_2)$, and $0 < r_1 < q_1$ from $0 < x^*(1/1) < 1$. Therefore, we have a Bayesian Nash equilibrium $(x^*(1/1) = r_1/q_1, x^*(1/2) = 0)$ which is an ESS if $0 < r_1 < q_1$ and $r_1 > \ell_3(r_2)$. In case of $0 < r_1 < q_1$ and $r_1 = \ell_3(r_2)$, it is an ESS if and only if $q_2 < g(q_1)$.

(h) $0 < (x^*(1/1) < 1, 0 < x^*(1/2)) < 1)$: (20) and (21) imply

$$f_1(x^*(1/1), x^*(1/2)) = r_1 - q_1x^*(1/1) - (1 - q_1)x^*(1/2) = 0 \dots (*),$$

$$f_2(x^*(1/1), x^*(1/2)) = r_2 - q_2x^*(1/1) - (1 - q_2)x^*(1/2) = 0 \dots (**).$$

Since $|Supp(\{x^*(s/1)\})| = 2$ and $|\{x^*(s/2)\}| = 2$ in this case, this Bayesian Nash equilibrium is an ESS if and only if $q_2 < g(q_1)$ by (iv) of Proposition 1. Now solving a linear system of equations (*) and (**), if $q_1 \neq q_2$, then we have a unique solution

$$x^*(1/1) = \frac{(1 - q_2)r_1 - (1 - q_1)r_2}{q_1 - q_2}, \quad x^*(1/2) = \frac{-q_2r_1 + q_1r_2}{q_1 - q_2}.$$

Besides being a solution, $x^*(1/1), x^*(1/2)$ must satisfy $0 < x^*(1/1), x^*(1/2) < 1$, i.e.

$$(22) \quad 0 < \frac{(1 - q_2)r_1 - (1 - q_1)r_2}{q_1 - q_2} < 1, \quad 0 < \frac{-q_2r_1 + q_1r_2}{q_1 - q_2} < 1.$$

In order to solve the inequalities (22) with respect to r_1 and r_2 , we investigate two cases, $0 < q_2 < q_1$ or $0 < q_1 < q_2$.

(h-1) $0 < q_2 < q_1 < 1$. In this case, we have $(1 - q_1)/(1 - q_2) < q_1/q_2$. Therefore, (22) implies $\{(r_1, r_2) ; r_1 < \ell_3(r_2) \wedge \ell_4(r_2), r_2 < \ell_1(r_1) \wedge \ell_2(r_1)\} - (1, 1)$ and by (iv) of Proposition 1, this Bayesian Nash equilibrium is an ESS if and only if $q_2 < g(q_1)$.

(h-2) $0 < q_1 < q_2 < 1$. In this case we have $q_1/q_2 < (1 - q_1)/(1 - q_2)$. Therefore, (22) implies $\{(r_1, r_2) ; r_1 > \ell_3(r_2) \vee \ell_4(r_2), r_2 > \ell_1(r_1) \vee \ell_2(r_1)\} - (1, 1)$ and by (iv) of Proposition 1, this Bayesian Nash equilibrium is not an ESS. (Notice that $0 < q_1 < q_2 < 1 \Rightarrow q_2 > g(q_1)$)

(h-3) If $q_1 = q_2 =: q$, there exist such solutions as

$$(23) \quad r - qx^*(1/1) - (1 - q)x^*(1/2) = 0$$

if and only if $r_1 = r_2 =: r \leq 1$. If $r < 1$, they are not ESS since $q \geq g(q)$. If $r = 1$, "H" strategy is a unique solution of this equation and an ESS by (v) of Proposition 1, but this result is a part of case (II).

Finally, combining all together, (VI-1) is obtained from the cases (c), (e), (f) and (h-1), (VI-2) is from (b), (d), (g) and (h-2), and (VI-3) is from (h-3), respectively. (Q.E.D.)

The author would like to thank Professor Higuchi of Kobe University for suggesting related literatures.

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