ON ORDERED SEMIGROUPS WHICH ARE COMPLETE CHAINS OF SEMIGROUPS

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Abstract. Complete chains of semigroups play an essential role in the decomposition of ordered semigroups. In this paper we prove that an ordered semigroup S is a complete chain of semigroups of a given type, say $\mathcal{T}$, if and only if it is decomposable into pairwise disjoint subsemigroups S_α of S indexed by a semilattice Y satisfying, for any $\alpha, \beta \in Y$, the two conditions $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ and whenever $S_\alpha \cap (S_\beta] \neq \emptyset$, then $\alpha = \alpha\beta (= \beta\alpha)$.

1. Introduction and prerequisites. Chains, also complete chains of semigroups play an essential role in the decomposition of ordered semigroups. Chains of semigroups have been already considered in [3, 4] dealing with the decomposition of ordered semigroups into their right simple subsemigroups. A *complete chain of semigroups* in an ordered semigroup S is a complete semilattice congruence σ on S such that the σ-class $(x)_\sigma$ of S containing x ($x \in S$) with the order " $\preceq$ " on the quotient set $S/\sigma := \{(x)_\sigma \mid x \in S\}$ defined by "$(x)_\sigma \preceq (y)_\sigma$ if and only if $(x)_\sigma = (xy)_\sigma$" is a chain. In the present paper we characterize the complete chains of semigroups of a given ordered semigroup S as partitions of S into its subsemigroups indexed by a semilattice (i.e. commutative and idempotent semigroup) Y. When we need to refer to Y, we also say that S is a complete chain Y of semigroups S_α ($\alpha \in Y$).

Let $(S, ., \leq)$ be an ordered semigroup. An equivalence relation σ on S is called *congruence* if $(a, b) \in \sigma$ implies $(ac, bc) \in \sigma$ and $(ca, cb) \in \sigma$ for every $c \in S$. A congruence σ on S is called *semilattice congruence* if $(a^2, a) \in \sigma$ and $(ab, ba) \in \sigma$ for every $a, b \in S$ [1]. A semilattice congruence σ on S is called *complete* if $a \leq b$ implies $(a, ab) \in \sigma$ [5]. If σ is a complete semilattice congruence on S, then the relation $a \leq a$ implies $(a, a^2) \in \sigma$. So a complete semilattice congruence on S can be also defined as a congruence σ on S such that $(ab, ba) \in \sigma$ for every $a, b \in S$ and whenever $x \leq y$, then $(x, xy) \in \sigma$. If σ is a semilattice congruence on S, then the σ-class $(x)_\sigma$ of S containing x is a subsemigroup of S for every $x \in S$ (cf. also [2]). For a subset H of S we denote by $(H]$ the subset of S defined by $(H] := \{t \in S \mid t \leq h \text{ for some } h \in H\}$.

2. Main Result

Definition 1. Let $(S, ., \leq)$ be an ordered semigroup. A congruence σ on S is called *complete semilattice congruence* if (1) $(ab, ba) \in \sigma$ for every $a, b \in S$ and (2) if $x \leq y$ implies $(x, xy) \in \sigma$.

Definition 2. An ordered semigroup S is called a *complete chain of semigroups of a given type, say $\mathcal{T}$*, if there exists a complete semilattice congruence σ on S such that the σ-class

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$(x)_\sigma$ of S containing x ($x \in S$) is a subsemigroup of S of type $\mathcal{T}$ for every $x \in S$, and the set $S/\sigma := \{(x)_\sigma \mid x \in S\}$ of all σ-classes of S endowed with the order

$$(x)_\sigma \preceq (y)_\sigma \Leftrightarrow (x)_\sigma = (xy)_\sigma$$

is a chain.

Theorem. *An ordered semigroup $(S, ., \leq)$ is a complete chain of semigroups of type $\mathcal{T}$ if and only if there exists a semilattice Y at the same time a chain with the order*

$$\alpha \preceq \beta \Leftrightarrow \alpha = \alpha\beta (= \beta\alpha)$$

$\alpha, \beta \in Y$, and a family $\{S_\alpha \mid \alpha \in Y\}$ of subsemigroups of S of type $\mathcal{T}$ such that the following assertions are satisfied:

(A) *$S_\alpha \cap S_\beta = \emptyset$ for every $\alpha, \beta \in Y$, $\alpha \neq \beta$*
(B) *$S = \bigcup\limits_{\alpha \in Y} S_\alpha$*
(C) $S_\alpha S_\beta \subseteq S_{\alpha\beta}$ for every $\alpha, \beta \in Y$
(D) *If $\alpha, \beta \in Y$ such that $S_\alpha \cap (S_\beta] \neq \emptyset$, then $\alpha \preceq \beta$.*

Proof. $\Longrightarrow$. Let σ be a complete semilattice congruence on S such that $(x)_\sigma$ is a subsemigroup of S of type $\mathcal{T}$ for every $x \in S$, and the set S/σ endowed with the order $(x)_\sigma \preceq (y)_\sigma \Leftrightarrow (x)_\sigma = (xy)_\sigma$ is a chain. Since σ is a congruence on S, the set $Y := S/\sigma$ with the multiplication $(x)_\sigma (y)_\sigma := (xy)_\sigma$ is a semigroup. Since σ is a semilattice congruence on S, the semigroup Y is commutative and idempotent i.e. Y is a semilattice. Let now $\alpha, \beta \in Y$, $\alpha = (x)_\sigma$, $\beta = (y)_\sigma$ for some $x, y \in S$. By hypothesis, we have $(x)_\sigma \preceq (y)_\sigma$ or $(y)_\sigma \preceq (x)_\sigma$, that is, $(x)_\sigma = (xy)_\sigma$ or $(y)_\sigma = (yx)_\sigma = (xy)_\sigma$. If $(x)_\sigma = (xy)_\sigma$, then $\alpha = (x)_\sigma = (xy)_\sigma = (x)_\sigma (y)_\sigma = \alpha\beta$. If $(y)_\sigma = (xy)_\sigma$, similarly we have $\beta = \alpha\beta$.

For every $\alpha \in Y$, $\alpha = (x)_\sigma$, $x \in S$, we put $S_\alpha := (x)_\sigma$.
By hypothesis, S_α is a subsemigroup of S of type $\mathcal{T}$ for every $\alpha \in Y$. Moreover, the family $\{S_\alpha \mid \alpha \in Y\}$ satisfies conditions (A)–(D). In fact:

(A) Let $\alpha, \beta \in Y$, $\alpha \neq \beta$. Suppose $\alpha = (x)_\sigma$, $\beta = (y)_\sigma$ for some $x, y \in S$. Then $S_\alpha := (x)_\sigma$, $S_\beta := (y)_\sigma$. Since $\alpha \neq \beta$, we have $(x)_\sigma \neq (y)_\sigma$, then $(x)_\sigma \cap (y)_\sigma = \emptyset$, so $S_\alpha \cap S_\beta = \emptyset$.

(B) $S = \bigcup\limits_{\alpha \in Y} S_\alpha$. Indeed:
S_α being a subsemigroup, is a subset of S for every $\alpha \in Y$, so $\bigcup\limits_{\alpha \in Y} S_\alpha \subseteq S$. Let now $a \in S$. Since $(a)_\sigma \in Y$, we have $S_{(a)_\sigma} := (a)_\sigma$. Then $a \in S_{(a)_\sigma} \subseteq \bigcup\limits_{\alpha \in Y} S_\alpha$.

(C) Let $\alpha, \beta \in Y$. Then $S_\alpha S_\beta \subseteq S_{\alpha\beta}$. Indeed:
Suppose $\alpha = (x)_\sigma$, $\beta = (y)_\sigma$ for some $x, y \in S$. Then $S_\alpha := (x)_\sigma$, $S_\beta := (y)_\sigma$, $\alpha\beta \in Y$, $\alpha\beta = (x)_\sigma (y)_\sigma := (xy)_\sigma$ and $S_{\alpha\beta} := (xy)_\sigma$. Thus we have $S_\alpha S_\beta = (x)_\sigma (y)_\sigma = (xy)_\sigma = S_{\alpha\beta}$.

(D) Let $\alpha, \beta \in Y$ such that $S_\alpha \cap (S_\beta] \neq \emptyset$. Then $\alpha = \alpha\beta$. Indeed:
Suppose $\alpha = (x)_\sigma$ and $\beta = (y)_\sigma$ for some $x, y \in S$. Then $S_\alpha := (x)_\sigma$, $S_\beta := (y)_\sigma$, and $(x)_\sigma \cap ((y)_\sigma] \neq \emptyset$. Let now $t \in (x)_\sigma \cap ((y)_\sigma]$. Then $t \in (x)_\sigma$ and $t \leq z$ for some $z \in (y)_\sigma$. Then we have $(t, x) \in \sigma$, $t \leq z$ and $(z, y) \in \sigma$. Since σ is complete, we have $(t, tz) \in \sigma$. Since σ is a semilattice congruence, we have $(tz, xz) \in \sigma$ and $(xz, xy) \in \sigma$. Then we have $(t, xy) \in \sigma$, and $\alpha = (x)_\sigma = (t)_\sigma = (xy)_\sigma = (x)_\sigma (y)_\sigma = \alpha\beta$.

$\Longleftarrow$. Let Y be a semilattice which is a chain under the order $\alpha \preceq \beta \Leftrightarrow \alpha = \alpha\beta (= \beta\alpha)$ and $\{S_\alpha \mid \alpha \in Y\}$ a family of subsemigroups of S of type $\mathcal{T}$ such that conditions (A)–(D) are satisfied. We consider the relation σ on S defined by

$$\sigma := \{(x, y) \in S \times S \mid \exists\, \alpha \in Y : x, y \in S_\alpha\}.$$

The relation σ is a complete semilattice congruence on S. In fact: The relation σ is clearly reflexive and symmetric. Let $(x, y) \in \sigma$ and $(y, z) \in \sigma$. Then $x, y \in S_\alpha$, $y, z \in S_\beta$ for some $\alpha, \beta \in Y$. If $\alpha \neq \beta$, then $y \in S_\alpha \cap S_\beta = \emptyset$ which is impossible. Thus we have $\alpha = \beta$, and $x, z \in S_\alpha$. Since $\alpha \in Y$ and $x, z \in S_\alpha$, we have $(x, z) \in \sigma$, so σ is transitive. Let $(x, y) \in \sigma$ and $z \in S$. Suppose $\alpha \in Y$ such that $x, y \in S_\alpha$ and $\beta \in Y$ such that $z \in S_\beta$. Then $xz, yz \in S_\alpha S_\beta \subseteq S_{\alpha\beta}$, where $\alpha\beta \in Y$, so $(xz, yz) \in \sigma$. Similarly we get $(zx, zy) \in \sigma$. Let $x, y \in S$, $x \in S_\alpha$, $y \in S_\beta$ for some $\alpha, \beta \in Y$. Then $xy \in S_\alpha S_\beta \subseteq S_{\alpha\beta}$ and $yx \in S_\beta S_\alpha \subseteq S_{\beta\alpha} = S_{\alpha\beta}$. Since $\alpha, \beta \in Y$ and $xy, yx \in S_{\alpha\beta}$, we have $(xy, yx) \in \sigma$. If $x \leq y$, then $(x, xy) \in \sigma$. Indeed: Let $x \in S_\alpha$, $y \in S_\beta$, $\alpha, \beta \in Y$. Then $xy \in S_\alpha S_\beta \subseteq S_{\alpha\beta}$ and $x \in (S_\beta]$. Since $S_\alpha \cap (S_\beta] \neq \emptyset$, by condition (D), we have $\alpha = \alpha\beta$. Since $x, xy \in S_\alpha$, $\alpha \in Y$, we have $(x, xy) \in \sigma$.

If $x \in S$ and $x \in S_\alpha$ for some $\alpha \in Y$, then $(x)_\sigma = S_\alpha$. In fact: Let $y \in (x)_\sigma$. Since $(y, x) \in \sigma$, there exists $\beta \in Y$ such that $y, x \in S_\beta$. If $\alpha \neq \beta$, then $x \in S_\alpha \cap S_\beta = \emptyset$ which is impossible. Thus we have $\alpha = \beta$, and $y \in S_\alpha$. Conversely, let $t \in S_\alpha$. Since $x, t \in S_\alpha$, $\alpha \in Y$, we have $(x, t) \in \sigma$, so $t \in (x)_\sigma$. As a consequence, $(x)_\sigma$ is a subsemigroup of S of type $\mathcal{T}$ for every $x \in S$.

Finally, let $x, y \in S$. Then $(x)_\sigma = (xy)_\sigma$ or $(y)_\sigma = (xy)_\sigma$. In fact: Let $x \in S_\alpha$, $y \in S_\beta$ for some $\alpha, \beta \in Y$. By hypothesis, we have $\alpha = \alpha\beta$ or $\beta = \alpha\beta$. If $\alpha = \alpha\beta$, then $S_\alpha = S_{\alpha\beta} \supseteq S_\alpha S_\beta$. On the other hand, since $x \in S_\alpha$, $y \in S_\beta$ and $\alpha, \beta \in Y$, we have $S_\alpha = (x)_\sigma$ and $S_\beta = (y)_\sigma$. Hence we have $(x)_\sigma \supseteq (x)_\sigma (y)_\sigma := (xy)_\sigma$, and $(x, xy) \in \sigma$. If $\beta = \alpha\beta$, similarly we obtain $(y)_\sigma = (xy)_\sigma$. □

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