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ON THE PROBLEM OF TOTAL STABILITY FOR PERIODIC DIFFERENTIAL SYSTEMS *

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ABSTRACT. Consider a smooth ω–periodic differential system in $\mathbf{R} \times \mathbf{R}^n$, say S_ω, of ordinary differential equations, and let E be an equilibrium for S_ω. Preliminarily it is shown that the total stability of E is equivalent to the existence of a fundamental family of asymptotically stable neighborhoods of E. Thus a known theorem of Seibert [8] concerning autonomous systems is extended to periodic systems. Let us assume now the existence of a smooth invariant manifold Φ in $\mathbf{R} \times \mathbf{R}^n$, containing $\mathbf{R} \times \{E\}$, ω–periodic in t, and asymptotically stable "near" E. By using the above extension of Seibert's theorem and some previous results in our paper [6], [7], we prove here that if E is totally stable on Φ (that is with respect to the solutions lying on Φ), then E is unconditionally totally stable.

1. Introduction. We have been concerned in [6], [7] with the stability problem of time dependent sets under a differential system $\dot{x} = f(t,x)$, $(t,x) \in \mathbf{R} \times \mathbf{R}^n$, and f smooth. More specifically, let M be a positively invariant, s–compact subset of $\mathbf{R} \times \mathbf{R}^n$ contained in a smooth invariant m–dimensional manifold $\Phi \subseteq \mathbf{R} \times \mathbf{R}^n$, $m < n$. We recall that a set A in $\mathbf{R} \times \mathbf{R}^n$ is said to be s–compact if for any t in $\mathbf{R}$ the section $A(t) = \{x \in \mathbf{R}^n \,:\, (t,x) \in A\}$ is nonempty, compact and there exists a compact set K in $\mathbf{R}^n$ such that $A(t) \subseteq K$ for all t in $\mathbf{R}$. Moreover the set A is said to be periodic or t–independent if the map $t \to A(t)$ is periodic or time independent respectively. Under the assumption that M is uniformly asymptotically stable on Φ (that is with respect to initial data lying on Φ) we have analyzed in [6], [7] the unconditional stable behavior of M. The results involve the stability properties of Φ "near" M.

Assume now that: (a) f is ω–periodic in t, for some $\omega > 0$ and M is time independent, that is $M = \mathbf{R} \times N$, N being a compact subset of $\mathbf{R}^n$; (b) Φ is ω–periodic in t, precisely $\Phi = \{(t,y,z) : t \in \mathbf{R}, y \in \mathbf{R}^m, z = g(t,y)\}$ where $(y,z) = x$ and g is ω–periodic in t and smooth; (c) Φ is asymptotically stable near M. In terms of the variables (y,u) with $u = z - g(t,y)$ we may write $\Phi = \mathbf{R} \times \Psi$ with Ψ an invariant subset of $\mathbf{R}^n$. The stability properties of any set $Q \subseteq \Psi$ (in particular for $Q = N$ or $Q = \Psi$) may be then viewed as stability properties of $\mathbf{R} \times Q$ and vice versa. Let $\mathcal{P}$ be a given stability property of N. We will say that $\mathcal{P}$ is transferable from Ψ to the whole space if the occurrence of $\mathcal{P}$ on Ψ implies its unconditional occurrence. On the basis of the results in [6],[7] recalled before we have that the asymptotic stability is one of these property. It seems very natural to conjecture that under conditions $(a), (b), (c)$ such transferability holds even if the asymptotic stability is replaced by the weaker property of total stability, that is the stability with respect to both the perturbations of the initial data and the perturbations of the differential system. As a first step we will prove here the conjecture in the case that N is an equilibrium E and then $M = \mathbf{R} \times E$. In other words we prove that if E is totally stable on Ψ, then E is unconditionally totally stable. The notion of total stability will be intended in the sequel always in a uniform sense.

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Preliminarly on the basis of the results in [5] we prove that the total stability of an equilibrium of a periodic system is equivalent to the existence of a fundamental family of asymptotically stable compact neighborhoods of the equilibrium. Thus a known statement due to Seibert [8] for autonomous systems is integrally extended to periodic systems. Our result is at last obtained by using this equivalence and the transferability from Ψ to the whole space of the asymptotic stability of compact subsets of Ψ.

We conclude the paper by observing that the assumptions $(a), (b), (c)$ are not sufficient to guarantee the transferability of the non–asymptotic stability of the origin from Ψ to the whole space, even in the autonomous case (Section 5). We notice at last that in a well known paper [3] , Kelley proved this transferability under conditions $(a), (b)$ but with (c) replaced by the stronger assumption that the asymptotic stability of Ψ near the origin is recognizable on the linear part of f, being then of the exponential type.

2 Preliminaries. Although our interest is mainly devoted to periodic differential systems, the results in this section (needed for our successive treatment) concern the more general system:

$$\dot{x} = f(t, x), \qquad (\dot{}) = \frac{d}{dt}, \tag{2.1}$$

where $f \in C(\mathbf{R} \times \mathbf{R}^n, \mathbf{R}^n)$ is continuous, locally Lipschitzian in x uniformly in t, that is for every compact set $K \subset \mathbf{R}^n$ there exists a constant $L(K) > 0$ such that $\|f(t,x) - f(t,y)\| \leq L(K)\|x-y\|$ for all t in $\mathbf{R}$ and x, y in K, Moreover we assume that $f(t, 0) \equiv 0$, so that (2.1) admits the solution $x \equiv 0$. By a suitable modification of f outside of a neighborhood of the origin we will assume without any restriction for our stability problems, that the solutions of (2.1) exists for any time t in $\mathbf{R}$. The solution of (2.1) passing through $(t_0, x_0) \in \mathbf{R} \times \mathbf{R}^n$ will be denoted by $x(t, t_0, x_0)$. Given $\mu > 0$, any function $\psi : [t_0, t_1] \to \mathbf{R}^n$, $t_0 < t_1$, will be called a μ–solution of (2.1) if ψ is absolutely continuous and sup$\|\psi(t) - f(t, \psi(t))\| < \mu$. For $a > 0$ we denote by $B^n(a)$ and by $B^n[a]$ the sets $\{x \in \mathbf{R}^n : \|x\| < a\}$ and $\{x \in \mathbf{R}^n : \|x\| \leq a\}$ respectively. We will assume as known the stability concepts of sets with respect to perturbations of the initial conditions (t_0, x_0). For the stability concepts under constantly acting perturbations, we fix our attention on the total stability of the null solution. Precisely we assume the following definition:

Definition 2.1 *The solution $x \equiv 0$ of (2.1) is said to be totally stable (in a uniform sense) if given any $\varepsilon > 0$, there exist $\delta_1 = \delta_1(\varepsilon) > 0$, $\delta_2 = \delta_2(\varepsilon) > 0$ such that if $t_0 \in \mathbf{R}$ and $\psi : [t_0, t_1] \to \mathbf{R}^n$ is any δ_2–solution of (2.1) with $\|\psi(t_0)\| < \delta_1$, then $\|\psi(t)\| < \varepsilon$ for all $t \in [t_0, t_1]$.*

This definition is stronger than the usual definition of uniform total stability of equilibrium due to Dubosin [1] , which involves differentiable perturbations. The two definitions coincide if the δ_2–solutions are all solutions of differential equations.

The property of total stability of the origin as in Definition 2.1 may be related to the existence of a suitable family of compact sets all contained in a bounded neighborhood of the origin. To do this we need the following definition.

Definition 2.2 *Let $U \subset \mathbf{R}^n$ be a compact set. U is said to be:*

(i) *contracting if (1) $x(t, t_0, x_0) \in$ int(U); for all $(t_0, x_0) \in \mathbf{R} \times U$ and $t > t_0$, (2) there exists a compact set V and $T > 0$ such that $x(t, t_0, x_0) \in V$ for all $(t_0, x_0) \in \mathbf{R} \times U$ and $t \geq t_0 + T$ (in the autonomous case (1) implies (2));*

(ii) *quasi-contracting, if there exists a compact set V and a divergent sequence $\{\theta_k\} \subset \mathbf{R}$, $\theta_{k+1} > \theta_k$, $k = 0, 1, 2, ...$, such that*

(a) *sup$(\theta_{k+1} - \theta_k) < +\infty$*

(b) *$x(\theta_{k+1}, \theta_k, x_0) \in V$ for all $x_0 \in U$ and $k = 0, 1, 2, ...$*

In [5] the following theorem was proven.

Theorem 2.1 *Assume that the solution $x \equiv 0$ of (2.1) is stable and that the origin admits a fundamental family F of quasi-contracting compact neighborhoods. Then the solution $x \equiv 0$ of (2.1) is totally stable.*

As a consequence of this theorem we may prove that the well known theorem of Malkin [4] and Gorsin [2] that the uniform asymptotic stability of the origin implies its total stability in the sense of Dubosin's definition [1] is still valid also in the sense of Definition 2.1. Precisely Theorem 2.1 admits the following corollary.

Corollary 2.1 *If the null solution of (2.1) is uniformly asymptotically stable, then it is (uniformly) totally stable.*

Proof. Since the null solution is uniformly asymptotically stable, there exists $\sigma > 0$ such that for every $\nu \in (0, \sigma)$ we can find $T > 0$ such that

$$t_0 \in \mathbf{R} \text{ and } \|x_0\| < \sigma \text{ imply } \|x(t, t_0, x_0)\| < \frac{\nu}{2} \text{ for all } t \geq t_0 + T.$$

Hence $B^n[\nu]$ is contracting and consequently quasi-contracting. The result follows by applying Theorem 2.1. ■

3. Total stability for periodic differential systems. Assume now that system (2.1) is periodic:

$$\dot{x} = f(t, x), \qquad f(t, 0) \equiv 0, \tag{3.1}$$

where $f \in C(\mathbf{R} \times \mathbf{R}^n, \mathbf{R}^n)$ is continuous, locally Lipschitzian in x, and periodic in t for a constant $\omega > 0$. Moreover as in Section 2 we assume that the solution $x(t, t_0, x_0)$ through (t_0, x_0) exists for all t in $\mathbf{R}$.

We emphasize that because of the periodicity of f, the stability and the asymptotic stability of any compact set in $\mathbf{R}^n$ are always uniform. Moreover we wish to emphasize that if the conditions for the occurrence of the stability properties with respect to the perturbation of the initial data (t_0, x_0) are satisfied for one fixed t_0, they are satisfied for any other t_0.

In [5] it was proven that for such a system Theorem 2.1 is invertible. More precisely the following theorem holds:

Theorem 3.1 *The solution $x \equiv 0$ of (3.1) is totally stable if and only if:*

(i) *the solution $x \equiv 0$ is stable;*

(ii) *there exists a fundamental family F of quasi-contracting compact neighborhoods of the origin.*

Moreover if $x \equiv 0$ is totally stable, then for any $U \in F$ one may choose

$$V = \{x(\omega, 0, x_0) : x_0 \in U\} \text{ and } \theta_0 = 0,\ \theta_k = k\omega \text{ for any } k \in \mathbf{N} \equiv \{1, 2, ...\}. \tag{3.2}$$

We prove now another characterization of total stability for the origin which, as we said in Section 1, is an extension of a theorem due to Seibert [8] . Precisely the following theorem holds:

Theorem 3.2 *The solution $x \equiv 0$ of (3.1) is totally stable if and only if there exists a fundamental family G of compact neighborhoods of the origin which are asymptotically stable.*

Proof. **(1) Necessity**. By virtue of Theorem 3.1, for any given $\sigma > 0$ there exist two compact neighborhoods of the origin $U = U(\sigma)$, $V = V(\sigma)$, such that $U \in F$, $U \subset B^n(\sigma)$, and $V \subset \text{int}(U)$ is expressed by (3.2) with $\theta_k = k\omega$ for all $k \in \mathbf{N}$. Thus

$$x(k\omega, 0, x_0) \in V \ \text{ for any } \ x_0 \in U \ \text{ and } \ k \in \mathbf{N}. \tag{3.3}$$

Indeed, along the solution $x(t, 0, x_0)$ we have $x_1 = x(\omega, 0, x_0) \in V \subset U$. Hence $x(2\omega, 0, x_0) = x(2\omega, \omega, x_1) \in V$ and so on. Choose now any $\varepsilon \in (0, \rho(\partial U, V))$. Since the function

$$(t, y) \to \rho(x(t, 0, y), V), t \in \mathbf{R}, y \in \mathbf{R}^n,$$

is uniformly continuous in $[0, \omega] \times U$, there exists $\delta \in (0, \varepsilon)$ such that

$$\rho(y, V) < \delta \text{ implies } \rho(x(t, 0, y), V) < \varepsilon \text{ for any } t \in [0, \omega].$$

Then, taking into account that $x(t, t_0, x_0) = x(t + \omega, t_0 + \omega, x_0)$ for any t_0, x_0, it follows

$$\rho(y, V) < \delta \text{ implies } \rho(x(t, k\omega, y), V) < \varepsilon \text{ for any } t \in [k\omega, (k + 1)\omega] \text{ and } k \in \mathbf{N}. \tag{3.4}$$

Given any y in $B^n(V, \delta)$ let us consider now the motion $x(t, 0, y)$ in the interval $[0, +\infty)$. Since $B^n(V, \varepsilon) \subset U$ and $\delta \in (0, \varepsilon)$, by virtue of (3.3), (3.4) it is immediate to recognize that

$$\rho(y, V) < \delta \text{ implies } \rho(x(t, 0, y), V) < \varepsilon \text{ for any } t \geq 0.$$

Thus V is (uniformly) stable. On the other hand V is weakly attracting by virtue of (3.3). Then V is asymptotically stable. Because of the arbitrariness of $\sigma > 0$, the family G exists and one has $G = \{V(\sigma)\}$.

(2) Sufficiency. For any given $\sigma > 0$, let $D = D(\sigma)$ be the compact set of the family G contained in $B^n(\sigma)$. Since D is asymptotically stable, then there exists a compact neighborhood $U = U(\sigma) \subset B^n(\sigma)$ of D such that for any compact neighborhood $V \subset \text{int}(U)$ of D one may find $\tau = \tau(V) > 0$ for which $x(t, t_0, x_0) \in V$ for all $t_0 \geq 0$, $t \geq t_0 + \tau$ and $x_0 \in U$. For a fixed V, and for an integer j such that $j\omega \geq \tau$ we then have $x(kj\omega, 0, x_0) \in V$ for all $x_0 \in U$ and $k \in \mathbf{N}$. Thus the family F exists with $F = \{U(\sigma)\}$. It remains to prove that the origin is stable for (3.1). Consider any $\varepsilon > 0$ and a member D of the family G contained in $B^n(\varepsilon)$. Let $\varepsilon_1 \in (0, \varepsilon)$ such that $B^n(D, \varepsilon_1) \subset B^n(\varepsilon)$. Since D is stable, there exists $\delta \in (0, \varepsilon_1)$ such that

$$x_0 \in B^n(D, \delta) \text{ implies } x(t, 0, x_0) \in B^n(D, \varepsilon_1) \text{ for any } t \geq 0,$$

that is $\|x(t, 0, x_0)\| < \varepsilon$ for any $t \geq 0$. Since $B^n(D, \delta)$ is a neighborhood of the origin, the stability of $x \equiv 0$ follows. The proof is complete. ∎

4 Conditional and unconditional total stability properties. Assume now that for an integer $m \in (0, n)$ system (3.1) admits a $(m + 1)$–invariant manifold

$$\Phi = \{(t, y, z) \, : t \in \mathbf{R}, y \in \mathbf{R}^m, z = g(t, y)\}, \tag{4.1}$$

where $g \in C^2(\mathbf{R} \times \mathbf{R}^m, \mathbf{R}^{n-m})$ is ω–periodic in t and $g(t, 0) \equiv 0$. Let $u = z - g(t, y)$. In terms of (y, u) system (3.1) may be written as

$$\begin{aligned} \dot{y} &= Y(t, y, u) \\ \dot{u} &= U(t, y, u), \end{aligned} \tag{4.2}$$

where Y, U are continuous, locally Lipschitzian in (y, u), ω–periodic in t, and $Y(t, 0, 0) \equiv 0$, $U(t, y, 0) \equiv 0$. Moreover in terms of (y, u) we may write $\Phi = \mathbf{R} \times \Psi$, where

$$\Psi = \{(y, u) \,:\, y \in \mathbf{R}^m, u = 0\}, \tag{4.3}$$

is an m–invariant manifold in $\mathbf{R}^n$. The y–part of the solutions of (4.2) lying on Ψ are the solutions of the system

$$\dot{y} = Y(t, y, 0). \tag{4.4}$$

Let N be a positively invariant compact set in $\mathbf{R}^n$. Since the sets $M = \mathbf{R} \times N$, $\Phi = \mathbf{R} \times \Psi$ are both time independent, as we pointed out in Section 1, the stability properties of M, Φ may be viewed as stability properties of N, Ψ respectively and vice versa. Then the Definition 2.1 and the Theorem 4.1 in [7] may be in the present case reformulated as follows:

Definition 4.1 *Let $N \subset \Psi$ be a compact set. We say that Ψ has a stability property near N if there exists a neighborhood $\mathcal{N}$ of N in $\mathbf{R}^n$ such that the stability property is satisfied with respect to the solutions $((y(t, t_0, y_0, u_0), u(t, t_0, y_0, u_0))$ of (4.2) for which $(y_0, u_0) \in \mathcal{N}$.*

Theorem 4.1 *Let $N \subset \Psi$ be a compact set. Then we have:*

(u) *the stability and the asymptotic stability of Ψ near N when occurring are always uniform;*

(v) *if N is asymptotically stable on Ψ, then N is unconditionally stable (asymptotically stable) if and only if Ψ is stable (asymptotically stable) near N.*

We are now in position to prove the following theorem.

Theorem 4.2 *Assume that:*

(i) *Ψ is asymptotically stable near $\{(0, 0)\}$;*

(ii) *$\{(0, 0)\}$ is totally stable on Ψ (i.e. $y \equiv 0$ is totally stable for (4.4)).*

Then $\{(0, 0)\}$ is unconditionally totally stable.

Proof. By virtue of Theorem 3.2 applied to system (4.4), condition (ii) implies the stability of $\{(0, 0)\}$ on Ψ and the existence of a fundamental family G of compact neighborhoods of $y = 0$ in the y–space which are asymptotically stable on Ψ. We choose the family G such that each member of the family is contained in the open set $\mathcal{N}$ associated as in Definition 4.1 with the asymptotic stability of Ψ near $\{(0, 0)\}$. Clearly then Ψ is asymptotically stable near each $D \in G$. By using Theorem 4.1, we recognize that each member of G is (unconditionally) asymptotically stable for system (4.2) . Given any $\sigma \in (0, \rho(\{(0, 0)\}, N))$ let $D = D(\sigma)$ be the member of G contained in $B^n(\sigma)$. Since D is asymptotically stable for (4.2), then there exists a compact neighborhood $U \subset B^n(\sigma)$ of D such that for any compact neighborhood $V \subset \text{int}(U)$ of D one may find $\tau = \tau(V) > 0$ for which $x(t, t_0, x_0) \in V$ for any $t_0 \geq 0$, $t \geq t_0 + \tau$ and $x_0 \in U$. Hence V is uniformly attracting and then stable and then asymptotically stable for (4.2). Thus choosing for the given σ one of the sets V, say $V(\sigma)$, we obtain a family $\{V(\sigma)\}$ which satisfies for system (4.2) the conditions in Theorem 3.2. The proof is complete. ■

5. Final considerations. Let us assume that Ψ is uniformly asymptotically stable near $x = 0$. Then from our results in [6], [7] it follows that the origin is asymptotically stable for (4.2) if the origin is asymptotically stable on Ψ. In other words the asymptotic stability is transferable from Ψ to the whole space. It is natural to ask if this happens even for the non-asymptotic stability. The answer is negative. This has been proved in [6] by the following simple counterexample that we report here. Let us consider the system

$$\begin{aligned} \dot{y} &= yz^2 \\ \dot{z} &= -z^3, \end{aligned} \tag{5.1}$$

with $y, z \in \mathbf{R}$. Clearly any solution of (5.1) exists for all $t \geq t_0$. Hence, by using $(5.1)_2$, we see that $\Psi = \{(y, z) \,:\, y \in \mathbf{R},\ z = 0\}$ is an asymptotically stable manifold in $\mathbf{R}^2$. Moreover with respect to the solutions lying on Ψ the origin is stable but non-asymptotically. It is immediate to see that the origin is unstable. Indeed $(5.1)_1$ by means of $(5.1)_2$ may be written as

$$\dot{y} = \frac{yz_0^2}{1 + 2z_0^2(t - t_0)}$$

from which it follows

$$y(t, t_0, y_0, z_0) = y_0[1 + 2z_0^2(t - t_0)]^{\frac{1}{2}}.$$

Thus $y(t, y_0, z_0) \to +\infty$ as $t \to +\infty$ for any choice of $y_0 \neq 0$, $z_0 \neq 0$. Hence our assert follows. Then it is clear that in order to have the transferability from Ψ to the whole space of the non–asymptotic stability of the origin, we have to enforce the property that Ψ is asymptotically stable near M. On this line it may be considered a well known result due to Kelley. In the case of autonomous systems, by requiring that the asymptotic stability of the manifold is recognizable by the first order term of the $r.h.s.$ of the equation (i.e. the manifold is exponentially asymptotically stable), he proved that the stability on the manifold ensures the unconditional stability [3] . Instead if we want to retain our assumption on the asymptotic behavior of Ψ, the non–asymptotic stability is not a transferable property. Hence it is natural to search for transferable properties in between non–asymptotic and asymptotic stability. The result on the transferability of the total stability of the origin that we obtained in Theorem 4.2 is exactly on this line.

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