MULTIPLIERS IN SUBTRACTION ALGEBRAS

Yong Ho Yon and Kyung Ho Kim

Received December 31, 2010

Abstract. In this paper, we introduce the concept of multiplier of subtraction algebras and obtained some properties of subtraction algebras. Also, we introduce the simple multiplier and characterized the kernel of multipliers of subtraction algebras.

1 Introduction In [4] a partial multiplier on a commutative semigroup $(A, \cdot)$ has been introduced as a function F from a nonvoid subset D_F of A into A such that $F(x)\cdot y = x\cdot F(y)$ for all $x, y \in D_F$. In this paper, we introduce the concept of multiplier of subtraction algebras and obtained some properties of subtraction algebras. Also, we introduce the simple multiplier and characterized the kernel of multipliers of subtraction algebras.

2 Preliminaries We first recall some basic concepts which are used in the paper.

By a *subtraction algebra* we mean an algebra $(X; -)$ with a single binary operation "$-$" that satisfies the following identities: for any $x, y, z \in X$,

(S1) $x - (y - x) = x$;

(S2) $x - (x - y) = y - (y - x)$;

(S3) $(x - y) - z = (x - z) - y$.

The last identity permits us to omit parentheses in expressions of the form $(x - y) - z$. The subtraction determines an order relation on X: $a \leq b \Leftrightarrow a - b = 0$, where $0 = a - a$ is an element that does not depend on the choice of $a \in X$. The ordered set $(X; \leq)$ is a semi-Boolean algebra in the sense of [1], that is, it is a meet semilattice with zero 0 in which every interval $[0, a]$ is a Boolean algebra with respect to the induced order. Here $a \wedge b = a - (a - b)$; the relative complement b' of an element $b \in [0, a]$ is $a - b$; and if $b, c \in [0, a]$, then

$$\begin{aligned} b \vee c &= (b' \wedge c')' = a - ((a - b) \wedge (a - c)) \\ &= a - ((a - b) - ((a - b) - (a - c))). \end{aligned}$$

In a subtraction algebra, the following properties are true:

(p1) $(x - y) - y = x - y$.

(p2) $x - 0 = x$ and $0 - x = 0$.

(p3) $(x - y) - x = 0$.

(p4) $x - (x - y) \leq y$.

(p5) $(x - y) - (y - x) = x - y$.

2000 *Mathematics Subject Classification.* 06F35, 03G25, 08A30.
Key words and phrases. Subtraction algebra, multiplier, simple multiplier.

(p6) $x - (x - (x - y)) = x - y$.

(p7) $(x - y) - (z - y) \leq x - z$.

(p8) $x \leq y$ if and only if $x = y - w$ for some $w \in X$.

(p9) $x \leq y$ implies $x - z \leq y - z$ and $z - y \leq z - x$ for all $z \in X$.

(p10) $x, y \leq z$ implies $x - y = x \wedge (z - y)$.

(p11) $(x \wedge y) - (x \wedge z) \leq x \wedge (y - z)$.

(p12) $(x - y) - z = (x - z) - (y - z)$.

A non-empty subset I of a subtraction algebra X is called a *subalgebra* if $x - y \in I$ for all $x, y \in I$. A mapping d from a subtraction algebra X to a subtraction algebra Y is called a *morphism* if $d(x - y) = d(x) - d(y)$ for all $x, y \in X$. A self map d of a subtraction algebra X which is a morphism is called an *endomorphism*.

A nonempty subset I of a subtraction algebra X is called an *ideal* of X if it satisfies

(I1) $0 \in I$,

(I2) for any $x, y \in X$, $y \in I$ and $x - y \in I$ implies $x \in I$.

For an ideal I of a subtraction algebra X, it is clear that $x \leq y$ and $y \in I$ imply $x \in I$ for any $x, y \in X$. If $x \leq y$ implies $d(x) \leq d(y)$, d is called an *isotone maping*.

A function f of a semilattice ($\wedge$-semilattice) L into itself is a *dual closure* if f is monotone, non-expansive(i.e., $f(x) \leq x$ for all $x \in L$) and idempotent(i.e., $f \circ f = f$).

3 Multipliers in subtraction algebra

Definition 3.1. Let $(X, -, 0)$ be a subtraction algebra. A self-map f is called a *multiplier* if

$$f(x - y) = f(x) - y$$

for all $x, y \in X$.

Example 3.2. Let $X = \{0, a, b\}$ be a subtraction algebra with the following Cayley table

$-$	0	a	b
0	0	0	0
a	a	0	a
b	b	b	0

Define a map $f : X \to X$ by

$$f(x) = \begin{cases} 0 & \text{if } x = 0, a \\ b & \text{if } x = b \end{cases}$$

Then it is easily checked that f is a multiplier of subtraction algebra X.

Example 3.3. Let $X = \{0, a, b, 1\}$ in which "$-$" is defined by

$-$	0	a	b	1
0	0	0	0	0
a	a	0	a	0
b	b	b	0	0
1	1	b	a	0

It is easy to check that $(X; -, 0)$ is a subtraction algebra. Define a map $f : X \to X$ by

$$f(x) = \begin{cases} 0 & \text{if } x = 0, a \\ b & \text{if } x = b, 1 \end{cases}$$

Then f is a multiplier of subtraction algebra X.

Example 3.4. The identity mapping ϵ, the unit mapping $\iota : a \longmapsto 1$ are multipliers of X.

Example 3.5. For $p \in X$, the mapping $\alpha_p(a) = a - p$ is multiplier of X.

Proof. Let $p \in X$. Then we have

$$\alpha_p(a - b) = (a - b) - p = (a - p) - b = \alpha_p(a) - b.$$

This completes the proof. □

Lemma 3.6. *Let f be a multiplier in subtraction algebra X. Then we have $f(0) = 0$.*

Proof. Since $0 = 0 - x$ for all $x \in X$, we have $f(0) = f(0 - f(0)) = f(0) - f(0) = 0$. □

Proposition 3.7. *Let f be a multiplier in subtraction algebra. Then $f(x) \leq x$ for all $x \in X$,*

Proof. Putting $x = y$ in Definition 3.1, we get $0 = f(0) = f(x - x) = f(x) - x$, that is, $f(x) \leq x$. □

Theorem 3.8. *Let f be a multiplier of subtraction algebra X. If $x \leq y$ for any $x, y \in X$, then $f(x) \leq f(y)$.*

Proof. Let $x \leq y$ for all $x, y \in X$. Then by (p8), $x = y - w$ for some $w \in X$. Hence we have

$$f(x) = f(y - w) = f(y) - w \leq f(y).$$

This completes the proof. □

Theorem 3.9. *Let f be a multiplier of a subtraction algebra X. Then we have $f^2 = f \circ f = f$.*

Proof. Let f be a multiplier of X. Then by definition of multiplier f and Proposition 3.6, we have

$$\begin{aligned} f^2(x) &= f(f(x)) = f(x \wedge f(x)) \\ &= f(x - (x - f(x))) \\ &= f(x) - (x - f(x)) \\ &= f(x) \end{aligned}$$

□

Corollary 3.10. *Let f be a multiplier of a subtraction algebra X. Then f is a dual closure operator on X.*

Proof. It is clear from Proposition 3.7 and Theorem 3.8 and 3.9. □

Proposition 3.11. *Let f is a non-expansive map on a subtraction algebra X, i.e., $f(x) \leq x$ for all $x \in X$. Then $f(x) - y \leq x - f(y)$ for all $x, y \in X$.*

Proof. Suppose that f is a non-expansive map on X and $x, y \in X$. Then $f(x) \leq x$ and $f(y) \leq y$. Hence $f(x) - y \leq x - y$ and $x - y \leq x - f(y)$ by (p9). It follows that $f(x) - y \leq x - f(y)$. □

Theorem 3.12. *Let X be a subtraction algebra. Every multiplier of X is an homomorphism.*

Proof. Suppose that f is a multiplier of X and $x, y \in X$. Then $f(y) \leq y$. It implies

$$f(x - y) = f(x) - y \leq f(x) - f(y)$$

by (p9). Also we have

$$\begin{aligned}
&(f(x) - f(y)) - (f(x) - y) \\
&\quad = (ff(x) - f(y)) - (f(x) - y) \quad \text{(by Theorem 3.9)} \\
&\quad = (ff(x) - (f(x) - y)) - f(y) \quad \text{(by (S3))} \\
&\quad = f(f(x) - (f(x) - y)) - f(y) \quad \text{(by Definition 3.1)} \\
&\quad = f(y - (y - f(x))) - f(y) \quad \text{(by (S2))} \\
&\quad \leq f(y) - f(y) \quad \text{(by p(3), Theorem 3.9 and p(9))} \\
&\quad = 0
\end{aligned}$$

It follows that $(f(x) - f(y)) - (f(x) - y) = 0$ and $f(x) - f(y) \leq f(x) - y = f(x - y)$. Hence $f(x) - f(y) = f(x - y)$. □

Let X be a subtraction algebra and f_1, f_2 two self-maps. We define $f_1 \circ f_2 : X \to X$ by

$$(f_1 \circ f_2)(x) = f_1(f_2(x))$$

for all $x \in X$.

Proposition 3.13. *Let X be a subtraction algebra and f_1, f_2 two multipliers. Then $f_1 \circ f_2$ is also a multiplier of X.*

Proof. Let X be a subtraction algebra and f_1, f_2 two multipliers. Then we have

$$\begin{aligned}
(f_1 \circ f_2)(a - b) &= f_1(f_2(a - b)) \\
&= (f_1(f_2(a) - b)) \\
&= f_1(f_2(a)) - b \\
&= (f_1 \circ f_2)(a) - b.
\end{aligned}$$

This completes the proof. □

Let X be a subtraction algebra and f_1, f_2 two self-maps. We define $(f_1 \wedge f_2)(x)$ by

$$(f_1 \wedge f_2)(x) = f_1(x) \wedge f_2(x)$$

for all $x \in X$.

Proposition 3.14. *Let X be a subtraction algebra and f_1, f_2 two multipliers. Then $f_1 \wedge f_2$ is also a multiplier of X.*

Proof. Let X be a subtraction algebra and f_1, f_2 two multipliers. Then we have

$$\begin{aligned}
(f_1 \wedge f_2)(a - b) &= f_1(a - b) \wedge f_2(a - b) \\
&= f_1(a) - b \wedge f_2(a) - b \\
&= f_1(a) - b - ((f_1(a) - b) - (f_2(a) - b)) \\
&= f_1(a) - b - ((f_1(a) - f_2(a)) - b) \\
&= (f_1(a) - (f_1(a) - f_2(a))) - b \\
&= (f_1(a) \wedge f_2(a)) - b \\
&= (f_1 \wedge f_2)(a) - b.
\end{aligned}$$

This completes the proof. □

Denote by $\mathcal{M}(X)$ the set of all multipliers of X.

Definition 3.15. For any $f \in \mathcal{M}(X)$, we define the *kernel* of f as follows:

$$K_\varphi := \{x \in X \mid \varphi(x) = 0\}.$$

Proposition 3.16. *Let f be a multiplier of X. Then K_φ is a subalgebra of X.*

Proof. Clearly, $0 \in K_\varphi$ and so K_φ is nonempty. For any $x, y \in K_\varphi$, we have $f(x - y) = f(x) - y = 0 - y = 0$, and so $x - y \in K_\varphi$. Hence K_φ is a subalgebra of X. □

Theorem 3.17. *Let φ be a multiplier of X. If $y \in K_\varphi$ and $x \leq y$, then $x \in K_\varphi$.*

Proof. Let $y \in K_\varphi$ and $x \leq y$. Then

$$\begin{aligned}
f(x) &= f(x - 0) = f(x - (x - y)) \\
&= f(y - (y - x)) = f(y) - (y - x) \\
&= 0 - (y - x) = 0
\end{aligned}$$

since $f(y) = 0$ and $x - y = 0$. This completes the proof. □

Theorem 3.18. *Let φ be a multiplier and a homomorphism of X. Then K_φ is an ideal of X.*

Proof. Clearly, $0 \in K_\varphi$ since $\varphi(0) = 0$. Let $y \in K_\varphi$ and $x - y \in K_\varphi$. Then $0 = \varphi(x - y) = \varphi(x) - \varphi(y) = \varphi(x) - 0 = \varphi(x)$, and so $x \in \varphi(y)$. This completes the proof. □

Proposition 3.19. *Let f be a multiplier in subtraction algebra and $F_f = \{x \in X \mid f(x) = x\}$. Then F_f is a subalgebra of X.*

Proof. Let $x, y \in F_f$. Then we have $f(x) = x$ and $f(y) = y$. Hence $f(x - y) = f(x) - y = x - y$, and so $x - y \in F_f$. This proves that F_f is a subalgebra of X. □

Proposition 3.20. *Let X be a subtraction algebra and f a multiplier. If $x \in F$, then $x \wedge y \in F$.*

Proof. Let $x \in F$. Then we have $f(x) = x$, and so

$$\begin{aligned} f(x \wedge y) &= f(x - (x - y)) \\ &= f(x) - x - y \\ &= x - (x - y) \\ &= x \wedge y. \end{aligned}$$

This completes the proof. □

Proposition 3.21. *Let X be a subtraction algebra and f a multiplier. If $y \in F$ and $x \leq y$, then $x \in F$.*

Proof. Let X be a subtraction algebra. If $y \in F$ and $x \leq y$, then we have

$$\begin{aligned} f(x) &= f(x - 0) = f((x - (x - y)) \\ &= f(y - (y - x)) = f(y) - (y - x) \\ &= y - (y - x) = x - (x - y) \\ &= x - 0 = x. \end{aligned}$$

This completes the proof. □

We call the derivation $\alpha_p(a) = a - p$ of Example 3.5 as *simple multiplier.*

Proposition 3.22. *For every $p \in X$, the simple multiplier α_p is an endomorphism of X.*

Proof. Let $a, b \in X$. Using (P12), we have

$$\alpha_p(a - b) = (a - b) - p = (a - p) - (b - p) = \alpha_p(a) - \alpha_p(b).$$

Hence α_p is an endomorphism of X.. □

Proposition 3.23. *The simple multiplier α_0 is an identity function of X.*

Proof. For every $a \in X, \alpha_0(a) = a - 0 = a$. This completes the proof. □

Proposition 3.24. *Let X be a subtraction algebra. Then, for each $p \in X$, we have $\alpha_p(x \wedge p) = 0$.*

Proof. For each $p \in X$, we have

$$\begin{aligned} \alpha_p(x \wedge p) &= \alpha_p((x - (x - p)) = (x - (x - p)) - p \\ &= (x - p) - (x - p) = 0. \end{aligned}$$

This completes the proof. □

Denote by $\mathcal{M}(X)$ by the set of all multipliers of X. That is,

$$\mathcal{M}(X) = \{f \mid f \text{ is a multiplier on } X\}.$$

Define a binary operation "$\circ$" and a map $\mathbf{1}$ on $\mathcal{M}(X)$ as follows:

$$(\alpha \circ \beta)(x) = \alpha(x)\beta(x) \text{ and } \mathbf{1}(x) = 1$$

for all $\alpha, \beta \in \mathcal{M}(X)$ and $x \in X$.

Proposition 3.25. *Let X be an subtraction algebra. Then $\mathcal{M}(X)$ is a subtraction algebra with the above binary operation "$\circ$".*

Proof. It is easy to prove that $\mathcal{M}(X)$ is a subtraction algebra with the above binary operation "$\circ$". □

Let $\mathcal{S}(X)$ be the set of all simple multipliers on X, i.e.,

$$\mathcal{S}(X) = \{\alpha_p \in \mathcal{M}(X) \mid p \in X\}$$

and $\mathcal{M}(X) = \{f \mid f : X \to X : \textit{multiplier}\}$. Then It is clear that $\mathcal{S}(X) \subseteq \mathcal{M}(X)$.

Proposition 3.26. *Let X be a subtraction algebra. If $\theta : X \to \mathcal{M}(X)$ is map defined by $\theta(p) = \alpha_p$ for all $p \in X$, where α_p is a simple map on X, then θ is a homomorphism of X.*

Proof. Let $f \in \mathcal{M}(X)$ and $p, q \in H$. Then we have $\theta(p - q) = \alpha_{p-q}$. But $\alpha_{p-q}(x) = (p-q)-x = (p-x)-(q-x) = \alpha_p(x)-\alpha_q(x) = (\alpha_p-\alpha_q)(x)$. Hence we have $\alpha_{p-q} = \alpha_p-\alpha_q$, i.e., $\theta(p - q) = \theta(p) - \theta(q)$. This completes the proof. □

Proposition 3.27. *Let X be a subtraction algebra and $\mathcal{M}(X) = \{f \mid f$ is a multiplier on $X\}$. If $\theta : X \to \mathcal{M}(X)$ is map defined by $\theta(p) = \alpha_p$ for all $p \in X$, where α_p is a simple map on X, then θ is an isotone mapping.*

Proof. Let $p \leq q$. Then we have $p - q = 0$, and $\theta(p) - \theta(q) = \theta(p - q) = \theta(0) = 0$ since θ is an homomorphism from Proposition 3.26. Hence $\theta(p) - \theta(q) = 0$, that is, $\theta(p) \leq \theta(q)$. This completes the proof. □

References

[1] J. C. Abbott, *Sets, Lattices and Boolean Algebras,* Allyn and Bacon, Boston 1969.

[2] G. Gierz, K. H. Hofmann, K. Keimel, J. D. Lawson, M. Mislove, D. S. Scott, A Compendium of Continuous Lattices, Springer-Verlag, New York, 2003.

[3] K. H. Kim and Y. H .Yon, *On derivations of subtraction algebras* (to be submitted)

[4] R. Larsen, An Introduction to the Theory of Multipliers, Berlin: Springer-Verlag, 1971.

[5] B. M. Schein, *Difference Semigroups,* Comm. in Algebra **20** (1992), 2153–2169.

[6] B. Zelinka, *Subtraction Semigroups,* Math. Bohemica, **120** (1995), 445–447.

Yong Ho Yon
Department of Mathematics, Chungbuk National University
Cheongju 361-763, Korea
yhyonkr@hanmail.net

Kyung Ho Kim
Department of Mathematics, Chungju National University
Chungju 380-702, Korea
ghkim@cjnu.ac.kr
corresponding author