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JENSEN TYPE INEQUALITIES ON QUASI-ARITHMETIC OPERATOR MEANS

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Abstract. As a continuation of our previous considerations about the operator order among quasi-arithmetic means [Linear Algebra Appl. 434 (2011), 1228–1237], we study this order with a different condition on the spectra. As an application we gave the order among some means. Also, we give similar results for F-order.

1 Introduction We recall some notations and definitions. Let $\mathcal{B}(H)$ be the C*-algebra of all bounded linear operators on a Hilbert space H and 1_H stands for the identity operator. We define bounds of a self-adjoint operator $A \in \mathcal{B}(H)$

$$m_A = \inf_{\|x\|=1} \langle Ax, x\rangle \quad \text{and} \quad M_A = \sup_{\|x\|=1} \langle Ax, x\rangle \tag{1.1}$$

for $x \in H$. If $\mathsf{Sp}(A)$ denotes the spectrum of A, then $\mathsf{Sp}(A)$ is real and $\mathsf{Sp}(A) \subseteq [m_A, M_A]$.

B. Mond and J. Pečarić in [6] proved the following version of Jensen's operator inequality

$$f\left(\sum_{i=1}^{n} w_i \Phi_i(A_i)\right) \le \sum_{i=1}^{n} w_i \Phi_i\left(f(A_i)\right), \tag{1.2}$$

for operator convex functions f defined on an interval I, where $\Phi_i : \mathcal{B}(H) \to \mathcal{B}(K)$, $i = 1, \dots, n$, are unital positive linear mappings, $A_1, \dots, A_n$ are self-adjoint operators with the spectra in I and $w_1, \dots, w_n$ are non-negative real numbers with $\sum_{i=1}^n w_i = 1$.

F. Hansen, J. Pečarić and I. Perić gave in [3] a generalization of (1.2) for unital field of positive linear mappings. Recently, J.Mićić, J.Pečarić and Y.Seo in [5] gave a generalization of this results for field of positive linear mappings such that the field $t \to \Phi_t(1_H)$ is integrable with $\int_T \Phi_t(1_H)\, d\mu(t) = k 1_K$ for some positive scalar k.

Very recently, J. Mićić, Z. Pavić and J. Pečarić [4, Theorem 1] gave the following version of Jensen's operator inequality without operator convexity.

Theorem A *Let $(A_1, \dots, A_n)$ be an n-tuple of self-adjoint operators $A_i \in \mathcal{B}(H)$ with bounds m_i and M_i, $m_i \le M_i$, $i = 1, \dots, n$. Let $(\Phi_1, \dots, \Phi_n)$ be an n-tuple of positive linear mappings $\Phi_i : \mathcal{B}(H) \to \mathcal{B}(K)$, $i = 1, \dots, n$, such that $\sum_{i=1}^n \Phi_i(1_H) = 1_K$. If*

$$(m_A, M_A) \cap [m_i, M_i] = \emptyset \quad \text{for } i = 1, \dots, n, \tag{1.3}$$

where m_A and M_A, $m_A \le M_A$, are bounds of the self-adjoint operator $A = \sum_{i=1}^n \Phi_i(A_i)$, then

$$f\left(\sum_{i=1}^{n} \Phi_i(A_i)\right) \le \sum_{i=1}^{n} \Phi_i\left(f(A_i)\right) \tag{1.4}$$

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holds for every continuous convex function $f : I \to \mathbf{R}$ *provided that the interval* I *contains all* m_i, M_i.

If $f : I \to \mathbf{R}$ *is concave, then the reverse inequality is valid in* (1.4).

In the same paper [4], we study the quasi-arithmetic operator mean

$$M_\varphi(\mathbf{A}, \mathbf{\Phi}, n) = \varphi^{-1}\left(\sum_{i=1}^{n} \Phi_i\left(\varphi(A_i)\right)\right), \tag{1.5}$$

where $(A_1, \ldots, A_n)$ is an n-tuple of self-adjoint operators in $\mathcal{B}(H)$ with the spectra in I, $(\Phi_1, \ldots, \Phi_n)$ is an n-tuple of positive linear mappings $\Phi_i : \mathcal{B}(H) \to \mathcal{B}(K)$ such that $\sum_{i=1}^{n} \Phi_i(1_H) = 1_K$, and $\varphi : I \to \mathbf{R}$ is a continuous strictly monotone function.

The following results about the monotonicity of this mean are proven in [4, Theorem 3].

Theorem B *Let* $(A_1, \ldots, A_n)$ *and* $(\Phi_1, \ldots, \Phi_n)$ *be as in the definition of the quasi-arithmetic mean* (1.5). *Let* m_i *and* M_i, $m_i \le M_i$ *be the bounds of* A_i, $i = 1, \ldots, n$. *Let* $\varphi, \psi : I \to \mathbf{R}$ *be continuous strictly monotone functions on an interval* I *which contains all* m_i, M_i. *Let* m_φ *and* M_φ, $m_\varphi \le M_\varphi$, *be the bounds of the mean* $M_\varphi(\mathbf{A}, \mathbf{\Phi}, n)$, *such that*

$$(m_\varphi, M_\varphi) \cap [m_i, M_i] = \emptyset \qquad \text{for } i = 1, \ldots, n. \tag{1.6}$$

If one of the following conditions

(i) $\psi \circ \varphi^{-1}$ *is convex and* ψ^{-1} *is operator monotone,*

(i') $\psi \circ \varphi^{-1}$ *is concave and* $-\psi^{-1}$ *is operator monotone,*

is satisfied, then

$$M_\varphi(\mathbf{A}, \mathbf{\Phi}, n) \le M_\psi(\mathbf{A}, \mathbf{\Phi}, n). \tag{1.7}$$

If one of the following conditions

(ii) $\psi \circ \varphi^{-1}$ *is concave and* ψ^{-1} *is operator monotone,*

(ii') $\psi \circ \varphi^{-1}$ *is convex and* $-\psi^{-1}$ *is operator monotone,*

is satisfied, then the reverse inequality is valid in (1.7).

In this paper we will study the monotonicity of the quasi-arithmetic mean with a condition on the bounds of operators $A_1, \ldots, A_n$ and $A = \sum_{i=1}^{n} \Phi_i(A_i)$, but without the bounds of means which partake in this order. As an application we gave the order among some means. Also, we give similar results for F-order under the same conditions.

2 Order among quasi-arithmetic means In Theorem B, we give the order among the quasi-arithmetic means under the conditions (1.6) which include the bounds of the mean in the LHS of (1.7). It is interesting to study the case when (1.7) holds only under the condition placed on the bounds of operators whose means we are considering. We study it in the following theorem.

Theorem 2.1 *Let $(A_1,\dots,A_n)$ be an n-tuple of self-adjoint operators $A_i \in B(H)$ with the bounds m_i and M_i, $m_i \le M_i$, $i=1,\dots,n$. Let $(\Phi_1,\dots,\Phi_n)$ be an n-tuple of positive linear mappings $\Phi_i : B(H) \to B(K)$, $i=1,\dots,n$, such that $\sum_{i=1}^n \Phi_i(1_H) = 1_K$. Let*

$$(m_A, M_A) \cap [m_i, M_i] = \emptyset \quad \textit{for } i = 1,\dots,n,$$

where m_A and M_A, $m_A \le M_A$, are the bounds of the operator $A = \sum_{i=1}^n \Phi_i(A_i)$. Let $f,g : I \to \mathbf{R}$ be continuous strictly monotone functions on an interval I which contains all m_i, M_i.
If one of the following conditions
(i) *f is convex, f^{-1} is operator monotone, g is concave, g^{-1} is operator monotone,*
(ii) *f is convex, f^{-1} is operator monotone, g is convex, $-g^{-1}$ is operator monotone,*
(iii) *f is concave, $-f^{-1}$ is operator monotone, g is convex, $-g^{-1}$ is operator monotone,*
(iv) *f is concave, $-f^{-1}$ is operator monotone, g is concave, g^{-1} is operator monotone,*
is satisfied, then

$$M_g(\mathbf{A},\mathbf{\Phi},n) \le M_f(\mathbf{A},\mathbf{\Phi},n). \tag{2.1}$$

But, if one of the following conditions
(i') *f is convex, $-f^{-1}$ is operator monotone, g is concave, $-g^{-1}$ is operator monotone,*
(ii') *f is convex, $-f^{-1}$ is operator monotone, g is convex, g^{-1} is operator monotone,*
(iii') *f is concave, f^{-1} is operator monotone, g is convex, g^{-1} is operator monotone,*
(iv') *f is concave, f^{-1} is operator monotone, g is concave, $-g^{-1}$ is operator monotone,*
is satisfied, then the reverse inequality is valid in (2.1).

Proof. We get this theorem by applying Theorem B. For example, by replacing ψ by f and φ by the identity function $\mathcal{I}$ in Theorem B(i) and by replacing ψ by g and φ by $\mathcal{I}$ in Theorem B(ii), we obtain (2.1) in the case (i).

But, we can give a direct and clear proof by using Theorem A.

Since f is a convex function and taking into account that $(m_A, M_A) \cap [m_i, M_i] = \emptyset$ for $i=1,\dots,n$, holds, we have by using Theorem A

$$f\left(\sum_{i=1}^n \Phi_i(A_i)\right) \le \sum_{i=1}^n \Phi_i\left(f(A_i)\right).$$

Since f^{-1} is an operator monotone function, it follows from the above inequality

$$\sum_{i=1}^n \Phi_i(A_i) \le f^{-1}\left(\sum_{i=1}^n \Phi_i\left(f(A_i)\right)\right). \tag{2.2}$$

Similarly to the above case, applying the concave case in Theorem A on g and next using that g^{-1} is operator monotone, we obtain

$$g^{-1}\left(\sum_{i=1}^n \Phi_i\left(g(A_i)\right)\right) \le \sum_{i=1}^n \Phi_i(A_i). \tag{2.3}$$

Combining the two inequalities (2.2) and (2.3), we obtain

$$g^{-1}\left(\sum_{i=1}^n \Phi_i\left(g(A_i)\right)\right) \le f^{-1}\left(\sum_{i=1}^n \Phi_i\left(f(A_i)\right)\right),$$

which is the desired inequality (2.1).

In the remaining cases the proof is essentially the same as in the above case. $\square$

Remark 2.2 *If we replace ψ by the identity function $\mathcal{I}$ and φ by f in Theorem B(i), then*

$$f^{-1}\left(\sum_{i=1}^{n}\Phi_i\left(f(A_i)\right)\right)\le\sum_{i=1}^{n}\Phi_i\left(A_i\right)$$

holds for every strictly monotone function $f: I\to\mathbf{R}$ provided that f^{-1} is convex and

$$(m_f, M_f)\cap[m_i, M_i]=\emptyset \qquad \text{for } i=1,\dots,n, \tag{2.4}$$

where

$$m_f\le f^{-1}\left(\sum_{i=1}^{n}\Phi_i\left(f(A_i)\right)\right)\le M_f \qquad \text{and} \qquad m_i 1_H\le A_i\le M_i 1_H,\ i=1,\dots,n.$$

We remark that if

$$(m_A, M_A)\cap[m_i, M_i]=\emptyset \quad \text{for } i=1,\dots,n, \tag{2.5}$$

holds, then (2.4) *does not hold for a general function f. (It is enough to put*

$$A_1=\begin{pmatrix}2 & 1/2\\ 1/2 & 1\end{pmatrix} \quad \text{and} \quad A_2=\begin{pmatrix}5 & 1\\ 1 & 6\end{pmatrix},$$

$\Phi_1(B)=\Phi_2(B)=\frac{1}{2}B$ *for $B\in M_2(\mathbf{C})$ and $f(t)=t^{-3}$. Then* (2.5) *stands and* (2.4) *does not stand.)*

So, if f^{-1} is a general convex function and g^{-1} is a general concave function, then we get the order (2.1) *under the condition* (2.4) *(see [4, Corollary 5]), but we can not get it under the condition* (2.5).

Next, we observe some examples. First, we study the following quasi-arithmetic mean

$$M_{G_{r,k}}(\mathbf{A},\mathbf{\Phi},n):=\left(\sqrt{k^2 1_H+\sum_{i=1}^{n}\Phi_i\left(A_i^r+2kA_i^{r/2}\right)}-k1_H\right)^{2/r},\ r\neq 0,\ k\ge 0, \tag{2.6}$$

where $(A_1,\dots,A_n)$ is an n-tuple of strictly positive operators in $\mathcal{B}(H)$ and $(\Phi_1,\dots,\Phi_n)$ is an n-tuple of positive linear mappings $\Phi_i:\mathcal{B}(H)\to\mathcal{B}(K)$ such that $\sum_{i=1}^{n}\Phi_i(1_H)=1_K$. This mean is induced by the function $G_{r,k}(t)=t^r+2kt^{r/2}$ for $t\in(0,\infty)$. Putting $k=0$ in (2.6), we obtain the power mean

$$M_r(\mathbf{A},\mathbf{\Phi},n)=\left(\sum_{i=1}^{n}\Phi_i\left(A_i^r\right)\right)^{1/r},\ r\neq 0, \tag{2.7}$$

and putting $n=2$, $k=1$, $\Phi_1(B)=(1-\lambda)B$ for $B\in\mathcal{B}(H)$ and $0\le\lambda\le 1$ in (2.6), we obtain a mean

$$A\mathbf{m}_{G_r,\lambda}B=\left(\sqrt{1_H+(1-\lambda)\left(A^r+2A^{r/2}\right)+\lambda\left(B^r+2B^{r/2}\right)}-1_H\right)^{2/r},\ 0\le\lambda\le 1.$$

This quasi-arithmetic mean is studied in [2, §4] for $-1\le r\le 1$.

In the following corollary we give the monotonicity of (2.6) on r by using [4, Theorem B] and Theorem B. We do not get anything new by applying Theorem 2.1.

Corollary 2.3 *Let $(A_1,\dots,A_n)$ and $(\Phi_1,\dots,\Phi_n)$ be as in the definition of the quasi-arithmetic mean* (2.6)*.*

If either $r \le s$, $r \notin (-2,2)$, $s \notin (-2,2)$, then

$$M_{G_{r,k}}(\mathbf{A},\mathbf{\Phi}) \le M_{G_{s,k}}(\mathbf{A},\mathbf{\Phi}). \tag{2.8}$$

Further, let m_i and M_i, $0<m_i\le M_i$ be the bounds of A_i, $i=1,\dots,n$, and $m_{p,k}$ and $M_{p,k}$, $m_{p,k}\le M_{p,k}$ be the bounds of $M_{G_{p,k}}(\mathbf{A},\mathbf{\Phi})$. If one of the following conditions

(i) $(m_{r,k},M_{r,k})\cap[m_i,M_i]=\emptyset$ for $i=1,\dots,n$, and either $s\ge 2$, $0\ne r\le s$ or $r\le s\le -2$

(ii) $(m_{s,k},M_{s,k})\cap[m_i,M_i]=\emptyset$ for $i=1,\dots,n$, and either $r\le -2$, $r\le s\ne 0$ or $2\le r\le s$

is satisfied, then (2.8) *holds.*

Proof. *Without the condition on the bounds of the operators:* If $r\le s\le -2$ we put $f(t)=t^r+2kt^{r/2}$ and $g(t)=t^s+2kt^{s/2}$ for $t>0$. Then $-g^{-1}(t)=-\left(\sqrt{k^2+t}-k\right)^{2/s}$ is operator monotone and $g\circ f^{-1}(t)=\left(\left(\sqrt{k^2+t}-k\right)^{s/r}+k\right)^2-k^2$ is operator concave (which we prove similarly to [2, Proof of Corollary 3.2]). It follows (by applying [4, Theorem B(i')]) that (2.8) is valid. If $2\le r\le s$ we put $f(t)=t^s+2kt^{s/2}$ and $g(t)=t^r+2kt^{r/2}$ for $t>0$. Then g^{-1} is operator monotone and $g\circ f^{-1}$ is operator concave and it follows (by applying [4, Theorem B(ii)]) that (2.8) is valid. Next, since the functions $t\mapsto t^{-1}$ and $t\mapsto t^{-1/2}$ are operator convex we have $M_{G_{-2,k}}(\mathbf{A},\mathbf{\Phi})<M_{G_{2,k}}(\mathbf{A},\mathbf{\Phi})$. Now, if $r\le -2$ and $s\ge 2$, then it follows from the above results

$$M_{G_{r,k}}(\mathbf{A},\mathbf{\Phi}) \le M_{G_{-2,k}}(\mathbf{A},\mathbf{\Phi}) < M_{G_{2,k}}(\mathbf{A},\mathbf{\Phi}) \le M_{G_{s,k}}(\mathbf{A},\mathbf{\Phi}),$$

which gives (2.8).

With the condition on the bounds of the operators: We prove only the case(i). If $(m_{r,k},M_{r,k})\cap[m_i,M_i]=\emptyset$, $i=1,\dots,n$, holds, then we put $f(t)=t^r+2kt^{r/2}$ and $g(t)=t^s+2kt^{s/2}$ for $t>0$. Since

$$\begin{aligned}(g\circ f^{-1}(t))'' &= \frac{s\left(\sqrt{k^2+t}-k\right)^{s/r-2}}{2r(k^2+t)^{3/2}}\left(k\left((s/r-1)\sqrt{k^2+t}-(\sqrt{k^2+t}-k)\right)\right.\\ &+ \left.\left(\sqrt{k^2+t}-k\right)^{s/r}\left((2s/r-1)\sqrt{k^2+t}-(\sqrt{k^2+t}-k)\right)\right),\end{aligned}$$

it follows that $g\circ f^{-1}(t)$ is convex for $s>0$, $r<0$ or $0<r\le s$. Further, $g^{-1}(t)=\left(\sqrt{k^2+t}-k\right)^{2/s}$ is operator monotone for $s\ge 2$. By applying Theorem B(i) we obtain (2.8) for $s\ge 2$, $0\ne r\le s$. But, $g\circ f^{-1}(t)$ is concave for $r\le s<0$ and $-g^{-1}(t)$ is operator monotone for $s\le -2$. By applying Theorem B(i') we obtain (2.8) for $r\le s\le -2$.

In the case (ii) we put $f(t)=t^s+2kt^{s/2}$ and $g(t)=t^r+2kt^{r/2}$ for $t>0$ and we use the same technique as in the case (i). □

The power mean $M_r(\mathbf{A},\mathbf{\Phi})$ is monotone on r in a wider region (see [4]).

On the other hand, we can apply Theorem 2.1 to obtain order among some means, as given in Corollary 2.4 and Corollary 2.5.

Corollary 2.4 *Let $(A_1,\dots,A_n)$ and $(\Phi_1,\dots,\Phi_n)$ be as in the definition of the quasi-arithmetic mean* (2.6)*. Let m_i and M_i, $0<m_i\le M_i$ be the bounds of A_i, $i=1,\dots,n$, and let m_A and M_A, $m_A\le M_A$, be the bounds of $A=\sum_{i=1}^n\Phi_i(A_i)$.*

If $k>0$ and $(m_A, M_A)\cap[m_i, M_i]=\emptyset$ for $i=1,\dots,n$, then

$$M_{G_{s,k}}(\mathbf{A},\mathbf{\Phi},n)\le M_r(\mathbf{A},\mathbf{\Phi},n) \tag{2.9}$$

holds for every $r\ge 1$ and $s\le -2$. The reverse inequality is valid in (2.9) for every $r\le -1$ and $s\ge 2$.

Proof. Since $f(t)=t^r$ is convex, $f^{-1}(t)=t^{1/r}$ is operator monotone for $r\ge 1$ and $G_{s,k}(t)=t^s+2kt^{s/2}$ is convex, $-G_{s,k}^{-1}(t)=-\left(\sqrt{k^2+t}-k\right)^{2/s}$ is operator monotone for $s\le -2$, then by applying Theorem 2.1(ii) we obtain (2.9). Similarly, by applying Theorem 2.1(ii'), we get the reverse inequality in (2.9) for $r<-1$ and $s\ge 2$. □

We can apply Theorem 2.1 to obtain order among other means. E.g. we observe quasi-arithmetic means induced by the functions $f(t)=\exp(t)$ for $t\in(0,\infty)$, $f(t)=\exp(1/t)$ for $t\in(0,\infty)$, $t\ne 1$ (also known as the radical mean) and $f(t)=\arctan(t)$ for $t\in(0,\infty)$, i.e.

$$M_{\exp}(\mathbf{A},\mathbf{\Phi},n) := \log\left(\sum_{i=1}^n \Phi_i\left(\exp(A_i)\right)\right), \tag{2.10}$$

$$M_{rad}(\mathbf{A},\mathbf{\Phi},n) := \left(\log\left(\sum_{i=1}^n \Phi_i\left(\exp(A_i^{-1})\right)\right)\right)^{-1}, \tag{2.11}$$

$$M_{\arctan}(\mathbf{A},\mathbf{\Phi},n) := \tan\left(\sum_{i=1}^n \Phi_i\left(\arctan(A_i)\right)\right), \tag{2.12}$$

where $(A_1,\dots,A_n)$ is an n-tuple of strictly positive operators in $\mathcal{B}(H)$ and $(\Phi_1,\dots,\Phi_n)$ is an n-tuple of positive linear mappings $\Phi_i:\mathcal{B}(H)\to\mathcal{B}(K)$ such that $\sum_{i=1}^n\Phi_i(1_H)=1_K$.

Corollary 2.5 *Let $(A_1,\dots,A_n)$ be an n-tuple of strictly positive operators in $\mathcal{B}(H)$ with the bounds m_i and M_i, $0<m_i\le M_i$, $i=1,\dots,n$. Let $(\Phi_1,\dots,\Phi_n)$ be an n-tuple of positive linear mappings $\Phi_i:\mathcal{B}(H)\to\mathcal{B}(K)$ such that $\sum_{i=1}^n\Phi_i(1_H)=1_K$. Let m_A and M_A, $m_A\le M_A$, be the bounds of $A=\sum_{i=1}^n\Phi_i(A_i)$.*

If $k>0$ and $(m_A, M_A)\cap[m_i, M_i]=\emptyset$ for $i=1,\dots,n$, then

$$M_{G_{s,k}}(\mathbf{A},\mathbf{\Phi},n)\le M_{\exp}(\mathbf{A},\mathbf{\Phi},n) \quad \text{for} \quad s\le -2, \tag{2.13}$$

$$M_r(\mathbf{A},\mathbf{\Phi},n)\le M_{\exp}(\mathbf{A},\mathbf{\Phi},n) \quad \text{for} \quad r\le -1, \tag{2.14}$$

$$M_{\arctan}(\mathbf{A},\mathbf{\Phi},n)\le M_{G_{s,k}}(\mathbf{A},\mathbf{\Phi},n) \quad \text{for} \quad s\ge 2, \tag{2.15}$$

$$M_{\arctan}(\mathbf{A},\mathbf{\Phi},n)\le M_r(\mathbf{A},\mathbf{\Phi},n) \quad \text{for} \quad r\ge 1. \tag{2.16}$$

Additionally if $\max\{M_1,\dots,M_n\}<1$ *or* $\min\{m_1,\dots,m_n\}>1$, *then*

$$M_{rad}(\mathbf{A},\mathbf{\Phi},n)\le M_{G_{s,k}}(\mathbf{A},\mathbf{\Phi},n) \quad \text{for} \quad s\ge 2, \tag{2.17}$$

$$M_{rad}(\mathbf{A},\mathbf{\Phi},n)\le M_r(\mathbf{A},\mathbf{\Phi},n) \quad \text{for} \quad r\ge 1. \tag{2.18}$$

Proof. Since $f(t)=\exp(t)$ is convex and $f^{-1}(t)=\log t$ is operator monotone we obtain (2.13) and (2.14) by applying Theorem 2.1(ii). Next, we obtain (2.15) and (2.16) by applying Theorem 2.1(iii'), since $f(t)=\arctan(t)$ is concave for $t\in(0,\infty)$ and $f^{-1}(t)=\tan(t)$ is operator monotone for $t\in(0,\pi/2)$. Finally, we obtain (2.17) and (2.18) by applying of Theorem 2.1(ii'), since $f(t)=\exp(\frac{1}{t})$ is convex and $-f^{-1}(t)=-(\log t)^{-1}$ is operator monotone for $t\in(0,1)$ or $t\in(1,\infty)$. □

Remark 2.6 *The inequalities* (2.13)–(2.18) *are not valid in general. E.g. we put* $r = 2$, $n = 2$, $k = 1$ *and* $\Phi_1(B) = \Phi_2(B) := \frac{1}{2}B$ *for* $B \in \mathcal{B}(H)$ *in* (2.6) *and* (2.10)*. Then*

$$\begin{aligned} M_{G_{2,1}}(\mathbf{A}, \mathbf{\Phi}, 2) &= \sqrt{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \frac{1}{2}A_1^2 + A_1 + \frac{1}{2}A_2^2 + A_2} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \\ M_{\exp}(\mathbf{A}, \mathbf{\Phi}, 2) &= \log\left(\frac{1}{2}\exp(A_1) + \frac{1}{2}\exp(A_2)\right). \end{aligned}$$

If

$$A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 4 \end{pmatrix} \quad \textit{and} \quad A_2 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix},$$

then

$$M_{\exp}(\mathbf{A}, \mathbf{\Phi}, 2) = \begin{pmatrix} 0.705353 & 0.0687674 \\ 0.0687674 & 3.33028 \end{pmatrix} \not\geq \begin{pmatrix} 0.797617 & 0.136284 \\ 0.136284 & 2.70559 \end{pmatrix} = M_{G_{2,1}}(\mathbf{A}, \mathbf{\Phi}, 2).$$

Given the above, there is no relation between $M_{\exp}(\mathbf{A}, \mathbf{\Phi}, 2)$ *and* $M_{G_{2,1}}(\mathbf{A}, \mathbf{\Phi}, 2)$*.*

3 Result for F-order In this section we give results for F-order similar to the ones given in the above section and [4] for the usual operator order.

We recall that F-order (denoted by $A \leq_F B$) defined for a monotone function F and self-adjoint operators A, B by $F(A) \leq F(B)$ in the usual operator order. Putting $F \equiv \log$ we have log-order $\ll$ (i.e. the chaotic order) among positive operators.

We also define that a function f is F-monotone if the composition $F \circ f$ is defined and operator monotone.

In [2, §2] a chaotic mean $\mathbf{m}$ is defined. In the same way, a mean $M(\mathbf{A}, n)$ is called *an F-type main* if $M(\mathbf{A}, n)$ is an $n-$ary operation on an n-tuple of strictly positive operators $(A_1, \ldots, A_n)$ satisfying:

monotonicity $A_i \leq B_i$, $i = 1, \ldots, n$, imply $M(\mathbf{A}, n) \leq_F M(\mathbf{B}, n)$,

semi-continuity $A_{1,j} \downarrow A_1, \ldots, A_{n,j} \downarrow A_n$, imply $M(\mathbf{A}_j, n) \downarrow\downarrow M(\mathbf{A}, n)$,

normalization $A_i = A$, $i = 1, \ldots, n$, imply $M(\mathbf{A}, n) = A$.

Here $\downarrow\downarrow$ denote F-decreasing convergence according to the following definition. A sequence $\{B_j\}$ of positive operators is called *F-decreasing* and denoted by $B_j \downarrow\downarrow$ if $B_j \leq_F B_{j+1}$ for all j. If a F-decreasing convergence sequence $\{B_j\}$ is lower bounded (i.e. $B_j \geq_F c1_H$ for some scalars c), then it converges to some positive operator B, which is denoted by $B_j \downarrow\downarrow B$.

If $F \equiv \log$ then log-type mean is called *a chaotic mean.*

The following obvious theorem is a generalization of [1, Theorem 2.3] and [2, Lemma 2.2].

Theorem 3.1 *Let* $(A_1, \ldots, A_n)$ *be an n-tuple of strictly positive operators in* $\mathcal{B}(H)$ *and* $(\Phi_1, \ldots, \Phi_n)$ *be an n-tuple of positive linear mappings* $\Phi_i : \mathcal{B}(H) \to \mathcal{B}(K)$ *such that* $\sum_{i=1}^n \Phi_i(1_H) = 1_K$*. If f is operator monotone and f^{-1} is F-monotone, then the quasi-arithmetic mean* $M_f(\mathbf{A}, \mathbf{\Phi}, n)$ *is an F-type mean.*

By using Theorem 3.1 and taking into account that $G_{r,k}(t) = t^r + 2kt^{\frac{r}{2}}$ is operator monotone for $0 < r \leq 1$ and $\log G_{r,k}^{-1}(t) = \frac{2}{r}\log\left(\sqrt{k^2+t} - k\right)$ is operator monotone for $r > 0$, we get the following corollary.

Corollary 3.2 *Let $(A_1,\dots,A_n)$ and $(\Phi_1,\dots,\Phi_n)$ be as in Theorem 3.1. The mean $M_{G_{r,k}}(\mathbf{A},\mathbf{\Phi},n)$ defined by* (2.6) *is a chaotic mean for $r\in[-1,1], r\neq 0$ and $k\geq 0$.*

Remark 3.3 *The mean $M_{G_{r,k}}(\mathbf{A},\mathbf{\Phi},n)$ defined by* (2.6) *is not a chaotic mean for $r\notin[-1,1]$. E.g. we put $A_1=\begin{pmatrix}7&0\\0&4\end{pmatrix}$, $B_1=\begin{pmatrix}3&2\\2&3\end{pmatrix}$, $A_2=2A_1$ and $B_2=2A_1$. Then $A_i<B_i$, $i=1,2$, but $A_i^2\not< B_i^2$. So $M_{G_{2,1}}(\mathbf{A},\mathbf{\Phi},2)\not\ll M_{G_{2,1}}(\mathbf{B},\mathbf{\Phi},2)$, where $\Phi_1(C)=\Phi_2(C):=\frac{1}{2}C$ for $C\in\mathcal{B}(H)$.*

Similarly to Theorem B, we obtain F-monotonicity of the quasi-arithmetic mean.

Theorem 3.4 *Let $(A_1,\dots,A_n)$ be an n-tuple of self-adjoint operators $A_i\in\mathcal{B}(H)$ with bounds m_i and M_i, $m_i\leq M_i$, $i=1,\dots,n$. Let $(\Phi_1,\dots,\Phi_n)$ be an n-tuple of positive linear mappings $\Phi_i:\mathcal{B}(H)\to\mathcal{B}(K)$, $i=1,\dots,n$, such that $\sum_{i=1}^n\Phi_i(1_H)=1_K$. Let $f,g:I\to\mathbf{R}$ be continuous strictly monotone functions on an interval I which contains all m_i,M_i. Let m_f and M_f, $m_f\leq M_f$, be the bounds of the mean $M_f(\mathbf{A},\mathbf{\Phi},n)$, such that*

$$(m_f,M_f)\cap[m_i,M_i]=\emptyset\qquad \textit{for } i=1,\dots,n.$$

If one of the following conditions

(i) $g\circ f^{-1}$ is convex and g^{-1} is F-monotone,

(i') $g\circ f^{-1}$ is concave and $-g^{-1}$ is F-monotone,

is satisfied, then

$$M_f(\mathbf{A},\mathbf{\Phi},n)\leq_F M_g(\mathbf{A},\mathbf{\Phi},n). \tag{3.1}$$

If one of the following conditions

(ii) $g\circ f^{-1}$ is concave and g^{-1} is F-monotone,

(ii') $g\circ f^{-1}$ is convex and $-g^{-1}$ is F-monotone,

is satisfied, then the reverse inequality is valid in (3.1).

Proof. The proof is quite similar to [4, Theorem 3]. We omit the details. □

As special cases of the above theorem, we get the following corollary.

Corollary 3.5 *Let $(A_1,\dots,A_n)$ and $(\Phi_1,\dots,\Phi_n)$ be as in Theorem 3.1. The mean $M_{G_{r,k}}(\mathbf{A},\mathbf{\Phi},n)$ defined by* (2.6) *is chaotic monotone on r, i.e.*

$$M_{G_{r,k}}(\mathbf{A},\mathbf{\Phi},n)\ll M_{G_{s,k}}(\mathbf{A},\mathbf{\Phi},n) \tag{3.2}$$

for $r,s\in\mathbf{R}$, $r\leq s$, $rs\neq 0$ and $k\geq 0$.

Proof. Let $0<r\leq s$ or $r<0<s$. We put $f(t)=t^r+2kt^{r/2}$ and $g(t)=t^s+2kt^{s/2}$. Then $g\circ f^{-1}=\left(\left(\sqrt{k^2+t}-k\right)^{s/r}+k\right)^2-k^2$ is convex and $\log\left(g^{-1}(t)\right)=\frac{1}{s}\log\left(\sqrt{k^2+t}-k\right)$ is operator monotone. Applying Theorem 3.4(i) we have (3.2). Similarly, if $r\leq s<0$ we get (3.2) putting $f(t)=t^s+2t^{s/2}$ and $g(t)=t^r+2t^{r/2}$ in Theorem 3.4(ii'). □

By using Theorem 3.4, we obtain the following result which is similar to Theorem 2.1.

Theorem 3.6 *Let* $(A_1,\dots,A_n)$ *and* $(\Phi_1,\dots,\Phi_n)$ *be as in Theorem 3.4. Let* m_i *and* M_i, $m_i \le M_i$ *be the bounds of* A_i, $i=1,\dots,n$, *and let* m_A *and* M_A, $m_A \le M_A$, *be the bounds of* $A=\sum_{i=1}^n \Phi_i(A_i)$, *such that*

$$(m_A, M_A) \cap [m_i, M_i] = \emptyset \quad \text{for } i=1,\dots,n.$$

Let $f,g : I \to \mathbf{R}$ *be continuous strictly monotone functions on an interval* I *which contains all* m_i, M_i.
If one of the following conditions
(i) f *is convex,* f^{-1} *is* F*-monotore,* g *is concave,* g^{-1} *is* F*-monotore,*
(ii) f *is convex,* f^{-1} *is* F*-monotore,* g *is convex,* $-g^{-1}$ *is* F*-monotore,*
(iii) f *is concave,* $-f^{-1}$ *is* F*-monotore,* g *is convex,* $-g^{-1}$ *is* F*-monotore,*
(iv) f *is concave,* $-f^{-1}$ *is* F*-monotore,* g *is concave,* g^{-1} *is* F*-monotore,*
is satisfied, then

$$M_g(\mathbf{A}, \mathbf{\Phi}, n) \le_F M_f(\mathbf{A}, \mathbf{\Phi}, n). \tag{3.3}$$

But, if one of the following conditions
(i') f *is convex,* $-f^{-1}$ *is* F*-monotore,* g *is concave,* $-g^{-1}$ *is* F*-monotore,*
(ii') f *is convex,* $-f^{-1}$ *is* F*-monotore,* g *is convex,* g^{-1} *is* F*-monotore,*
(iii') f *is concave,* f^{-1} *is* F*-monotore,* g *is convex,* g^{-1} *is* F*-monotore,*
(iv') f *is concave,* f^{-1} *is* F*-monotore,* g *is concave,* $-g^{-1}$ *is* F*-monotore,*
is satisfied, then the reverse inequality is valid in (2.1).

By applying Theorem 3.6 we can obtain the chaotic order among the means (2.6), (2.10)–(2.12).

Corollary 3.7 *Let* $(A_1,\dots,A_n)$ *be an* n*-tuple of strictly positive operators in* $\mathcal{B}(H)$ *with the bounds* m_i *and* M_i, $0 < m_i \le M_i$, $i=1,\dots,n$. *Let* $(\Phi_1,\dots,\Phi_n)$ *be an* n*-tuple of positive linear mappings* $\Phi_i : \mathcal{B}(H) \to \mathcal{B}(K)$ *such that* $\sum_{i=1}^n \Phi_i(1_H) = 1_K$. *Let* m_A *and* M_A, $m_A \le M_A$, *be the bounds of* $A=\sum_{i=1}^n \Phi_i(A_i)$.
If $k \ge 0$ *and* $(m_A, M_A) \cap [m_i, M_i] = \emptyset$ *for* $i=1,\dots,n$, *then*

$$\begin{aligned}
M_{G_{s,k}}(\mathbf{A}, \mathbf{\Phi}, n) &\ll M_r(\mathbf{A}, \mathbf{\Phi}, n) && \text{for} \quad r \ge 1 \text{ and } s \le 1,\ s \ne 0,\\
M_r(\mathbf{A}, \mathbf{\Phi}, n) &\ll M_{G_{s,k}}(\mathbf{A}, \mathbf{\Phi}, n) && \text{for} \quad r \le 1,\ r \ne 0 \text{ and } s \ge 1,\\
M_{\arctan}(\mathbf{A}, \mathbf{\Phi}, n) &\ll M_{G_{s,k}}(\mathbf{A}, \mathbf{\Phi}, n) && \text{for} \quad s \ge 1,\\
M_{G_{s,k}}(\mathbf{A}, \mathbf{\Phi}, n) &\ll M_{\exp}(\mathbf{A}, \mathbf{\Phi}, n) && \text{for} \quad s \le 1,\ s \ne 0,.
\end{aligned}$$

Additionally if $\max\{M_1,\dots,M_n\} < 1$ *or* $\min\{m_1,\dots,m_n\} > 1$, *then*

$$M_{rad}(\mathbf{A}, \mathbf{\Phi}, n) \ll M_{G_{s,k}}(\mathbf{A}, \mathbf{\Phi}, n) \qquad \text{for} \quad s \ge 1.$$

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