1-SAFE PETRI NETS GENERATING EVERY BINARY n-VECTOR EXACTLY ONCE

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Received March 17, 2011 ; revised June 26, 2011

Abstract. Petri nets have been among the most succinct models that can describe the structure and dynamics of discrete event-driven systems. In this paper, a necessary condition for a 1-safe Petri net generating all the binary n-vectors and the existence of 1-safe Petri nets which generate every binary n-vector as one of their marking vectors exactly once in the smallest possible number of steps have been established.

1 INTRODUCTION

Petri nets, originally proposed by C.A.Petri[4], have been widely recognized as one of the most promising mathematical tools for describing and analyzing the structure and dynamics of discrete event-driven dynamic systems. They have the advantage of being used as a visual communication tool as well, similar to flow-chart, block diagram, or a network. Theoretically, Petri nets have been used as a powerful and convenient tool for representing and studying the structure of decision making processes which can often be tricky or complex. The development of high-end computers has greatly enhanced the use of Petri nets in diverse fields.

Kansal *et al.*[3] proposed a 1-safe *star Petri net* S_n, having n places and $(n+1)$ transitions, that generates all the binary n-vectors, as its marking vectors, in which the initial marking has been taken as (1, 1, 1,$\cdots$, 1). After constructing S_n, many fundamental questions arose, e.g., (i) " Do there exist Petri nets that generate every binary n-vector exactly once? " (ii) "Is it not possible to take any marking other than (1, 1, 1,$\cdots$, 1) " as an initial marking for such a Petri net? In this paper, we shall answer both these questions.

2 PRELIMINARIES

For standard terminology and notation on Petri nets, we refer the reader to Peterson[5]. Jenson[2] has given the following more operative definition of a Petri net, which we shall adopt in this paper.

A *Petri net* is a 5-tuple $C = (P, T, I^-, I^+, \mu^0)$, where

(a) P is a nonempty set of 'places',

(b) T is a nonempty set of 'transitions',

(c) $P \cap T = \emptyset$,

(d) I^-, I^+:P × T $\longrightarrow \mathbb{N}$, where $\mathbb{N}$ is the set of nonnegative integers, are called the *negative* and the *positive* 'incidence functions' (or, 'flow functions') respectively,

2000 *Mathematics Subject Classification.* 37, 93, 94 .
Key words and phrases. 1-safe Petri net, reachability tree, binary n-vector .

(e) $\forall\, p \in P, \exists t \in T : I^-(p,t) \neq 0$ or $I^+(p,t) \neq 0$ and
$\forall t \in T, \exists p \in P : I^-(p,t) \neq 0$ or $I^+(p,t) \neq 0$,

(f) $\mu^0 : P \to \mathbb{N}$ is the *initial marking.*

In fact, $I^-(p,t)$ and $I^+(p,t)$ represent the number of arcs from p to t and t to p respectively. I^-, I^+ and μ^0 can be viewed as matrices of size $|P| \times |T|$, $|P| \times |T|$ and $|P| \times 1$, respectively.

As in many standard books (e.g., see Reiseg[6]), Petri nets have a well known graphical representation in which transitions are represented as boxes and places as circles with directed arcs interconnecting places and transitions to represent the flow relation. The initial marking is represented by placing a token in the circle representing a place p_i as a black dot whenever $\mu^0(p_i) = 1$, $1 \leq i \leq n = |P|$. In general, a *marking* μ is a mapping $\mu : P \longrightarrow \mathbb{N}$. A marking μ can hence be represented as a vector $\mu \in \mathbb{N}^n$, $n = |P|$, such that the i^{th} component of μ is the value $\mu(p_i)$.

Let $C = (P, T, I^-, I^+, \mu)$ be a Petri net. A transition $t \in T$ is said to 'fire' at the marking μ (or it is '*enabled* at μ') iff $I^-(p,t) \leq \mu(p)$, $\forall p \in P$. After firing at μ, the new marking μ' is given by the rule

$$\mu'(p) = \mu(p) - I^-(p,t) + I^+(p,t), \text{ for all } p \in P.$$

We say *t fires at* μ to yield μ' (or *t fires* μ to μ'), and we write $\mu \xrightarrow{t} \mu'$, whence μ' is said to be *directly reachable* from μ. Hence, it is clear, what is meant by a sequence like

$$\mu^0 \xrightarrow{t_1} \mu^1 \xrightarrow{t_2} \mu^2 \xrightarrow{t_3} \mu^3 \cdots \xrightarrow{t_k} \mu^k,$$

which simply represents the fact that the transitions $t_1, t_2, t_3, \ldots, t_k$ have been successively fired to transform the marking μ^0 into the marking μ^k. The whole of this sequence of transformations is also written in short as $\mu^0 \xrightarrow{\sigma} \mu^k$, where $\sigma = t_1, t_2, t_3, \ldots, t_k$.

A marking μ is said to be *reachable from* μ^0, if there exists a sequence of transitions which can be successively fired to obtain μ from μ^0. The set of all markings of a Petri net C reachable from a given marking μ is denoted by $R(C, \mu)$ and, together with the arcs of the form $\mu^i \xrightarrow{t_r} \mu^j$, represents what in standard terminology called the *reachability tree* of the Petri net C.

A place in a Petri net is *safe* if the number of tokens in that place never exceeds one. A Petri net is *safe* if all its places are safe. The *pre-set* of a transition t is the set of all input places to t, i.e., ${}^\bullet t = \{p \in P : I^-(p,t) > 0\}$. The *post-set* of t is the set of all output places from t, i.e., $t^\bullet = \{p \in P : I^+(p,t) > 0\}$. Similarly, p's pre-set is ${}^\bullet p = \{t \in T : I^+(p,t) > 0\}$ and p's post-set is $p^\bullet = \{t \in T : I^-(p,t) > 0\}$.

3 MAIN RESULTS

In this section, we start by first answering the second question raised in the Introduction.

Proposition 1. *If a* 1*-safe Petri net generates all the binary n-vectors then* $\mu^0(p) = 1, \forall p \in P$.

Proof. Suppose $C = (P, T, I^-, I^+, \mu^0)$ is a 1-safe Petri net which generates all the binary n-vectors and $\mu^0(p_i) \neq 1$ for some $p_i \in P$. By the definition of a Petri net, no place can

be isolated. Therefore p_i has to be connected to some $t_i \in T$. Now, three cases arise for consideration:
Case-1: $p_i \in t_i^\bullet$,
Case-2: $p_i \in {}^\bullet t_i \cap t_i^\bullet$, and
Case-3: $p_i \in {}^\bullet t_i$
In Case 1, since the given Petri net C is safe, ${}^\bullet t_i \neq \emptyset$ [1]. Therefore, $\exists\ p_j \in {}^\bullet t_i$ for some $p_j \in P$. p_j will have either one token or no token. If p_j has one token then t_i is enabled and hence fires. After firing of t_i, p_j will have no token and p_i will receive one token. So, both the places cannot have one token simultaneously. Hence, we will not get the marking vector whose components are all equal to 1. Again, if p_j has no token then t_i cannot fire, whence p_i will never receive a token, which contradicts the assumption of the case.

Proof of Case 2 follows from that of Case 1 since, in particular, $p_i \in t_i^\bullet$.

Also, in Case 3, as in the proof of Case 1 $p_i \in {}^\bullet t_i$ implies that we cannot have the marking vector whose components are all equal to 1.
Thus, if a Petri net generates all the binary n-vectors then $\mu^0(p_i) = 1\ \forall\ p_i \in P$. □

Theorem 1. *There exists a 1-safe Petri net with the initial marking $\mu^0(p) = 1, \forall p \in P$ which generates each of the 2^n binary n-vectors*

$$(a_1, a_2, a_3, \cdots, a_n),\ a_i \in \{0, 1\},\ n = |P|,$$

as one of its marking vectors, exactly once.

Proof. We shall prove this result by using the Principle of Mathematical Induction (PMI) on $n = |P|$.

For $n = 1$, we construct a Petri net C_1 as shown in Figure 1.

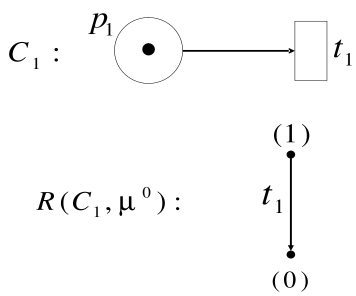

Figure 1

In this Petri net C_1,

the total number of transitions $= 2^1 - 1 = 1$,
$|p_1^\bullet| = 2^1 - 1 = 1$,
$|{}^\bullet p_1| = 2^{1-1} - 1 = 0$,
$|{}^\bullet t_1| = 1$.

Total number of transitions whose post-sets having no element $= {}^1C_0 = 1$ and this transition is t_1. Clearly, $R(C_1, \mu^0)$ of C_1 generates both the binary 1-vectors (1) and (0) as shown in Figure-1 in the first step and after this step, transition becomes dead.

Next, for $n = 2$, the Petri net C_2 shown in Figure-2 has two places.

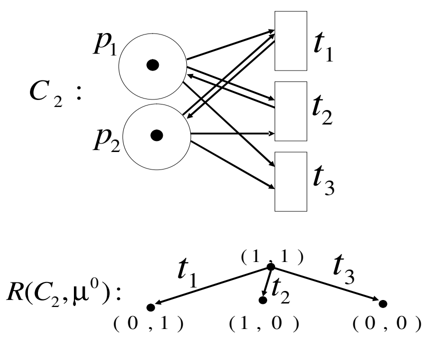

Figure 2

In C_2, we have

the total number of transitions $= 2^2 - 1 = 4 - 1 = 3$,
$|p^\bullet| = 2^2 - 1 = 3,\ \forall\ p$,
$|{}^\bullet p| = 2^{2-1} - 1 = 1,\ \forall\ p$,
$|{}^\bullet t| = 2,\ \forall\ t$.

The total number of transitions whose post-sets have one element $= {}^2C_1 = 2$ and these transitions are t_1, t_2.

The total number of transitions whose post-sets have no element $= {}^2C_0 = 1$ and this transition is t_3.

It is clear from Figure-2 that $R(C_2, \mu^0)$ has exactly $4 = 2^2$ binary 2-vectors (a_1, a_2), $a_1, a_2 \in \{0, 1\}$ in the first step and after this step, all the transitions become dead.

We can construct $R(C_2, \mu^0)$ from $R(C_1, \mu^0)$ as follows: Take two copies of $R(C_1, \mu^0)$. In the first copy, augment each vector of $R(C_1, \mu^0)$ by the adjunction of a '0' entry at the second coordinate of every marking vector and denote the resulting labeled tree as $R_0(C_1, \mu^0)$. Similarly, in the second copy, augment each vector by the adjunction of a '1' at the second coordinate of every marking vector and let $R_1(C_1, \mu^0)$ be the resulting labeled tree (see Figure-3).
Now, using the following steps we construct the reachability tree $R(C_2, \mu^0)$ of C_2 from $R_0(C_1, \mu^0)$ and $R_1(C_1, \mu^0)$.

Step-1. Clearly, the binary 2-vectors in $R_0(C_1, \mu^0) \cup R_1(C_1, \mu^0)$ are all distinct and are exactly $2^2 = 4$ in number.

Step-2. In $R_0(C_1, \mu^0)$, none of the transitions t_j is enabled at $(1, 0)$.

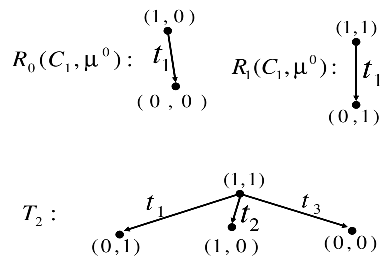

Figure 3

Step-3. In $R_0(C_1, \mu^0)$, the root node $(1, 0)$ has the marking obtained after firing of transition t_2 in C_2. Hence, we join the root node $(1, 0)$ of $R_0(C_1, \mu^0)$ to the root node $(1, 1)$ of $R_1(C_1, \mu^0)$ by an arc labeled t_2 so that $(1, 0)$ would become the 'child node' obtained by firing t_2 in C_2. Next, we join the child node $(0, 0)$ of $R_0(C_1, \mu^0)$ to the root node $(1, 1)$ of $R_1(C_1, \mu^0)$ by an arc labeled t_3 so that $(0, 0)$ would become the child node obtained by firing t_3 in C_2. Then, the resulting labeled tree T_2 has exactly 2^2 binary 2-vectors as its set of nodes. T_2 is indeed the reachability tree of C_2 because in C_2 all the transitions t_1, t_2 and t_3 are enabled at the initial marking $(1, 1)$ and fire. Further, after firing of each transition, the new markings obtained by the rule

$$\mu'(p_i) = \mu^0(p_i) - I^-(p_i, t_j) + I^+(p_i, t_j)$$

are $(0, 1), (1, 0)$ and $(0, 0)$ respectively and no further firing takes place as the enabling condition fails to hold for these marking vectors; i.e., we get exactly $2^2 = 4$ binary 2-vectors in the first step only.

Next, suppose this result is true for $n = k$. That is, C_k is the 1-safe Petri net having k-places and $2^k - 1$ transitions $t_1, t_2, t_3, \cdots$, generating each of the 2^k binary k-vectors exactly once and having the structure as schematically shown in Figure-4 which has the following parameters:
$|p^\bullet| = 2^k - 1, \ \forall\, p$,
$|^\bullet p| = 2^{k-1} - 1, \ \forall\, p$,
$|^\bullet t| = k, \ \forall\, t$.

The total number of transitions whose post-sets have $k-1$ elements $= {}^kC_{k-1} = {}^kC_1 = k$ and these transitions are $t_1, t_2, t_3, \cdots, t_k$.

The total number of transitions whose post-sets have $k-2$ elements $= {}^kC_{k-2} = {}^kC_2 = \frac{k(k-1)}{2}$ and these transitions are $t_{k+1}, t_{k+2}, t_{k+3}, \cdots, t_{\frac{k^2+k}{2}}$.

The total number of transitions whose post-sets have $k-3$ elements $= {}^kC_{k-3} = {}^kC_3 = \frac{k(k-1)(k-2)}{6}$ and these transitions are $t_{\frac{k^2+k+2}{2}}, t_{\frac{k^2+k+4}{2}}, t_{\frac{k^2+k+6}{2}}, \cdots, t_{\frac{k^3+5k}{6}}$.

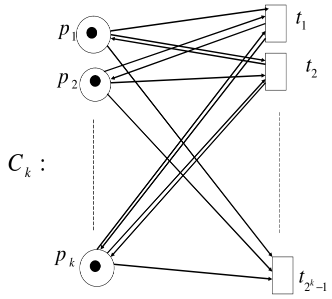

Figure 4

$\vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots \qquad\qquad \vdots$

The total number of transitions whose post-sets have one element $= {}^kC_1 = k$ and these transitions are $t_{2^k-k-1}, t_{2^k-k}, t_{2^k-k+1}, \cdots, t_{2^k-2}$.

The total number of transitions whose post-sets have no element $= {}^kC_0 = 1$ and this transition is t_{2^k-1}.

We will now prove the result for the 1-safe Petri net C_{k+1} having $k+1$ places and $t_{2^{k+1}}-1$ transitions and having the structure shown schematically in Figure-4. For this purpose, take two copies of $R(C_k, \mu^0)$. In the first copy, augment each vector of $R(C_k, \mu^0)$ by the adjunction of a '0' entry at the $(k+1)^{th}$ coordinate of every marking vector and denote the resulting labeled tree as $R_0(C_k, \mu^0)$. Similarly, in the second copy, augment each vector by the adjunction of a '1' at the $(k+1)^{th}$ coordinate of every marking vector and let $R_1(C_k, \mu^0)$ be the resulting labeled tree. Now, using the following steps we construct the reachability tree $R(C_{k+1}, \mu^0)$ of C_{k+1} from $R_0(C_k, \mu^0)$ and $R_1(C_k, \mu^0)$.
Step-1. The induction hypothesis implies that the binary $(k+1)$-vectors in $R_0(C_k, \mu^0) \cup R_1(C_k, \mu^0)$ are all distinct and they are exactly $2^k + 2^k = 2^{k+1}$ in number.

Step-2. In $R_0(C_k, \mu^0)$, none of the transitions is enabled at $(1, 1, 1, \cdots, 0)$.

Step-3. In $R_0(C_k, \mu^0)$, the root node $(1, 1, 1, \cdots, 0)$ is the marking obtained after firing of transition t_{k+1} in C_{k+1}. Hence, we join the root node $(1, 1, 1, \cdots, 0)$ of $R_0(C_k, \mu^0)$ to the root node $(1, 1, 1, \cdots, 1)$ of $R_1(C_k, \mu^0)$ by an arc labeled t_{k+1} so that $(1, 1, 1, \cdots, 0)$ would become the child node obtained by firing t_{k+1} in C_{k+1} and in $R_1(C_k, \mu^0)$ the child node $(0, 0, 0, \cdots, 1)$ is the marking obtained after firing of the transition t_{k+2} at the root node $(1, 1, 1, \cdots, 1)$ of $R_1(C_k, \mu^0)$; so, we replace the arc labeled as t_{k+1} by t_{k+2} in $R_1(C_k, \mu^0)$. Next, we join the remaining $(2^{k+1} - 1) - \overline{k+2}$ child nodes $(0, 1, 0, \cdots, 0)$, $(1, 0, 0, \cdots, 0)$, $\cdots, (0, 0, 0, \cdots, 0)$ of $R_0(C_k, \mu^0)$ to the root node $(1, 1, 1, \cdots, 1)$ of $R_1(C_k, \mu^0)$ by an arc each, labeled $t_{k+3}, t_{k+4}, t_{k+5}, \cdots, t_{2^k-1}$ respectively, so that $(0, 1, 0, \cdots, 0)$, $(1, 0, 0, \cdots, 0)$, $\cdots$, $(0, 0, 0, \cdots, 0)$ would become the marking obtained after firing of t_{k+3}, t_{k+4}, t_{k+5}, $\cdots$, t_{2^k-1} respectively in C_{k+1}. Then the resulting labeled tree T_{k+1} has exactly 2^{k+1} binary $(k+1)$-vectors. T_{k+1} is indeed the reachability tree of C_{k+1} because in C_{k+1} all the transitions are enabled at the initial marking $(1, 1, 1, \cdots, 1)$ and fire. After firing, the new markings obtained by the rule

$$\mu'(p_i) = \mu^0(p_i) - I^-(p_i, t_j) + I^+(p_i, t_j)$$

are

$$(0, 1, 1, \cdots, 1), (1, 0, 1, \cdots, 1), (1, 1, 0, \cdots, 1), \cdots, (0, 0, 0, \cdots, 0)$$

respectively and no further firing takes place as the enabling condition fails to hold for these marking vectors; i.e., we get exactly 2^{k+1} binary $(k+1)$-vectors each generated exactly once in the first step itself.

It is clear that the Petri net constructed above generates each of the 2^n binary n-vectors exactly once in the very first step and, hence, is the smallest number of steps because no firing will take place after that step.

Hence, the result follows by PMI. □

It may be observed from the above proof that the Petri net constructed therein yields all the binary n-vectors as marking vectors in the least possible number of steps.

4 Conclusions and scope

We have shown in this paper that there exists a Petri net that generates every binary n-vector exactly once. A computationally good characterization of such Petri nets in general is highly desirable since the instances where we need such Petri nets for applications are imaginably (as well as arguably) large in number as pointed out in Kansal *et al.*[3]. Optimization of the *order* (i.e., $|P| + |T|$) and *size* (the number of arcs) in such a Petri net is the next desired goal. Lastly, one can think of the need to minimize the number of firing of transitions in such a Petri net. Characterization of the subclass of such Petri nets with least possible order, size and number of enabled transitions would be essential for application purposes.

Acknowledgement
The authors deeply acknowledge the valuable suggestions made by Dr. B.D. Acharya from time to time in carrying out the work reported in this paper. Especially, we are thankful to him for giving us his perspectives detailed in the last section.

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