FIXED POINT THEOREMS FOR GENERAL CONTRACTIVE MAPPINGS IN METRIC SPACES AND ESTIMATING EXPRESSIONS

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Abstract.In this paper, we first consider a broad class of nonlinear mappings containing the class of contractive mappings in a metric space. Let (X, d) be a metric space. A mapping $T : X \to X$ is called contractively generalized hybrid if there are $\alpha, \beta \in \mathbb{R}$ and $r \in [0, 1)$ such that

$$\alpha d(Tx, Ty) + (1 - \alpha)d(x, Ty) \leq r\{\beta d(Tx, y) + (1 - \beta)d(x, y)\}$$

for all $x, y \in X$. Then, we deal with fixed point theorems for these nonlinear mappings in a complete metric space. Using the results, we prove well-known fixed point theorems in a complete metric space. Furthermore, we obtain an estimating expression for contractively generalized hybrid mappings in a Banach space.

1 Introduction Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. A mapping $T : C \to C$ is said to be *nonexpansive* [10], *nonspreading* [8], and *hybrid* [11] if

$$\|Tx - Ty\| \leq \|x - y\|,$$

$$2\|Tx - Ty\|^2 \leq \|Tx - y\|^2 + \|Ty - x\|^2$$

and

$$3\|Tx - Ty\|^2 \leq \|x - y\|^2 + \|Tx - y\|^2 + \|Ty - x\|^2$$

for all $x, y \in C$, respectively. These mappings are deduced from a firmly nonexpansive mapping in a Hilbert space; see [11]. A mapping $F : C \to C$ is said to be *firmly nonexpansive* if

$$\|Fx - Fy\|^2 \leq \langle x - y, Fx - Fy\rangle$$

for all $x, y \in C$; see, for instance, Browder [1], Goebel and Kirk [3], and Kohsaka and Takahashi [7]. Motivated by these nonlinear mappings, Kocourek, Takahashi and Yao [6] introduced a broad class of mappings $T : C \to C$ called *generalized hybrid* such that for some $\alpha, \beta \in \mathbb{R}$,

$$\alpha\|Tx - Ty\|^2 + (1 - \alpha)\|x - Ty\|^2 \leq \beta\|Tx - y\|^2 + (1 - \beta)\|x - y\|^2$$

for all $x, y \in C$. Such a mapping is also called (α, β)-*generalized hybrid.* We observe that the class of the mappings above covers several classes of well-known mappings. An (α, β)-generalized hybrid mapping is nonexpansive for $\alpha = 1$ and $\beta = 0$, nonspreading for $\alpha = 2$ and $\beta = 1$, and hybrid for $\alpha = \frac{3}{2}$ and $\beta = \frac{1}{2}$. On the other hand, we know important classes of mappings in a metric space. Let X be a metric space with metric d. A mapping $T : X \to X$ is said to be *contractive* if there exists $r \in [0, 1)$ such that $d(Tx, Ty) \leq rd(x, y)$

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for all $x, y \in X$. Such a mapping is also called *r-contractive*. A mapping $T : X \to X$ is said to be *Kannan* [5] if there exists $\alpha \in [0, \frac{1}{2})$ such that $d(Tx, Ty) \leq \alpha(d(x, Tx) + d(y, Ty))$ for all $x, y \in X$. A mapping $T : X \to X$ is said to be *contractively nonspreading* [2], [4] and [13] if there exists $\beta \in [0, \frac{1}{2})$ such that $d(Tx, Ty) \leq \beta(d(x, Ty) + d(y, Tx))$ for all $x, y \in X$. A mapping $T : X \to X$ is said to be *contractively hybrid* [11] if there exists a real number γ with $0 \leq \gamma < \frac{1}{3}$ and

$$d(Tx, Ty) \leq \gamma\{d(Tx, y) + d(Ty, x) + d(x, y)\}$$

for all $x, y \in X$.

In this paper, we first consider a broad class of nonlinear mappings containing the classes of contractive mappings and contractively nonspreading mappings in a metric space. Then, we deal with fixed point theorems for these nonlinear mappings in a complete metric space. Using the results, we prove well-known fixed point theorems in a complete metric space. Furthermore, we obtain an estimating expression for contractively generalized hybrid mappings in a Banach space.

2 Preliminaries Throughout this paper, we denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{R}$ the set of real numbers. Let X be a metric space with metric d. We denote the convergence of $\{x_n\}$ to $x \in X$ by $x_n \to x$. A sequence $\{x_n\}$ in X is said to be *Cauchy* [10] if there exists a sequence $\{\alpha_n\}$ of real numbers such that for $m, n \in \mathbb{N}$ with $m \geq n$, $d(x_m, x_n) \leq \alpha_n$ and $\alpha_n \to 0$. A metric space X is called *complete* if every Cauchy sequence $\{x_n\}$ is convergent, i.e., $\{x_n\} \to u$ for some $u \in X$. In 1972, Zamfirescu [13] proved the following theorem which is one of generalizations of the Banach contraction principle.

Theorem 2.1. *Let X be a complete metric space with metric d and let $T : X \to X$ be a mapping which satisfies one of the following:*

(i) T is contractive;

(ii) T is Kannan;

(iii) T is contractively nonspreading.

Then T has a unique fixed point in X.

Let l^∞ be the Banach space of bounded sequences with supremum norm. Let μ be an element of $(l^\infty)^*$ (the dual space of l^∞). Then, we denote by $\mu(f)$ the value of μ at $f = (x_1, x_2, x_3, \dots) \in l^\infty$. Sometimes, we denote by $\mu_n(x_n)$ the value $\mu(f)$. A linear functional μ on l^∞ is called a *mean* if $\mu(e) = \|\mu\| = 1$, where $e = (1, 1, 1, \dots)$. A mean μ is called a *Banach limit* on l^∞ if $\mu_n(x_{n+1}) = \mu_n(x_n)$. We know that there exists a Banach limit on l^∞. If μ is a Banach limit on l^∞, then for $f = (x_1, x_2, x_3, \dots) \in l^\infty$,

$$\liminf_{n\to\infty} x_n \leq \mu_n(x_n) \leq \limsup_{n\to\infty} x_n.$$

In particular, if $f = (x_1, x_2, x_3, \dots) \in l^\infty$ and $x_n \to a \in \mathbb{R}$, then we have $\mu(f) = \mu_n(x_n) = a$. For the proof of existence of a Banach limit and its other elementary properties, see [9]. We also know a fixed point theorem for generalized hybrid mappings in a Hilbert space which was proved by using Banach limits; see also [12].

Theorem 2.2 (Kocourek, Takahashi and Yao [6])**.** *Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let $T : C \to C$ be a generalized hybrid mapping. Then T has a fixed point in C if and only if $\{T^n x\}$ is bounded for some $x \in C$.*

3 Fixed Point Theorems In this section, we start with defining a broad class of mappings in a metric space. Let (X, d) be a metric space. A mapping $T : X \to X$ is called *contractively generalized hybrid* if there are $\alpha, \beta \in \mathbb{R}$ and $r \in [0, 1)$ such that

$$\alpha d(Tx, Ty) + (1 - \alpha)d(x, Ty) \le r\{\beta d(Tx, y) + (1 - \beta)d(x, y)\} \tag{3.1}$$

for all $x, y \in X$. We call such a mapping an (α,β,r)-*contractively generalized hybrid* mapping. We observe that the class of the mappings above covers classes of well-known mappings in a metric space. For example, an (α,β,r)-contractively generalized hybrid mapping T is r-contractive for $\alpha = 1$ and $\beta = 0$, i.e.,

$$d(Tx, Ty) \le rd(x, y), \quad \forall x, y \in X.$$

Now, we prove fixed point theorems in a metric space. Before proving the fixed point theorems, we show the following lemma.

Lemma 3.1. *Let (X, d) be a metric space, let $\{x_n\}$ be a bounded sequence in X and let μ be a mean on l^∞. If $g : X \to \mathbb{R}$ is defined by*

$$g(z) = \mu_n d(x_n, z), \quad \forall z \in X,$$

then g is a continuous function on X.

Proof. Since $\{x_n\}$ is bounded, we have that for any $y \in X$, $\{d(x_n, y)\}$ is an element of l^∞. So, using a mean μ on l^∞, we can define a function $g : X \to \mathbb{R}$ as follows:

$$g(y) = \mu_n d(x_n, y), \quad \forall y \in X.$$

Let $z, y \in X$. Then, we have that for any $n \in \mathbb{N}$,

$$d(x_n, z) \le d(x_n, y) + d(y, z).$$

Since μ is a mean on l^∞, we have

$$g(z) = \mu_n d(x_n, z) \le \mu_n d(x_n, y) + \mu_n d(y, z) = g(y) + d(y, z). \tag{3.2}$$

Similarly, we have

$$g(y) \le g(z) + d(z, y) = g(z) + d(y, z). \tag{3.3}$$

Therefore, we have from (3.2) and (3.3) that

$$|g(y) - g(z)| \le d(y, z).$$

This implies that $g : X \to \mathbb{R}$ is a continuous function on X. □

Theorem 3.2. *Let X be a complete metric space and let T be a mapping of X into itself. Suppose that there exist a real number r with $0 \le r < 1$ and an element $x \in X$ such that $\{T^n x\}$ is bounded and*

$$\mu_n d(T^n x, Ty) \le r\mu_n d(T^n x, y), \quad \forall y \in X$$

for some mean μ on l^∞. Then, the following hold:

(i) T has a unique fixed point u in X;

(ii) for every $z \in X$, the sequence $\{T^n z\}$ converges to u in X.

Proof. Since $\{T^n x\}$ is bounded, we have that for any $y \in X$, $\{d(T^n x, y)\}$ is an element of l^∞. So, using a mean μ on l^∞, we can define a function $g : X \to \mathbb{R}$ as follows:

$$g(y) = \mu_n d(T^n x, y), \quad \forall y \in X.$$

From Lemma 3.1, $g : X \to \mathbb{R}$ is a continuous function on X. For any $z \in X$, consider a sequence $\{T^n z\}$ in X. Then, we have that for any $m, n \in \mathbb{N}$,

$$d(T^m z, T^{m+1} z) \leq d(T^m z, T^n x) + d(T^n x, T^{m+1} z).$$

Since μ is a mean on l^∞, we have that for any $m \in \mathbb{N}$,

$$\begin{aligned} d(T^m z, T^{m+1} z) &\leq \mu_n d(T^m z, T^n x) + \mu_n d(T^n x, T^{m+1} z) \\ &= \mu_n d(T^n x, T^m z) + \mu_n d(T^n x, T^{m+1} z) \\ &\leq r\mu_n d(T^n x, T^{m-1} z) + r\mu_n d(T^n x, T^m z) \\ &\leq \dots \\ &\leq r^m \mu_n d(T^n x, z) + r^m \mu_n d(T^n x, Tz) \\ &\leq r^m \mu_n d(T^n x, z) + r^{m+1} \mu_n d(T^n x, z) \\ &= r^m (1 + r) \mu_n d(T^n x, z) \\ &= r^m (1 + r) g(z). \end{aligned}$$

So, we have that for any $l, m \in \mathbb{N}$ with $m \geq l$,

$$\begin{aligned} d(T^l z, T^m z) &\leq d(T^l z, T^{l+1} z) + d(T^{l+1} z, T^{l+2} z) + \dots + d(T^{m-1} z, T^m z) \\ &\leq r^l (1 + r) g(z) + r^{l+1} (1 + r) g(z) + \dots + r^{m-1} (1 + r) g(z) \\ &\leq r^l (1 + r) g(z) + r^{l+1} (1 + r) g(z) + \dots + r^{m-1} (1 + r) g(z) + \dots \\ &= r^l (1 + r) g(z) (1 + r + r^2 + r^3 + \dots) \\ &= r^l (1 + r) g(z) \frac{1}{1 - r} \end{aligned}$$

and $r^l (1 + r) g(z) \frac{1}{1-r} \to 0$ as $l \to \infty$. So, $\{T^m z\}$ is a Cauchy sequence in X. Since X is complete, $\{T^m z\}$ converges. Let $T^m z \to u$. Since

$$g(T^{m+1} z) = \mu_n d(T^n x, T^{m+1} z) \leq r\mu_n d(T^n x, T^m z) = r g(T^m z)$$

and g is continuous from Lemma 3.1, we obtain that $g(u) \leq r g(u)$. So, we have

$$\mu_n d(T^n x, u) = g(u) \leq r g(u) = r\mu_n d(T^n x, u).$$

From $0 \leq r < 1$, we have $\mu_n d(T^n x, u) = 0$. Since

$$d(Tu, u) \leq d(Tu, T^n x) + d(T^n x, u)$$

for all $n \in \mathbb{N}$, we have

$$\begin{aligned} d(Tu, u) &\leq \mu_n d(T^n x, Tu) + \mu_n d(T^n x, u) \\ &\leq r\mu_n d(T^n x, u) + \mu_n d(T^n x, u) \\ &= r0 + 0 = 0. \end{aligned}$$

So, we have $d(Tu, u) = 0$ and hence $Tu = u$. We show that such a fixed point is unique. Let $Tu = u$ and $Tv = v$. Since

$$\mu_n d(T^n x, u) = \mu_n d(T^n x, Tu) \leq r\mu_n d(T^n x, u),$$

we obtain $\mu_n d(T^n x, u) = 0$. Similarly, we have $\mu_n d(T^n x, v) = 0$. Since

$$d(u, v) \leq d(u, T^n x) + d(T^n x, v)$$

for all $n \in \mathbb{N}$, we have

$$\begin{aligned} d(u, v) &\leq \mu_n d(T^n x, u) + \mu_n d(T^n x, v) \\ &= 0 + 0 = 0. \end{aligned}$$

So, we have $d(u, v) = 0$ and hence $u = v$. This completes the proof. □

Next, using Theorem 3.2, we prove a fixed point theorem for contractively generalized hybrid mappings in a metric space.

Theorem 3.3. *Let (X, d) be a complete metric space and let $T : X \to X$ be a contractively generalized hybrid mapping. Then T has a fixed point in X if and only if $\{T^n x\}$ is bounded for some $x \in X$. In this case, the following hold:*

(i) T has a unique fixed point u in X;

(ii) for every $z \in X$, the sequence $\{T^n z\}$ converges to u in X.

Proof. Since $T : X \to X$ is a contractively generalized hybrid mapping, there are $\alpha, \beta \in \mathbb{R}$ and $r \in [0, 1)$ such that

$$\alpha d(Tx, Ty) + (1 - \alpha)d(x, Ty) \leq r\{\beta d(Tx, y) + (1 - \beta)d(x, y)\} \tag{3.4}$$

for all $x, y \in X$. If $F(T) \neq \emptyset$, then $\{T^n u\} = \{u\}$ for $u \in F(T)$. So, $\{T^n u\}$ is bounded. We show the reverse. Take $x \in X$ such that $\{T^n x\}$ is bounded. Then we have from (3.4) that for any $y \in X$ and $n \in \mathbb{N}$,

$$\begin{aligned} \alpha d(T^{n+1}x, Ty) &+ (1 - \alpha)d(T^n x, Ty) \\ &\leq r\{\beta d(T^{n+1}x, y) + (1 - \beta)d(T^n x, y)\}. \end{aligned}$$

Since $\{T^n x\}$ is bounded, we can apply a Banach limit μ to both sides of the inequality. Then, we have

$$\begin{aligned} \mu_n(\alpha d(T^{n+1}x, Ty) &+ (1 - \alpha)d(T^n x, Ty)) \\ &\leq \mu_n(r\{\beta d(T^{n+1}x, y) + (1 - \beta)d(T^n x, y)\}). \end{aligned}$$

So, we obtain

$$\begin{aligned} \alpha\mu_n d(T^{n+1}x, Ty) &+ (1 - \alpha)\mu_n d(T^n x, Ty) \\ &\leq \beta r\mu_n d(T^{n+1}x, y) + r(1 - \beta)\mu_n d(T^n x, y) \end{aligned}$$

and hence

$$\begin{aligned} \alpha\mu_n d(T^n x, Ty) &+ (1 - \alpha)\mu_n d(T^n x, Ty) \\ &\leq \beta r\mu_n d(T^n x, y) + r(1 - \beta)\mu_n d(T^n x, y). \end{aligned}$$

This implies

$$\mu_n d(T^n x, Ty) \leq r\mu_n d(T^n x, y)$$

for all $y \in X$. By Theorem 3.2, T has a unique fixed point u in X. Furthermore, for any $z \in X$, the sequence $\{T^n z\}$ converges to u in X. □

Using Theorem 3.3, we have the following fixed point theorem.

Theorem 3.4. *Let (X,d) be a complete metric space and let $T : X \to X$ be an (α,β,r)-contractively generalized hybrid mapping such that*

$$\beta \geq 0,\ \ \alpha - r\beta > 0 \text{ and } r < \frac{\alpha}{1+\beta}.$$

Then, the following hold:

(i) T has a unique fixed point u in X;

(ii) for every $z \in X$, the sequence $\{T^n z\}$ converges to u in X.

Proof. Since $T : X \to X$ is an (α,β,r)-contractively generalized hybrid mapping, we have that

$$\alpha d(Tx,Ty) + (1-\alpha)d(x,Ty) \leq r\{\beta d(Tx,y) + (1-\beta)d(x,y)\} \tag{3.5}$$

for all $x, y \in X$. We note that $0 \leq r < 1$. Fix $x \in X$ and $n \in \mathbb{N}$. Replacing x by $T^n x$ and y by $T^{n-1}x$ in (3.5), we have

$$\begin{aligned} \alpha d(T^{n+1}x, T^n x) &+ (1-\alpha)d(T^n x, T^n x) \\ &\leq r\{\beta d(T^{n+1}x, T^{n-1}x) + (1-\beta)d(T^n x, T^{n-1}x)\}. \end{aligned} \tag{3.6}$$

From $\beta \geq 0$ and (3.6), we have

$$\begin{aligned} \alpha d(T^{n+1}x, T^n x) \leq r\{\beta(&d(T^{n+1}x, T^n x) \\ &+ d(T^n x, T^{n-1}x)) + (1-\beta)d(T^n x, T^{n-1}x)\} \end{aligned} \tag{3.7}$$

and hence

$$(\alpha - r\beta)d(T^{n+1}x, T^n x) \leq rd(T^n x, T^{n-1}x). \tag{3.8}$$

From $\alpha - r\beta > 0$ we have

$$d(T^{n+1}x, T^n x) \leq \frac{r}{\alpha - r\beta} d(T^n x, T^{n-1}x). \tag{3.9}$$

From $r < \frac{\alpha}{1+\beta}$, we have $r < \alpha - r\beta$ and

$$0 \leq \frac{r}{\alpha - r\beta} < 1.$$

Putting $\lambda = \frac{r}{\alpha - r\beta}$, we have that for any $n \in \mathbb{N}$,

$$\begin{aligned} d(x, T^n x) &\leq d(x,Tx) + d(Tx, T^2x) + \cdots + d(T^{n-1}x, T^n x) \\ &\leq d(x,Tx) + \lambda d(x,Tx) + \cdots + \lambda^{n-1} d(x,Tx) \\ &\leq d(x,Tx) + \lambda d(x,Tx) + \cdots + \lambda^{n-1} d(x,Tx) + \ldots \\ &= d(x,Tx)(1 + \lambda + \cdots + \lambda^{n-1} + \ldots) \\ &= d(x,Tx)\frac{1}{1-\lambda}. \end{aligned}$$

So, the sequence $\{T^n x\}$ is bounded. We have from Theorem 3.3 that T has a unique fixed point u in X and for every $z \in X$, the sequence $\{T^n z\}$ converges to u in X. □

Using Theorem 3.4, we can also prove the following well-known fixed point theorems. We first prove a fixed point theorem for contractive mappings in a complete metric space.

Theorem 3.5. *Let (X, d) be a complete metric space and let $T : X \to X$ be a contractive mapping, i.e., there exists a real number r with $0 \le r < 1$ such that*

$$d(Tx, Ty) \le rd(x, y)$$

for all $x, y \in X$. Then, the following hold:

(i) T has a unique fixed point u in X;

(ii) for every $z \in X$, the sequence $\{T^n z\}$ converges to u in X.

Proof. Putting $\alpha = 1$ and $\beta = 0$ in (3.1), we have that

$$d(Tx, Ty) \le rd(x, y)$$

for all $x, y \in X$. Furthermore, we have that

$$\beta = 0 \ge 0,\ \alpha - r\beta = 1 > 0 \text{ and } \frac{\alpha}{1+\beta} = \frac{1}{1} = 1 > r.$$

From Theorem 3.4, we have the desired result. □

Theorem 3.6. *Let (X, d) be a complete metric space and let $T : X \to X$ be contractively nonspreading, i.e., there exists a real number γ with $0 \le \gamma < \frac{1}{2}$ such that*

$$d(Tx, Ty) \le \gamma\{d(Tx, y) + d(Ty, x)\}$$

for all $x, y \in X$. Then, the following hold:

(i) T has a unique fixed point u in X;

(ii) for every $z \in X$, the sequence $\{T^n z\}$ converges to u in X.

Proof. Setting $r = \frac{\gamma}{1-\gamma}$, we have $r - r\gamma = \gamma$ and hence $\gamma = \frac{r}{1+r}$. From $0 \le \gamma < \frac{1}{2}$, we have $0 \le r$. We have also

$$r < 1 \Leftrightarrow \frac{r}{1+r} = \gamma < \frac{1}{2}.$$

So, we have $0 \le r < 1$. Furthermore, we have

$$(1 + r)d(Tx, Ty) \le r\{d(Tx, y) + d(Ty, x)\}$$

for all $x, y \in X$. This implies that

$$(1 + r)d(Tx, Ty) - rd(x, Ty) \le rd(Tx, y)$$

for all $x, y \in X$. So, T is a $(1 + r, 1, r)$-contractively generalized hybrid mapping. Furthermore, we have that

$$\beta = 1 > 0,\ \alpha - r\beta = 1 > 0 \text{ and } \frac{\alpha}{1+\beta} = \frac{1+r}{2} > r.$$

From Theorem 3.4, we have the desired result. □

Theorem 3.7. *Let (X, d) be a complete metric space and let $T : X \to X$ be contractively hybrid, i.e., there exists a real number γ with $0 \leq \gamma < \frac{1}{3}$ and*

$$d(Tx, Ty) \leq \gamma\{d(Tx, y) + d(Ty, x) + d(x, y)\}$$

for all $x, y \in X$. Then, the following hold:

(i) T has a unique fixed point u in X;

(ii) for every $z \in X$, the sequence $\{T^n z\}$ converges to u in X.

Proof. Setting $r = \frac{2\gamma}{1-\gamma}$, we have $r - r\gamma = 2\gamma$ and hence $\gamma = \frac{r}{2+r}$. From $0 \leq \gamma < \frac{1}{3}$, we have $0 \leq r$. We have also

$$r < 1 \Leftrightarrow \frac{r}{2+r} = \gamma < \frac{1}{3}.$$

So, we have $0 \leq r < 1$. Furthermore, we have

$$(2 + r)d(Tx, Ty) \leq r\{d(Tx, y) + d(Ty, x) + d(x, y)\}$$

for all $x, y \in X$. This implies that

$$(2 + r)d(Tx, Ty) - rd(x, Ty) \leq r\{d(Tx, y) + d(x, y)\}$$

for all $x, y \in X$. So, we have that

$$(1 + \frac{r}{2})d(Tx, Ty) - \frac{r}{2}d(x, Ty) \leq r\{\frac{1}{2}d(Tx, y) + \frac{1}{2}d(x, y)\}$$

for all $x, y \in X$. This means that T is a $(1 + \frac{r}{2}, \frac{1}{2}, r)$-contractively generalized hybrid mapping. Furthermore, we have that

$$\beta = \frac{1}{2} > 0,\ \alpha - r\beta = 1 + \frac{r}{2} - r\frac{1}{2} = 1 > 0 \ and\ \frac{\alpha}{1+\beta} = \frac{1+\frac{r}{2}}{1+\frac{1}{2}} > r.$$

From Theorem 3.4, we have the desired result. □

4 Estimating Expressions Let (X, d) be a metric space. Let $T : X \to X$ be a mapping. We denote by $F(T)$ the set of fixed points of T. Let α, β, r be real numbers with $0 \leq r < 1$. Let $T : X \to X$ be an (α,β,r)-contractively generalized hybrid mapping. Observe that if $F(T) \neq \emptyset$, then T is *quasi-contractive*, i.e.,

$$d(u, Ty) \leq rd(u, y)$$

for all $u \in F(T)$ and $y \in X$. Indeed, putting $x = u \in F(T)$ in (3.1), we obtain

$$\alpha d(u, Ty) + (1 - \alpha)d(u, Ty) \leq r\{\beta d(u, y) + (1 - \beta)d(u, y)\}.$$

So, we have that

$$d(u, Ty) \leq rd(u, y) \tag{4.1}$$

for all $u \in F(T)$ and $y \in X$. This fact is used in the proof of Theorem 4.2 below. Before proving our main theorem in this section, we show the following basic lemma.

Lemma 4.1. *Let r and γ be real numbers with $0 < r < 1$ and $0 < \gamma < 1$, respectively. For any $P_0, P_1 \in \mathbb{R}$, define a sequence $\{P_n\}$ of real numbers as follows:*

$$P_{n+2} = r(\gamma P_{n+1} + (1-\gamma)P_n), \quad \forall n \in \mathbb{N}.$$

Then,

$$P_n = \frac{P_1 - P_0 v}{u-v} u^n + \frac{P_0 u - P_1}{u-v} v^n, \quad \forall n \in \mathbb{N}, \tag{4.2}$$

where

$$u = \frac{r\gamma + \sqrt{r^2\gamma^2 + 4r(1-\gamma)}}{2}, \quad v = \frac{r\gamma - \sqrt{r^2\gamma^2 + 4r(1-\gamma)}}{2}.$$

Proof. It is obvious that $u > 0$ and $v < 0$. We know also that u, v are two solutions of the following quadratic equation of λ:

$$\lambda^2 - r\gamma\lambda - r(1-\gamma) = 0.$$

So, we have

$$u + v = r\gamma, \quad uv = -r(1-\gamma). \tag{4.3}$$

Putting $f(\lambda) = \lambda^2 - r\gamma\lambda - r(1-\gamma)$ for all $\lambda \in \mathbb{R}$, we have $f(1) > 0$ and $f(0) < 0$. So, we have $0 < u < 1$. Next, if $v \le -1$, we have $u + v < 0$. This contadicts (4.3). So, we have $-1 < v < 0$. Let us prove (4.2). In the case of $n = 0$, we have

$$\frac{P_1 - P_0 v}{u-v} u^0 + \frac{P_0 u - P_1}{u-v} v^0 = \frac{P_0(u-v)}{u-v} = P_0.$$

Similarly, in the case of $n = 1$, we have

$$\frac{P_1 - P_0 v}{u-v} u^1 + \frac{P_0 u - P_1}{u-v} v^1 = \frac{P_1(u-v)}{u-v} = P_1.$$

Suppose

$$P_n = \frac{P_1 - P_0 v}{u-v} u^n + \frac{P_0 u - P_1}{u-v} v^n$$

for $n = k, k+1$. Then, we have from (4.3) that

$$\begin{aligned}
P_{k+2} &= r(\gamma P_{k+1} + (1-\gamma)P_k) \\
&= r(\gamma(\frac{P_1 - P_0 v}{u-v} u^{k+1} + \frac{P_0 u - P_1}{u-v} v^{k+1}) \\
&\quad + (1-\gamma)(\frac{P_1 - P_0 v}{u-v} u^k + \frac{P_0 u - P_1}{u-v} v^k)) \\
&= (u+v)(\frac{P_1 - P_0 v}{u-v} u^{k+1} + \frac{P_0 u - P_1}{u-v} v^{k+1}) \\
&\quad + r\frac{P_1 - P_0 v}{u-v} u^k + r\frac{P_0 u - P_1}{u-v} v^k - (u+v)(\frac{P_1 - P_0 v}{u-v} u^k + \frac{P_0 u - P_1}{u-v} v^k) \\
&= \frac{P_1 - P_0 v}{u-v}((u+v)u^{k+1} + ru^k - (u+v)u^k) \\
&\quad + \frac{P_0 u - P_1}{u-v}((u+v)v^{k+1} + rv^k - (u+v)v^k)
\end{aligned}$$

$$
\begin{aligned}
&= \frac{P_1 - P_0 v}{u - v}((u+v)u^{k+1} + (u+v-uv)u^k - (u+v)u^k) \\
&\quad + \frac{P_0 u - P_1}{u - v}((u+v)v^{k+1} + (u+v-uv)v^k - (u+v)v^k) \\
&= \frac{P_1 - P_0 v}{u - v}u^k((u+v)u - uv) + \frac{P_0 u - P_1}{u - v}v^k((u+v)v - uv) \\
&= \frac{P_1 - P_0 v}{u - v}u^k u^2 + \frac{P_0 u - P_1}{u - v}v^k v^2 \\
&= \frac{P_1 - P_0 v}{u - v}u^{k+2} + \frac{P_0 u - P_1}{u - v}v^{k+2}.
\end{aligned}
$$

By induction, we have

$$P_n = \frac{P_1 - P_0 v}{u - v}u^n + \frac{P_0 u - P_1}{u - v}v^n$$

for all $n \in \mathbb{N}$. This completes the proof. □

Using Lemma 4.1, we obtain the following estimating expression for contractively generalized hybrid mappings in a Banach space.

Theorem 4.2. *Let E be a Banach space and let C be a nonempty closed convex subset of E. Let α, β, r be real numbers with $0 < r < 1$ and let $T : C \to C$ be an (α,β,r)-contractively generalized hybrid mapping such that $F(T)$ is nonempty. Let $\gamma \in (0, 1)$ and define a sequence $\{x_n\}$ of C as follows: $x_0, x_1 \in C$ and*

$$x_{n+2} = T(\gamma x_{n+1} + (1-\gamma)x_n), \quad \forall n \in \mathbb{N}.$$

Then, $\{x_n\}$ converges a unique fixed point z of T. Furthermore,

$$\|x_n - z\| \le \frac{P_1 - P_0 v}{u - v}u^n + \frac{P_0 u - P_1}{u - v}v^n,$$

where $P_0 = \|x_0 - z\|$, $P_1 = \|x_1 - z\|$ and $u, v \in \mathbb{R}$ are two solutions of the quadratic equation of λ:

$$\lambda^2 - r\gamma\lambda - r(1-\gamma) = 0.$$

Proof. We know from Theorem 3.3 that T has a unique fixed point z in C. Let $P_0 = \|x_0 - z\|$ and $P_1 = \|x_1 - z\|$. Define a sequence $\{P_n\}$ of real numbers as follows:

$$P_{n+2} = r(\gamma P_{n+1} + (1-\gamma)P_n), \quad \forall n \in \mathbb{N}.$$

Then, we know from Lemma 4.1 that

$$P_n = \frac{P_1 - P_0 v}{u - v}u^n + \frac{P_0 u - P_1}{u - v}v^n, \quad \forall n \in \mathbb{N}.$$

So, for finishing the proof, it is sufficient to show that

$$\|x_n - z\| \le P_n, \quad \forall n \in \mathbb{N}.$$

From $P_0 = \|x_0 - z\|$ and $P_1 = \|x_1 - z\|$, we have $\|x_0 - z\| \le P_0$ and $\|x_1 - z\| \le P_1$. Suppose

$$\|x_n - z\| \le P_n$$

for $n = k, k+1$. Then, we have from (4.1) that

$$\|x_{k+2} - z\| = \|T(\gamma x_{k+1} + (1-\gamma)x_k) - z\|$$

$$\begin{aligned}
&\le r\|\gamma x_{k+1} + (1-\gamma)x_k - z\| \\
&= r\|\gamma(x_{k+1} - z) + (1-\gamma)(x_k - z)\| \\
&\le r(\gamma\|x_{k+1} - z\| + (1-\gamma)\|x_k - z\|) \\
&\le r(\gamma P_{k+1} + (1-\gamma)P_k) \\
&= P_{k+2}.
\end{aligned}$$

By induction, we have $\|x_n - z\| \le P_n$ for all $n \in \mathbb{N}$. Since

$$P_n = \frac{P_1 - P_0 v}{u - v} u^n + \frac{P_0 u - P_1}{u - v} v^n, \quad \forall n \in \mathbb{N},$$

we have from $0 < u < 1$ and $-1 < v < 0$ that $P_n \to 0$ as $n \to \infty$. This completes the proof. □

Using Theorem 4.2, we give estimating expressions for well-known mappings in a Banach space.

Theorem 4.3. *Let E be a Banach space and let C be a nonempty closed convex subset of E. Let $T : C \to C$ be a r-contractive mapping with $0 < r < 1$, i.e., there exists a real number r with $0 < r < 1$ such that*

$$\|Tx - Ty\| \le r\|x - y\|$$

for all $x, y \in C$. Let $\gamma \in (0,1)$ and define a sequence $\{x_n\}$ of C as follows: $x_0, x_1 \in C$ and

$$x_{n+2} = T(\gamma x_{n+1} + (1-\gamma)x_n), \quad \forall n \in \mathbb{N}.$$

Then, $\{x_n\}$ converges a unique fixed point z of T. Furthermore,

$$\|x_n - z\| \le \frac{P_1 - P_0 v}{u - v} u^n + \frac{P_0 u - P_1}{u - v} v^n,$$

where $P_0 = \|x_0 - z\|$, $P_1 = \|x_1 - z\|$ and $u, v \in \mathbb{R}$ are two solutions of the quadratic equation of λ:

$$\lambda^2 - r\gamma\lambda - r(1-\gamma) = 0.$$

Proof. Putting $\alpha = 1$ and $\beta = 0$ in (3.1), we have that

$$\|Tx - Ty\| \le r\|x - y\|$$

for all $x, y \in C$. Furthermore, as in the proof of Theorem 3.5, we have that

$$\beta \ge 0, \ \alpha - r\beta > 0 \text{ and } \frac{\alpha}{1+\beta} > r.$$

From Theorem 3.4, we have $F(T) \ne \emptyset$. So, from Theorem 4.2, we have the desired result. □

Theorem 4.4. *Let E be a Banach space and let C be a nonempty closed convex subset of E. Let $T : C \to C$ be contractively nonspreading with $0 < k < \frac{1}{2}$, i.e., there exists a real number k with $0 < k < \frac{1}{2}$ such that*

$$\|Tx - Ty\| \le k\{\|Tx - y\| + \|Ty - x\|\}$$

for all $x, y \in C$. Let $\gamma \in (0,1)$ and define a sequence $\{x_n\}$ of C as follows: $x_0, x_1 \in C$ and

$$x_{n+2} = T(\gamma x_{n+1} + (1-\gamma)x_n), \quad \forall n \in \mathbb{N}.$$

Then, $\{x_n\}$ converges a unique fixed point z of T. Furthermore,

$$\|x_n - z\| \leq \frac{P_1 - P_0 v}{u - v} u^n + \frac{P_0 u - P_1}{u - v} v^n,$$

where $P_0 = \|x_0 - z\|$, $P_1 = \|x_1 - z\|$ and $u, v \in \mathbb{R}$ are two solutions of the quadratic equation of λ:

$$(1-k)\lambda^2 - k\gamma\lambda - k(1-\gamma) = 0.$$

Proof. Setting $r = \frac{k}{1-k}$ as in the proof of Theorem 3.6, we have $r - rk = k$ and hence $k = \frac{r}{1+r}$. From $0 < k < \frac{1}{2}$, we have $0 < r$. We have also

$$r < 1 \Leftrightarrow \frac{r}{1+r} = k < \frac{1}{2}.$$

So, we have $0 < r < 1$. Furthermore, as in the proof of Theorem 3.6, we have that

$$(1+r)\|Tx - Ty\| - r\|x - Ty\| \leq r\|Tx - y\|$$

for all $x, y \in C$, that is, T is a $(1+r, 1, r)$-contractively generalized hybrid mapping. Finally, we have that

$$\beta > 0,\ \ \alpha - r\beta > 0 \text{ and } \frac{\alpha}{1+\beta} > r.$$

From Theorem 3.4, we have $F(T) \neq \emptyset$. So, from Theorem 4.2, we have the desired result. □

Theorem 4.5. *Let E be a Banach space and let C be a nonempty closed convex subset of E. Let $T : C \to C$ be contractively hybrid with $0 < s < \frac{1}{3}$, i.e., there exists a real number s with $0 < s < \frac{1}{3}$ and*

$$\|Tx - Ty\| \leq s\{\|Tx - y\| + \|Ty - x\| + \|x - y\|\}$$

for all $x, y \in C$. Let $\gamma \in (0,1)$ and define a sequence $\{x_n\}$ of C as follows: $x_0, x_1 \in C$ and

$$x_{n+2} = T(\gamma x_{n+1} + (1-\gamma)x_n), \quad \forall n \in \mathbb{N}.$$

Then, $\{x_n\}$ converges a unique fixed point z of T. Furthermore,

$$\|x_n - z\| \leq \frac{P_1 - P_0 v}{u - v} u^n + \frac{P_0 u - P_1}{u - v} v^n,$$

where $P_0 = \|x_0 - z\|$, $P_1 = \|x_1 - z\|$ and $u, v \in \mathbb{R}$ are two solutions of the quadratic equation of λ:

$$(1-s)\lambda^2 - 2s\gamma\lambda - 2s(1-\gamma) = 0.$$

Proof. Setting $r = \frac{2s}{1-s}$ as in the proof of Theorem 3.7, we have $r - rs = 2s$ and hence $s = \frac{r}{2+r}$. From $0 < s < \frac{1}{3}$, we have $0 < r$. We have also

$$r < 1 \Leftrightarrow \frac{r}{2+r} = s < \frac{1}{3}.$$

So, we have $0 < r < 1$. Furthermore, as in the proof of Theorem 3.7, we have that

$$(1 + \frac{r}{2})\|Tx - Ty\| - \frac{r}{2}\|x - Ty\| \leq r\{\frac{1}{2}\|Tx - y\| + \frac{1}{2}\|x - y\|\}$$

for all $x, y \in C$, that is, T is a $(1+\frac{r}{2}, \frac{1}{2}, r)$-contractively generalized hybrid mapping. Finally, we have that

$$\beta > 0,\ \ \alpha - r\beta > 0 \text{ and } \frac{\alpha}{1+\beta} > r.$$

From Theorem 3.4, we have $F(T) \neq \emptyset$. So, from Theorem 4.2, we have the desired result. □

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