The Replenishment Policy for an Inventory System with a Fixed Ordering Cost and a Proportional Penalty Cost under Poisson Arrival Demands

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Abstract. A store with a bounded tank sells a random resource to customers arriving according to the Poisson process. It reasonably has to be managed in a balance between a fixed ordering cost and a proportional penalty cost. The store should lay down the safety stock level so as to keep these losses as minimal as possible. Then he adopts an ordering policy in which the tank is filled with resource when the stock level falls to the safety stock level. We decide the optimal safety stock level so as to minimize the expected cost per unit time in the infinite periods.

1 Introduction Inventory control problems have been discussed in the academic literature since the 1950s. There are several policies such as order-up-to-level policy, (Q, r) policy, (s, S) policy, continuous review reorder policy, and so on. In recent studies Hariga [6] examined a single item continuous review inventory problem with stochastic demand and a bounded capacity. Babai, Jemai, Dallery [3] investigated a continuous ouder-up-to level policy for a single echelon inventory system where the demand is modelled as a compound Poisson process. On the study of reorder level Hohjo, Teraoka [8] suggested a single item continuous review inventory model, assuming that consumers arrive at a store according to the Poisson process and buy a resource expressed by continuous quantity, with the constant ordering cost and the constant penalty cost for stocking-out and analyzed on the optimal safety stock level under minimizing of the expected cost per unit time in the infinite period. However the penalty cost is often suitable to be given by a function related to cumulative quantity of shortages in some situations. This paper extends it to a model with the constant ordering cost and a proportional penalty cost. Our model doesn't deal with the holding cost but the ordering cost and the penalty one.

This paper is constructed as follows. Section 2 describes an inventory control model with a fixed ordering cost and a proportional penalty cost under Poisson arrival demands. Section 3 provides the calculations of the expected cost per unit time in the infinite period by using a renewal reward theorem [14], which leads to the optimal replenishment policy. Numerical examples illustrate some examinations for our result in Section 4. Section 5 gives some conclusions.

2 Model We consider an inventory control model on a resource expressed by continuous quantity. The model is described as follows: A store with an existing tank of maximum permissible quantity U deals in a liquescent resource. Customers arrive at the store according to the Poisson process with intensity function λ. The j-th customer purchases Y_j units of resources, where Y_j $(j = 1, 2, \dots)$ is a sequence of identical and independent nonnegative random variable having a general distribution function $G(x) = Pr\{Y_j \le x\}$. For such customers' behavior, the store should lay down the safety stock level so as to keep these losses

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as minimal as possible. Then he adopts an ordering policy in which the tank is filled with resource when the stock level falls to the safety stock level u $(0 \leq u \leq U)$. The ordered resource arrives instantaneously and the ordering cost is charged by C_r regardless of the order quantity. In being out of stock, the penalty cost is charged by p for one unit.

Under these assumptions, the objective is to decide the optimal safety stock level u^* so as to minimize the expected cost, denoted by $C(u)$, per unit time in the infinite period.

3 Formulation and Analysis We begin our analysis by calculating the expected cycle cost $R(u)$ and the expected cycle length $L(u)$.

Let Z_j $(j = 0, 1, 2, \cdots)$ denote the cumulative quantity when the j customers have purchased resources, where $Z_0 = 0$. Then the cumulative process Z_j can be expressed as

$$Z_j = \sum_{i=1}^{j} Y_i \quad \text{for} \quad j = 1, 2, 3, \cdots \tag{1}$$

with

$$Pr\{Z_j \leq x\} \equiv G^{(j)}(x) \quad \text{for} \quad j = 0, 1, 2, \cdots, \tag{2}$$

where $G^{(j)}(x)$ is the j-fold Stieltjes convolution of $G(x)$ with itself, and

$$G^{(0)}(x) \equiv \begin{cases} 1, & x \geq 0 \\ 0, & x < 0. \end{cases}$$

The probability $\alpha(u)$, of replenishing resources when the stock level reaches between 0 and u, is given by

$$\begin{aligned} \alpha(u) &= \sum_{j=1}^{\infty} Pr\{Z_{j-1} \leq U - u < Z_j \leq U\} \\ &= \sum_{j=1}^{\infty} \int_0^{U-u} Pr\{U - u - Z_{j-1} < Y_j \leq U - Z_{j-1} | Z_{j-1} = x\} \, dG^{(j-1)}(x) \\ &= \sum_{j=1}^{\infty} \int_0^{U-u} [G(U - x) - G(U - u - x)] \, dG^{(j-1)}(x) \\ &= \sum_{j=1}^{\infty} \int_0^{U-u} G(U - x) \, dG^{(j-1)}(x) - M(U - u), \end{aligned} \tag{3}$$

where

$$M(u) \equiv \sum_{j=1}^{\infty} G^{(j)}(u). \tag{4}$$

Using Eq.(3), the expected cycle cost $R(u)$ can be written as

$$\begin{aligned} R(u) &= \sum_{j=1}^{\infty} \{C_r + p(Z_j - U)\} Pr\{Z_{j-1} \leq U - u, Z_j > U\} + C_r \alpha(u) \\ &= \sum_{j=1}^{\infty} \int_0^{U-u} \{C_r + p(x + Y_j - U)\} Pr\{Y_j > U - x\} dG^{(j-1)}(x) + C_r \alpha(u) \end{aligned}$$

$$
\begin{aligned}
&= C_r \sum_{j=1}^{\infty} \int_0^{U-u} [1 - G(U-u-x)] dG^{(j-1)}(x) \\
&\quad + p \sum_{j=1}^{\infty} \int_0^{U-u} (x + Y_j - U) Pr\{Y_j > U - x\} dG^{(j-1)}(x) \\
&= C_r + p \sum_{j=1}^{\infty} \int_0^{U-u} \int_{U-x}^{\infty} (x + y - U) dG(y) dG^{(j-1)}(x). \qquad (5)
\end{aligned}
$$

The probability density function that the j-th customer arrives at time t is given by

$$
\frac{\lambda(\lambda t)^{j-1}}{(j-1)!} e^{-\lambda t} \qquad (j = 1, 2, \cdots), \tag{6}
$$

which is known as the Erlang distribution.

Letting $\bar{G}(x)$ denote the survivor function for a continuous distribution function $G(x)$, *i.e.* $\bar{G}(x) = 1 - G(x) = Pr\{Y_j > x\}$, the expected cycle length $L(u)$ can be written as

$$
\begin{aligned}
L(u) &= \sum_{j=1}^{\infty} \int_0^{\infty} t\ Pr\{Z_{j-1} \le U - u, Z_j > U\} \frac{\lambda(\lambda t)^{j-1}}{(j-1)!} e^{-\lambda t} dt \\
&\quad + \sum_{j=1}^{\infty} \int_0^{\infty} t\ Pr\{Z_{j-1} \le U - u < Z_j \le U\} \frac{\lambda(\lambda t)^{j-1}}{(j-1)!} e^{-\lambda t} dt \\
&= \sum_{j=1}^{\infty} \int_0^{\infty} t \frac{\lambda(\lambda t)^{j-1}}{(j-1)!} e^{-\lambda t} dt \int_0^{U-u} \bar{G}(U-x) dG^{(j-1)}(x) \\
&\quad + \sum_{j=1}^{\infty} \int_0^{\infty} t \frac{\lambda(\lambda t)^{j-1}}{(j-1)!} e^{-\lambda t} dt \int_0^{U-u} [G(U-x) - G(U-u-x)] dG^{(j-1)}(x) \\
&= \sum_{j=1}^{\infty} \int_0^{\infty} t \frac{\lambda(\lambda t)^{j-1}}{(j-1)!} e^{-\lambda t} dt \int_0^{U-u} [1 - G(U-u-x)] dG^{(j-1)}(x) \\
&= \sum_{j=1}^{\infty} \frac{j}{\lambda} [G^{(j-1)}(U-u) - G^{(j)}(U-u)] \\
&= \frac{1 + M(U-u)}{\lambda}. \qquad (7)
\end{aligned}
$$

The stock level is renewed by replenishing resources. When we define an interval of two adjacent renewal time as a cycle for the renewal process, the renewal reward theorem leads us to the expected cost $C(u)$ per unit time in the infinite period. From Eqs.(5) and (7) it can be expressed as

$$
\begin{aligned}
C(u) &= \frac{R(u)}{L(u)} \\
&= \frac{\lambda}{1 + M(U-u)} \left[C_r + p \sum_{j=1}^{\infty} \int_0^{U-u} \int_{U-x}^{\infty} (x + y - U) dG(y) dG^{(j-1)}(x) \right]. \qquad (8)
\end{aligned}
$$

To find the safety stock level u of minimizing the expected cost per unit time, we differentiate $C(u)$ with respect to u and set it to be equal to 0. Then we obtain

$$
\sum_{j=1}^{\infty} \int_0^{U-u} \left\{ \int_u^{\infty} (y - u) dG(y) - \int_{U-x}^{\infty} (x + y - U) dG(y) \right\} dG^{(j-1)}(x) = \frac{C_r}{p}. \tag{9}
$$

See Appendix A for the detailed calculation to derive Eq.(9). Putting the left side of Eq.(9) by $V(u)$ and differentiating it with respect to u, we have

$$\begin{aligned}\frac{dV(u)}{du} &= \sum_{j=1}^{\infty}\int_0^{U-u}\left\{-\int_u^{\infty} dG(y)\right\} dG^{(j-1)}(x)\\ &= -\bar{G}(u)[1+M(U-u)]. \end{aligned}\tag{10}$$

Then the derivative function $\frac{dV(u)}{du}$ is negative for all $0 \le u \le U$. Hence, $V(u)$ is a monotone decreasing function in $0 \le u \le U$, and it satisfies that

$$V(0) = \sum_{j=1}^{\infty}\int_0^{U}\left\{\int_0^{U-x} ydG(y) + (U-x)\bar{G}(U-x)\right\} dG^{(j-1)}(x) > 0 \tag{11}$$

and

$$V(U) \quad = \quad 0. \tag{12}$$

As a result, if $V(0) > C_r/p$, there exists a unique root $u = u^*$ satisfying Eq.(9). It holds the inequalities $V(u) > C_r/p$ for $0 \le u < u^*$ and $V(u) < C_r/p$ for $u^* < u \le U$, which are implied $C(u)$ is a decreasing function in u over $0 \le u < u^*$ and an increasing one over $u^* < u \le U$. On the other hand, if $V(0) \le C_r/p$, it holds the inequality $V(u) < C_r/p$ for all $0 \le u \le U$, which is implied the function $C(u)$ increases in u. Therefore the sufficient condition of the minimizing problem is guaranteed. From these arguments, we obtain the following theorem.

Theorem. 1 The optimal replenishment policy is given as follows:

(i) If $V(0) > C_r/p$, then there exists the unique root $u = u^*$ satisfying Eq.(9). The optimal replenishment policy is to order up to U as soon as the stock level falls below the safety level u^*. And the corresponding expected cost per unit time is given by

$$C(u^*) = \lambda p \int_{u^*}^{\infty} (y-u^*)dG(y).$$

(ii) If $V(0) \le C_r/p$, then $u = 0$ minimizes the value of the objective function $C(u)$. It means that the optimal replenishment policy is to order up to U after stocking out. The optimal expected cost per unit time is given by

$$C(u^*) = C(0) = \frac{\lambda}{1+M(U)}\left[C_r + p\sum_{j=1}^{\infty}\int_0^{U}\int_{U-x}^{\infty}(x+y-U)dG(y)dG^{(j-1)}(x)\right].$$

4 Numerical Examples We give a sensitive analysis in this section. Suppose that the function $G(x)$ is exponentially distributed with its mean $1/\theta$, *i.e.* $G(x) = 1 - e^{-\theta x}$, and let $C_r = 10, \lambda = 10$. Then Eq.(9) can be rewritten as

$$\frac{1}{\theta}\sum_{j=1}^{\infty}\left[e^{-\theta u} - \sum_{i=0}^{j-1}\frac{\{\theta(U-u)\}^i}{i!}e^{-\theta U}\right] = \frac{C_r}{p}. \tag{13}$$

We give the optimal safety stock level u^* and its corresponding expected cost $C(u^*)$ for each p and U on fixed value $\theta = 0.01$ in Figure 1. The optimal safety stock level u^* takes value 0 in $p = 0.01$ and $p = 0.02$ on $U = 500$, which results from a small tank. As

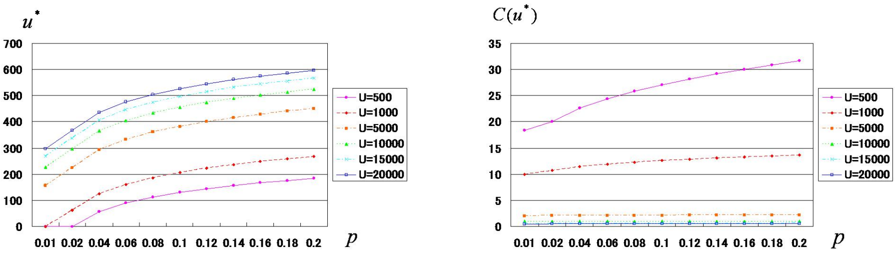

Figure 1: the optimal safety stock level and the corresponding total cost for p

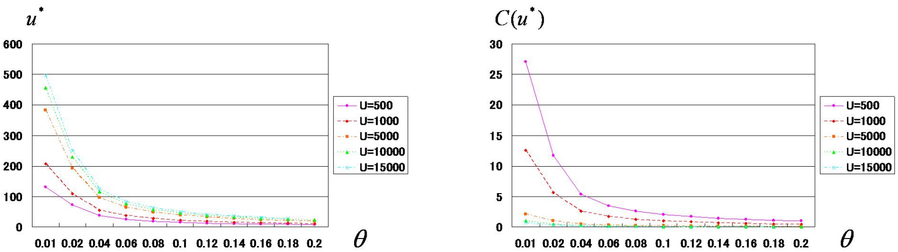

Figure 2: the optimal safety stock level and the corresponding total cost for θ

the unit penalty cost p increases, the optimal safety stock level u^* also increases. If the maximum permissible quantity U has large enough, the optimal expected cost can keep cheap. For more than 5000 of the maximum permissible quantity U, the corresponding expected costs almost takes the same value. There is no meaning that enlarges the limited capacity because it can suppress the optimal expectation cost to few with some size of the maximum permissible quantity.

Figure 2 represents the optimal safety stock level u^* and its coresponding expected cost $C(u^*)$ for each θ and U on fixed value $p = 0.1$. When θ is small, the optimal total cost is affected by value of U. Then establishing the optimal safety stock level is valid for cutting down the total cost.

5 Concluding Remarks This paper considered the optimal replenishment policy in an inventory control problem with customers arriving according to the Poisson process. Under assumptions with a fixed ordering cost and a proportional penalty cost, we showed an effective result on the setting of the safety stock level.

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Appendix A Let $g^{(i)}(u)$ denote the probability density function corresponding to the cumulative distribution function $G^{(i)}(u)$. The first derivative of function $C(u) = \frac{R(u)}{L(u)}$ is obtained by

$$\frac{dC(u)}{du} = \frac{R'(u)L(u) - R(u)L'(u)}{\{L(u)\}^2}.$$

The numerator is calculated as

$$
\begin{aligned}
&R'(u)L(u) - R(u)L'(u) \\
&= -p\sum_{i=1}^{\infty} g^{(i)}(U-u)\int_u^{\infty}(y-u)dG(y) \times \frac{1+M(U-u)}{\lambda} \\
&\quad - \left[C_r + p\sum_{j=1}^{\infty}\int_0^{U-u}\int_{U-x}^{\infty}(x+y-U)dG(y)dG^{(j-1)}(x)\right] \times \left(-\frac{1}{\lambda}\sum_{i=1}^{\infty} g^{(i)}(U-u)\right) \\
&= \frac{1}{\lambda}\sum_{i=1}^{\infty} g^{(i)}(U-u)\left[-p\int_u^{\infty}(y-u)dG(y) \times \sum_{j=1}^{\infty}\int_0^{U-u} dG^{(j-1)}(x)\right. \\
&\quad \left. + C_r + p\sum_{j=1}^{\infty}\int_0^{U-u}\int_{U-x}^{\infty}(x+y-U)dG(y)dG^{(j-1)}(x)\right] \\
&= \frac{1}{\lambda}\sum_{i=1}^{\infty} g^{(i)}(U-u)\left[C_r - p\sum_{j=1}^{\infty}\int_0^{U-u}\left\{\int_u^{\infty}(y-u)dG(y) - \int_{U-x}^{\infty}(x+y-U)dG(y)\right\}\right. \\
&\quad \left. dG^{(j-1)}(x)\right].
\end{aligned}
$$

By setting the numerator to be equal to 0, we can obtain Eq.(9).

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