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C^*-algebras arising from a matched pair of locally compact groupoids

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Abstract. We introduce a notion of a matched pair of locally compact groupoids and construct several C^*-algebras from a matched pair of locally compact groupoids without assuming the existence of quasi-invariant measures on the unit space. We also show that there exist natural representations of the above C^*-algebras when there exists an invariant measure.

1 Introduction A matched pair of groups has been studied in the theory of operator algebras. S. Majid studied bicrossed product Hopf-von Neumann algebras constructed from a matched pair of locally compact groups in [4]. T. Yamanouchi studied W^*-quantum groups arising from matched pairs of locally compact groups in [13]. S. Baaj, G. Skandalis and S. Vaes studied C^*-algebraic quantum groups obtained through the bicrossed product construction from a matched pair of locally compact groups in [1]. We remark that the definition of a matched pair of locally compact groups by Baaj, Skandalis and Vaes is different from the definition by Majid and Yamanouchi. The author studied C^*-algebras arising from a mathced pair of r-discrete groupoids in [7]. Recently J.-M. Vallin studied measured quantum groupoids associated with matched pairs of locally compact groupoids in the setting of von Neumann algebras in [12].

In [7], we assume the existence of an invariant measure on the unit space of groupoids. In this paper, we construct several C^*-algebras from a matched pair of locally compact groupoids without assuming the existence of quasi-invariant measures on the unit space. We also show that there exist natural representations of the C^*-algebras when there exists an invarianat measure.

We do not know yet how to formulate structures of quantum groupoids on these C^*-algebras. As for quantum groupoids in the setting of C^*-algebras, there are works by T. Timmermann [10, 11].

The paper is organized as follows: In Section 2, we introduce a notion of a matched pair (G_1, G_2) of locally compact groupoids where G_1 and G_2 are subgroupoids of a locally compact groupoids G and we also introduce two conditions for Harr systems. We prove that these conditions are satisfied if G is r-discrete in Proposition 2.6. In the previous paper [7], we did not know this fact and assumed that these conditions are satisfied for a matched pair of r-discrete groupoids. We also introduce a groupoid $\mathcal{T}$ which is isomorphic to G. In Section 3, we describe several representations of groupoid C^*-algebras on Hilbert C^*-modules. In Section 4, we study six Hilbert C^*-modules associated with $\mathcal{T}$, which are isomorphic with each other, and representations on these Hilbert C^*-modules. In Section 5, we introduce two C^*-algebras B and $\hat{B}$ associated with $\mathcal{T}$. In Section 6, we introduce four C^*-algebras $C_r^*(G_1) \bowtie C_r^*(G_2)$, $C_r^*(G_2) \bowtie C_r^*(G_1)$, $C_r^*(G_1) \ltimes C_0(G_2)$ and $C_r^*(G_2) \ltimes C_0(G_1)$

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and study representations of these C^*-algebras on Hilbert C^*-modules associated with $\mathcal{T}$. By construction, $C_r^*(G_1) \bowtie C_r^*(G_2)$ and $C_r^*(G_2) \bowtie C_r^*(G_1)$ are isomorphic to $C_r^*(G)$, which is isomorphic to $C_r^*(\mathcal{T})$, $C_r^*(G_1) \ltimes C_0(G_2)$ is isomorphic to B and $C_r^*(G_2) \ltimes C_0(G_1)$ is isomorphic to $\hat{B}$. In Section 7, we assume that there exists a G_1- and G_2-invariant measure μ on the unit space $G^{(0)}$. Then we show that there exist natural representations of the above C^*-algebras on a Hilbert space H in Theorem 7.2. In Section 8, we give two examples of actions of matched pairs.

2 A matched pair of locally compact groupoids Let G be a second countable locally compact Hausdorff groupoid. We denote by r_G (resp. s_G) the range (resp. source) map of G, by $G^{(0)}$ the unit space of G and by $G^{(2)}$ the set of composable pairs. The map r_G (resp. s_G) is also denoted by r (resp. s) to simplify a notation. For details of groupoids, we refer the reader to [8] and [9].

Definition 2.1. Let G_1 and G_2 be closed subgroupoids of G. A pair (G_1, G_2) is called a matched pair if $G_1G_2 = G$, $G_1 \cap G_2 = G^{(0)}$ and there exist continuous maps $p_1 : G \to G_1$ and $p_2 : G \to G_2$ such that $g = p_1(g)p_2(g)$ for all $g \in G$.

Let (G_1, G_2) be a matched pair. For $i = 1, 2$, set $G_{i,x} = s^{-1}(x) \cap G_i$ and $G_i^x = r^{-1}(x) \cap G_i$ for $x \in G^{(0)}$. For $(g_2, g_1) \in G^{(2)} \cap (G_2 \times G_1)$, set $g_2 \triangleright g_1 = p_1(g_2g_1)$ and $g_2 \triangleleft g_1 = p_2(g_2g_1)$. Note that we have $r(g_2 \triangleright g_1) = r(g_2)$, $s(g_2 \triangleleft g_1) = s(g_1)$ and $s(g_2 \triangleright g_1) = r(g_2 \triangleleft g_1)$. As in the group case, we have the following lemma (cf [2],[6]).

Lemma 2.2. *The following equations hold:*

(1) $h_2 \triangleright (g_2 \triangleright g_1) = (h_2g_2) \triangleright g_1 \quad (g_1 \in G_1,\ g_2 \in G_{2,r(g_1)},\ h_2 \in G_{2,r(g_2)})$.

(2) $(g_2 \triangleleft g_1) \triangleleft h_1 = g_2 \triangleleft (g_1h_1) \quad (g_2 \in G_2,\ g_1 \in G_1^{s(g_2)},\ h_1 \in G_1^{s(g_1)})$.

(3) $g_2 \triangleright (g_1h_1) = (g_2 \triangleright g_1)((g_2 \triangleleft g_1) \triangleright h_1)) \quad (g_2 \in G_2,\ g_1 \in G_1^{s(g_2)},\ h_1 \in G_1^{s(g_1)})$.

(4) $(h_2g_2) \triangleleft g_1 = (h_2 \triangleleft (g_2 \triangleright g_1))(g_2 \triangleleft g_1) \quad (g_1 \in G_1,\ g_2 \in G_{2,r(g_1)},\ h_2 \in G_{2,r(g_2)})$.

Let $\mathcal{T}$ be the fibered product

$$G_1 {}_s\times_s G_2 = \{(g_1, g_2) \in G_1 \times G_2; s(g_1) = s(g_2)\}.$$

Define maps κ, κ_1 and $\kappa_2 : \mathcal{T} \to \mathcal{T}$ by $\kappa(g_1, g_2) = (g_2 \triangleright g_1^{-1}, (g_2 \triangleleft g_1^{-1})^{-1})$, $\kappa_1(g_1, g_2) = (g_1^{-1}, g_2 \triangleleft g_1^{-1})$ and $\kappa_2(g_1, g_2) = ((g_2 \triangleright g_1^{-1})^{-1}, g_2^{-1})$ respectively. Then κ^2, κ_1^2 and κ_2^2 are the identity map, in particular, κ, κ_1 and κ_2 are homeomorphisms. Note that we have $\kappa = \kappa_1\kappa_2 = \kappa_2\kappa_1$. Define a homeomorphism $\omega : G \longrightarrow \mathcal{T}$ by $\omega(g) = (p_1(g^{-1}), p_2(g^{-1})^{-1})$. Then we have $\omega^{-1}(g_1, g_2) = g_2g_1^{-1}$. We introduce a structure of groupoid into $\mathcal{T}$ as follows: Let $\mathcal{T}^{(0)}$ be the set $\{(x, x); x \in G^{(0)}\}$, which we identify with $G^{(0)}$. The range and source maps $r_{\mathcal{T}}$, $s_{\mathcal{T}} : \mathcal{T} \longrightarrow G^{(0)}$ is defined by $r_{\mathcal{T}}(g_1, g_2) = r(g_2)$ and $s_{\mathcal{T}}(g_1, g_2) = r(g_1)$ respectively. The product is defined by

$$(g_1, g_2)(h_1, h_2) = (h_1(h_2^{-1} \triangleright g_1), g_2(h_2^{-1} \triangleleft g_1)^{-1})$$

for $((g_1, g_2), (h_1, h_2)) \in \mathcal{T}^{(2)}$. The inverse is defined by $(g_1, g_2)^{-1} = \kappa(g_1, g_2)$. Then ω is an isomorphism of groupoids.

We suppose that there exists a right Haar system $\{\lambda_{i,x}; x \in G^{(0)}\}$ on G_i for $i = 1, 2$. We denote by $\mathbb{R}_{>0}$ the multiplicative group of positive real numbers. Suppose that there exists a continuos homomorphism $\Delta_2 : G_2 \longrightarrow \mathbb{R}_{>0}$ such that

$$\text{(C1)} \qquad \begin{aligned} &\int_{G_1} \xi \circ \kappa(g_1, g_2)\, d\lambda_{1,s(g_2)}(g_1) \\ &= \int_{G_1} \xi \circ \kappa_1(g_1, g_2^{-1}) \Delta_2(g_2)\, d\lambda_{1,r(g_2)}(g_1) \end{aligned}$$

for every $g_2 \in G_2$ and every positive Borel function ξ on $\mathcal{T}$. Suppsoe that there exists a continuos homomorphism $\Delta_1 : G_1 \longrightarrow \mathbb{R}_{>0}$ such that

$$\text{(C2)} \qquad \begin{aligned} &\int_{G_2} \xi \circ \kappa(g_1, g_2)\, d\lambda_{2,s(g_1)}(g_2) \\ &= \int_{G_2} \xi \circ \kappa_2(g_1^{-1}, g_2)\Delta_1(g_1)^{-1}\, d\lambda_{2,r(g_1)}(g_2) \end{aligned}$$

for every $g_1 \in G_1$ and every positive Borel function ξ on $\mathcal{T}$. Note that equations (C1) and (C2) imply the following equations (D1) and (D2) respectively:

$$\text{(D1)} \qquad \int_{G_1} \xi(g_2 \triangleright g_1^{-1})\, d\lambda_{1,s(g_2)}(g_1) = \int_{G_1} \xi(g_1^{-1})\Delta_2(g_2)\, d\lambda_{1,r(g_2)}(g_1)$$

for every $g_2 \in G_2$ and every positive Borel function ξ on G_1 and

$$\text{(D2)} \qquad \int_{G_2} \xi(g_2 \triangleleft g_1)\, d\lambda_{2,r(g_1)}(g_2) = \int_{G_2} \xi(g_2)\Delta_1(g_1)\, d\lambda_{2,s(g_1)}(g_2)$$

for every $g_1 \in G_1$ and every positive Borel function ξ on G_2.

If G is an r-discrete groupoid, then the equations (C1) and (C2) hold for $\Delta_1 = \Delta_2 = 1$ (Proposition 2.6). If G is a groupoid arising from an action of a semidirect product group on a topological space, then the equations (C1) and (C2) hold for $\Delta_1 = 1$ (see §8).

Lemma 2.3. *The following equations hold:*
(1) $\Delta_1(g_2 \triangleright g_1) = \Delta_1(g_1)$,
(2) $\Delta_2(g_2 \triangleleft g_1) = \Delta_2(g_2)$,
for $(g_2, g_1) \in G^{(2)} \cap (G_2 \times G_1)$.

Proof. (1) For $\xi \in C_c(G_2)$ and $(g_2, g_1) \in G^{(2)} \cap (G_2 \times G_1)$, we have

$$\begin{aligned} &\int_{G_2} \xi(h_2)\Delta_1(g_2 \triangleright g_1)\, d\lambda_{2,s(g_2 \triangleright g_1)}(h_2) \\ &= \int_{G_2} \xi(h_2 \triangleleft (g_2 \triangleright g_1))\, d\lambda_{2,r(g_2 \triangleright g_1)}(h_2) \qquad \text{by (D2)} \\ &= \int_{G_2} \xi(((h_2 g_2) \triangleleft g_1)(g_2 \triangleleft g_1)^{-1})\, d\lambda_{2,r(g_2)}(h_2) \qquad \text{by Lemma2.2(4)} \\ &= \int_{G_2} \xi((h_2 \triangleleft g_1)(g_2 \triangleleft g_1)^{-1})\, d\lambda_{2,r(g_1)}(h_2) \\ &= \int_{G_2} \xi(h_2(g_2 \triangleleft g_1)^{-1})\Delta_1(g_1)\, d\lambda_{2,s(g_1)}(h_2) \qquad \text{by (D2)} \\ &= \int_{G_2} \xi(h_2)\Delta_1(g_1)\ d\lambda_{2,s(g_2 \triangleright g_1)}(h_2), \end{aligned}$$

where the last equation follows from the fact that $r(g_2 \triangleleft g_1) = s(g_2 \triangleright g_1)$. The statement (2) is proved similarly. $\square$

Using the equation (D2), we can prove the following:

Theorem 2.4. *There exists a right Haar system* $\{\nu_x; x \in G^{(0)}\}$ *of* $\mathcal{T}$ *such that*

$$\begin{aligned}&\int_{\mathcal{T}} f(u)\, d\nu_x(u)\\ &= \int_{G_2}\int_{G_1} f\circ\kappa(g_1, g_2^{-1})\Delta_1(g_1)\, d\lambda_{1,r(g_2)}(g_1) d\lambda_{2,x}(g_2)\end{aligned}$$

for $f \in C_c(\mathcal{T})$ *and* $x \in G^{(0)}$.

Proof. We define measures $\{\nu_x\,;\, x \in G^{(0)}\}$ by the equation in the theorem. We will show that the right invariance of $\{\nu_x\}$. For $f \in C_c(\mathcal{T})$ and $u = (g_1, g_2) \in \mathcal{T}$, we have

$$\begin{aligned}&\int_{\mathcal{T}} f(vu) d\nu_{r_{\mathcal{T}}(u)}(v)\\ &= \iint f(g_1((h_2g_2)^{-1} \triangleright h_1^{-1}), ((h_2g_2)^{-1} \triangleleft h_1^{-1})^{-1})\Delta_1(h_1)\, d\lambda_{1,r(h_2)}(h_1) d\lambda_{2,r(g_2)}(h_2)\\ &= \iint f(g_1((h_2^{-1} \triangleright h_1^{-1}), (h_2^{-1} \triangleleft h_1^{-1})^{-1})\Delta_1(h_1)\, d\lambda_{1,r(h_2)}(h_1) d\lambda_{2,s(g_1)}(h_2). \qquad (*)\end{aligned}$$

Since we have

$$\begin{aligned} g_1(h_2^{-1} \triangleright h_1^{-1}) &= p_1(\{h_1(h_2 \triangleright g_1^{-1})(h_2 \triangleleft g_1^{-1})\}^{-1}),\\ h_2^{-1} \triangleleft h_1^{-1} &= p_2(\{h_1(h_2 \triangleright g_1^{-1})(h_2 \triangleleft g_1^{-1})\}^{-1}),\end{aligned}$$

the equation $(*)$ is equal to

$$\begin{aligned}&\iint f(p_1(\{h_1(h_2 \triangleleft g_1^{-1})\}^{-1}), p_2(\{h_1(h_2 \triangleleft g_1^{-1}\}^{-1})^{-1})\Delta_1(h_1g_1)\\ &\qquad \times d\lambda_{1,r(h_2 \triangleleft g_1^{-1})}(h_1) d\lambda_{2,r(g_1)}(h_2)\\ &= \iint f(p_1(\{h_1h_2\}^{-1}), p_2(\{h_1h_2\}^{-1})^{-1})\\ &\qquad \times \Delta_1(h_1)\, d\lambda_{1,r(h_2)}(h_1) d\lambda_{2,r(g_1)}(h_2) \qquad \text{by (D2)}\\ &= \int_{\mathcal{T}} f(v)\, d\nu_{s_{\mathcal{T}}(u)}(v).\end{aligned}$$

□

Corollary 2.5. *There exists a right Haar system* $\{\lambda_x; x \in G^{(0)}\}$ *of* G *such that*

$$\int_G f(g)\, d\lambda_x(g) = \int_{G_2}\int_{G_1} f(g_1g_2)\Delta_1(g_1)\, d\lambda_{1,r(g_2)}(g_1) d\lambda_{2,x}(g_2)$$

for $f \in C_c(G)$ *and* $x \in G^{(0)}$.

Proof. Let $\{\lambda_x\}$ be the image of $\{\nu_x\}$ by ω. Then $\{\lambda_x\}$ has the desired property. □

As for an r-discrete groupoid, we have the following results:

Proposition 2.6. *Suppose that* (G_1, G_2) *is a matched pair of an* r*-discrete groupoid* G.

(1) G_i *is* r*-discrete* $(i = 1, 2)$.

(2) *If each of* G_1 *and* G_2 *has a right Haar system, then the equations* (C1) *and* (C2) *are hold with* $\Delta_1 = \Delta_2 = 1$.

(3) *If each of* G_1 *and* G_2 *has a right Haar system, then* G_i *is open* $(i = 1, 2)$.

Proof. (1) The groupoid G is said to be r-discrete if $G^{(0)}$ is open in G ([9], p.18, Definition 2.6, see also [8], p.44). Since we have $G_i^{(0)} = G^{(0)}$, G_i is r-discrete.

(2) If G_i has a right Haar system, then it is essentially the counting measure system ([9], p.18, Lemma 2.7). It follows from Lemma 2.2 that, for every $g_2 \in G_2$, the map $g_1 \in G_1^{s(g_2)} \mapsto g_2 \triangleright g_1 \in G_1^{r(g_2)}$ is a bijection. This implies that the equation (C1) holds. It follows from Lemma 2.2 that, for every $g_1 \in G_1$, the map $g_2 \in G_{2,r(g_1)} \mapsto g_2 \triangleleft g_1 \in G_{2,s(g_1)}$ is a bijection. This implies that the equation (C2) holds.

(3) It follows from the above (2) and Corollary 2.5 that G has a right Haar system. For every subset U of G, we donte by $r|U$ (resp. $s|U$) the restriction of r (resp. s) to U. We denote by $U \in G^{op}$ when U is an open set of G and $r|U$ and $s|U$ are homeomorphisms from U into G. Then G^{op} is a basis for the toplogy of G ([9], p.19, Proposition 2.8 and [8], p.44). Similary G_i^{op} is a basis for the topology of G_i. For every $g_i \in G_i$, there exist $U \in G_i^{op}$ and $V \in G^{op}$ such that $g_i \in U \cap V$. Since $U \cap V \in G_i^{op}$, $r(U \cap V)$ is open in G. Set $W = r^{-1}(r(U \cap V)) \cap V$, which is an open neighborhood of g_i in G. Since r is one-to-one on V, W is a subset of G_i. Therefore G_i is open in G. $\square$

3 Representations of groupoid C^*-algebras We denote by $C_c(G)$ the set of complex valued continuous functions on G with compact supports. Then, $C_c(G)$ is a $*$-algebra with the following product and involution:

$$(ab)(g) = \int_G a(gh^{-1})b(h)\, d\lambda_{s(g)}(h),$$
$$a^*(g) = \overline{a(g^{-1})}$$

for a, $b \in C_c(G)$ and $g \in G$. For $x \in G^{(0)}$, let $E_{G,x}$ be the Hilbert space $L^2(G, \lambda_x)$, where we assume that the inner product is linear in the second variable. Define a $*$-representation $\pi_{G,x} : C_c(G) \to \mathcal{L}(E_{G,x})$ by

$$(\pi_{G,x}(a)\xi)(g) = \int_G a(gh^{-1})\xi(h)\, d\lambda_x(h)$$

for $a \in C_c(G)$, $\xi \in E_{G,x}$ and $g \in G_x$. Define the reduced norm $\|a\|$ by

$$\|a\| = \sup\{\|\pi_{G,x}(a)\|\, ;\, x \in G^{(0)}\}.$$

The reduced groupoid C^*-algebra $C_r^*(G)$ is the completion of $C_c(G)$ by the reduced norm. We can extend $\pi_{G,x}$ to the $*$-representation of $C_r^*(G)$ on $E_{G,x}$, which we denote again by $\pi_{G,x}$.

We denote by $C_0(G^{(0)})$ the commutative C^*-algebra of complex valued continuous functions on $G^{(0)}$ vanishing at infinity. Set $A_0 = C_0(G^{(0)})$. Let E_G be a Hilbert A_0-module obtained by the completion of a pre-Hilbert A_0-module $C_c(G)$ with the following structure:

$$(\xi a_0)(g) = \xi(g)a_0(s(g)),$$
$$\langle \xi, \eta \rangle(x) = \int_G \overline{\xi(g)}\eta(g)\, d\lambda_x(g)$$

for ξ, $\eta \in C_c(G)$, $a_0 \in A_0$, $g \in G$ and $x \in G^{(0)}$. We denote by $\mathcal{L}_{A_0}(E_G)$ be the C^*-algebra of bounded adjointable operators from E_G to itself. Define an injective $*$-representation $\pi_G : C_r^*(G) \to \mathcal{L}_{A_0}(E_G)$ by

$$(\pi_G(a)\xi)(g) = \int_G a(gh^{-1})\xi(h)\, d\lambda_{s(g)}(h)$$

for $a \in C_c(G) \subset C_r^*(G)$, $\xi \in C_c(G) \subset E_G$ and $g \in G$. We can similarly define a representation (π_{G_i}, E_{G_i}) of $C_r^*(G_i)$ with respect to $\{\lambda_{i,x}\}$ $(i = 1, 2)$ and a representation $(\pi_{\mathcal{T}}, E_{\mathcal{T}})$ of $C_r^*(\mathcal{T})$ with respect to $\{\nu_x\}$.

Lemma 3.1. *For a, $\xi \in C_c(\mathcal{T})$ and $(g_1, g_2) \in \mathcal{T}$, the following equation holds:*

$$\begin{aligned}
&(\pi_{\mathcal{T}}(a)\xi)(g_1, g_2) \\
&= \iint a(h_1, h_2^{-1})\xi(p_1(g_1 g_2^{-1}(h_1 h_2)^{-1}), p_2(g_1 g_2^{-1}(h_1 h_2)^{-1})^{-1}) \\
&\quad \times \Delta_1(h_1)\, d\lambda_{1,r(h_2)}(h_1) d\lambda_{2,r(g_2)}(h_2).
\end{aligned}$$

Proof. It follows from Theorem 2.4 that we have

$$\begin{aligned}
&(\pi_{\mathcal{T}}(a)\xi)(g_1, g_2) \\
&= \int a((h_1, h_2)^{-1})\xi((h_1, h_2)(g_1, g_2))\, d\nu_{r_{\mathcal{T}}(g_1, g_2)}(h_1, h_2) \\
&= \iint a \circ \kappa^2(h_1, h_2^{-1})\xi(\kappa(h_1, h_2^{-1})(g_1, g_2)) \\
&\quad \times \Delta_1(h_1)\, d\lambda_{1,r(h_2)}(h_1) d\lambda_{2,r(g_2)}(h_2).
\end{aligned}$$

It follows from Lemma 2.2 (1) and (4) that we have

$$\begin{aligned}
&\kappa(h_1, h_2^{-1})(g_1, g_2) \\
&= (g_1(g_2^{-1} \triangleright (h_2^{-1} \triangleright h_1^{-1})), (h_2^{-1} \triangleleft h_1^{-1})^{-1}\{g_2^{-1} \triangleleft (h_2^{-1} \triangleright h_1^{-1})\}^{-1}) \\
&= (g_1((h_2 g_2)^{-1} \triangleright h_1^{-1}), ((h_2 g_2)^{-1} \triangleleft h_1^{-1})^{-1}) \\
&= (p_1(g_1 g_2^{-1}(h_1 h_2)^{-1}), p_2(g_1 g_2^{-1}(h_1 h_2)^{-1}).
\end{aligned}$$

□

Let $\tilde{E}_G$ be a Hilbert A_0-module obtained by the completion of a pre-Hilbert A_0-module $C_c(G)$ with the following structure:

$$\begin{aligned}
(\xi a_0)(g) &= \xi(g) a_0(r(g)), \\
\langle \xi, \eta \rangle(x) &= \int_G \overline{\xi(g^{-1})}\eta(g^{-1})\, d\lambda_x(g)
\end{aligned}$$

for ξ, $\eta \in C_c(G)$, $a_0 \in A_0$, $g \in G$ and $x \in G^{(0)}$. Define an isomorphism $J_G : E_G \to \tilde{E}_G$ by $(J_G \xi)(g) = \xi(g^{-1})$ for $\xi \in C_c(G) \subset E_G$ and $g \in G$. Define an injective $*$-representaion $\tilde{\pi}_G : C_r^*(G) \to \mathcal{L}_{A_0}(\tilde{E}_G)$ by $\tilde{\pi}_G(a) = J_G \pi_G(a) J_G^*$. We can similarly define a representation $(\tilde{\pi}_{G_i}, \tilde{E}_{G_i})$ of $C_r^*(G_i)$ $(i = 1, 2)$ and a representation $(\tilde{\pi}_{\mathcal{T}}, \tilde{E}_{\mathcal{T}})$ of $C_r^*(\mathcal{T})$.

Define a $*$-homomorphism $\phi : A_0 \to \mathcal{L}_{A_0}(E_{G_2})$ by $\phi(a_0)\xi = \xi a_0$ for $a_0 \in A_0$ and $\xi \in E_{G_2}$. We denote by E the interior tensor product $E_{G_1} \otimes_\phi E_{G_2}$ (cf. [3]). Note that A_0-valued inner product of E is given by

$$\langle \xi, \eta \rangle(x) = \int_{G_2} \int_{G_1} \overline{\xi(g_1, g_2)}\eta(g_1, g_2)\, d\lambda_{1,x}(g_1) d\lambda_{2,x}(g_2)$$

for $\xi, \eta \in C_c(\mathcal{T}) \subset E$ and $x \in G^{(0)}$. Define an injective $*$-homomorphism $\pi_{G_1} \otimes \iota : C_r^*(G_1) \to \mathcal{L}_{A_0}(E)$ by $(\pi_{G_1} \otimes \iota)(a) = \pi_{G_1}(a) \otimes_\phi I_{E_{G_2}}$ for $a \in C_r^*(G_1)$. Since ϕ and π_{G_2} commute, we can define an injective $*$-homomorphism $\iota \otimes \pi_{G_2} : C_r^*(G_2) \to \mathcal{L}_{A_0}(E)$ by $(\iota \otimes \pi_{G_2})(a) = I_{E_{G_1}} \otimes_\phi \pi_{G_2}(a)$ for $a \in C_r^*(G_2)$.

Define an injective $*$-homomorphism $\rho_G : C_0(G) \to \mathcal{L}_{A_0}(E_G)$ by $(\rho_G(a)\xi)(g) = a(g)\xi(g)$. We can define an injective $*$-homomorphism $\rho_{G_1} \otimes \iota : C_0(G_1) \to \mathcal{L}_{A_0}(E)$ by $(\rho_{G_1} \otimes \iota)(a) = \rho_{G_1}(a) \otimes_\phi I_{E_{G_2}}$ and an injective $*$-homomorphism $\iota \otimes \rho_{G_2} : C_0(G_2) \to \mathcal{L}_{A_0}(E)$ by $(\iota \otimes \rho_{G_2})(a) = I_{E_{G_1}} \otimes_\phi \rho_{G_2}(a)$.

4 Hilbert A_0-modules associated with $\mathcal{T}$ In this section, we introduce several Hilbert A_0-modules which are completion of $C_c(\mathcal{T})$. We have already introduced $E_\mathcal{T}$, $\tilde{E}_\mathcal{T}$ and E in Section 3. In the following, let $\xi, \eta \in C_c(\mathcal{T})$, $a \in A_0$, $(g_1, g_2) \in \mathcal{T}$ and $x \in G^{(0)}$.

The Hilbert A_0-module E is the completion of $C_c(\mathcal{T})$ with the following structure:

$$(\xi a)(g_1, g_2) = \xi(g_1, g_2)a(s(g_1))$$
$$\langle \xi, \eta \rangle(x) = \iint \overline{\xi(g_1, g_2)}\eta(g_1, g_2)\, d\lambda_{1,x}(g_1)d\lambda_{2,x}(g_2).$$

The Hilbert A_0-module E_1 is the completion of $C_c(\mathcal{T})$ with the following structure:

$$(\xi a)(g_1, g_2) = \xi(g_1, g_2)a(r(g_1))$$
$$\langle \xi, \eta \rangle(x) = \iint \overline{\xi(g_1^{-1}, g_2)}\eta(g_1^{-1}, g_2)\, d\lambda_{2,r(g_1)}(g_2)d\lambda_{1,x}(g_1).$$

The Hilbert A_0-module E_2 is the completion of $C_c(\mathcal{T})$ with the following structure:

$$(\xi a)(g_1, g_2) = \xi(g_1, g_2)a(r(g_2))$$
$$\langle \xi, \eta \rangle(x) = \iint \overline{\xi(g_1, g_2^{-1})}\eta(g_1, g_2^{-1})\, d\lambda_{1,r(g_2)}(g_1)d\lambda_{2,x}(g_2).$$

The Hilbert A_0-module $\tilde{E}$ is the completion of $C_c(\mathcal{T})$ with the following structure:

$$(\xi a)(g_1, g_2) = \xi(g_1, g_2)a(s(g_2 \triangleright g_1^{-1}))$$
$$\langle \xi, \eta \rangle(x) = \iint \overline{\xi \circ \kappa_2(g_1^{-1}, g_2)}\eta \circ \kappa_2(g_1^{-1}, g_2)\, d\lambda_{2,r(g_1)}(g_2)d\lambda_{1,x}(g_1).$$

The Hilbert A_0-module $E_\mathcal{T}$ is the completion of $C_c(\mathcal{T})$ with the following structure:

$$(\xi a)(g_1, g_2) = \xi(g_1, g_2)a(r(g_1))$$
$$\langle \xi, \eta \rangle(x) = \iint \overline{\xi \circ \kappa(g_1, g_2^{-1})}\eta \circ \kappa(g_1, g_2^{-1})\Delta_1(g_1)\, d\lambda_{1,r(g_2)}(g_1)d\lambda_{2,x}(g_2).$$

The Hilbert A_0-module $\tilde{E}_\mathcal{T}$ is the completion of $C_c(\mathcal{T})$ with the following structure:

$$(\xi a)(g_1, g_2) = \xi(g_1, g_2)a(r(g_2))$$
$$\langle \xi, \eta \rangle(x) = \iint \overline{\xi(g_1, g_2^{-1})}\eta(g_1, g_2^{-1})\Delta_1(g_1)\, d\lambda_{1,r(g_2)}(g_1)d\lambda_{2,x}(g_2).$$

Using the equations (C1), (C2) and (D2), we have the following equations:

$$\langle \xi, \eta \rangle_{\tilde{E}}(x) = \iint \overline{\xi \circ \kappa(g_1, g_2)}\eta \circ \kappa(g_1, g_2)\Delta_1(g_1)\, d\lambda_{1,x}(g_1)d\lambda_{2,x}(g_2),$$
$$\langle \xi, \eta \rangle_{E_\mathcal{T}}(x) = \iint \overline{\xi(g_1^{-1}, g_2)}\eta(g_1^{-1}, g_2)\Delta_2(g_2)^{-1}\, d\lambda_{2,r(g_1)}(g_2)d\lambda_{1,x}(g_1).$$

There exist the following isomorphisms between Hilbert A_0-modules:

$T : E_2 \longrightarrow E_{\mathcal{T}}$ defined by $(T\xi)(g_1, g_2) = \Delta_1(g_1)^{1/2}\xi \circ \kappa(g_1, g_2)$,
$T_1 : E \longrightarrow E_{\mathcal{T}}$ defined by $(T_1\xi)(g_1, g_2) = \Delta_1(g_1)^{1/2}\Delta_2(g_2)^{1/2}\xi \circ \kappa_1(g_1, g_2)$,
$T_2 : E \longrightarrow \tilde{E}_{\mathcal{T}}$ defined by $(T_2\xi)(g_1, g_2) = \Delta_1(g_1)^{-1/2}\Delta_2(g_2)^{-1/2}\xi \circ \kappa_2(g_1, g_2)$,
$\tilde{T}_2 : \tilde{E} \longrightarrow E_{\mathcal{T}}$ defined by $(\tilde{T}_2\xi)(g_1, g_2) = \Delta_2(g_2)^{1/2}\xi \circ \kappa_2(g_1, g_2)$.

We also have the following isomorphisms between Hilbert A_0-modules:

$S_1 : E_1 \longrightarrow E_{\mathcal{T}}$ defined by $(S_1\xi)(g_1, g_2) = \Delta_2(g_2)^{1/2}\xi(g_1, g_2)$,
$S_2 : E_2 \longrightarrow \tilde{E}_{\mathcal{T}}$ defined by $(S_2\xi)(g_1, g_2) = \Delta_1(g_1)^{-1/2}\xi(g_1, g_2)$.

Therefore the above Hilbert A_0-modules are isomorphic with each other.

Theorem 4.1. *The following equations hold:*

(1)
$$\begin{aligned}&(\tilde{T}_2^*\pi_{\mathcal{T}}(a)\tilde{T}_2\xi)(g_1, g_2)\\ &= \int_{G_2}\int_{G_1} a(h_1, h_2^{-1})\xi(\theta(g_1, g_2; h_1, h_2))\\ &\quad \times \Delta_1(h_1)\Delta_2(h_2)^{1/2}\, d\lambda_{1,r(h_2)}(h_1) d\lambda_{2,s(g_1)}(h_2)\end{aligned}$$

for $a \in C_c(\mathcal{T}) \subset C_r^*(\mathcal{T})$, $\xi \in C_c(\mathcal{T}) \subset \tilde{E}$ *and* $(g_1, g_2) \in \mathcal{T}$, *where*

$$\theta(g_1, g_2; h_1, h_2) = (p_1(h_1h_2g_1^{-1})^{-1},\ p_2(g_2(h_1h_2)^{-1})).$$

(2)
$$\begin{aligned}&(T_1(\pi_{G_1} \otimes \iota)(a)T_1^*\xi)(g_1, g_2)\\ &= \int_{G_1} a(h_1^{-1})\xi(g_1h_1^{-1}, g_2 \triangleleft h_1^{-1})\Delta_1(h_1)^{1/2}\, d\lambda_{1,s(g_1)}(h_1)\end{aligned}$$

for $a \in C_c(G_1) \subset C_r^*(G_1)$, $\xi \in C_c(\mathcal{T}) \subset E_{\mathcal{T}}$ *and* $(g_1, g_2) \in \mathcal{T}$.

(3)
$$\begin{aligned}&(T_2(\iota \otimes \pi_{G_2})(a)T_2^*\xi)(g_1, g_2)\\ &= \int_{G_2} a(h_2^{-1})\xi((h_2 \triangleright g_1^{-1})^{-1}, g_2h_2^{-1})\Delta_2(h_2)^{-1/2}\, d\lambda_{2,s(g_1)}(h_2)\end{aligned}$$

for $a \in C_c(G_2) \subset C_r^*(G_2)$, $\xi \in C_c(\mathcal{T}) \subset \tilde{E}_{\mathcal{T}}$ *and* $(g_1, g_2) \in \mathcal{T}$.

Proof. (1) Note that we have $(\tilde{T}_2^*\xi)(g_1, g_2) = \Delta_2(g_2)^{1/2}\, \xi \circ \kappa_2(g_1, g_2)$. Put $\gamma(g_1, g_2; h_1, h_2) = p_1(g_2g_1^{-1})^{-1}g_2(h_1h_2)^{-1}$. It follows from Lemma 3.1 that we have

$$\begin{aligned}&(\tilde{T}_2{}^*\pi_{\mathcal{T}}(a)\tilde{T}_2\xi)(g_1, g_2)\\ &= \Delta_2(g_2)^{1/2} \iint a(h_1, h_2^{-1})\Delta_2(p_2(\gamma(g_1, g_2; h_1, h_2))^{-1})^{1/2}\\ &\quad \times \xi \circ \kappa_2(p_1(\gamma(g_1, g_2; h_1, h_2)), p_2(\gamma(g_1, g_2; h_1, h_2))^{-1})\\ &\quad \times \Delta_1(h_1)\, d\lambda_{1,r(h_2)}(h_1) d\lambda_{2,s(g_2)}(h_2).\end{aligned}$$

Since we have

$$p_1[p_2\{g_2^{-1}(h_1h_2)^{-1}\}p_1\{h_1h_2p_1(g_2g_1^{-1})\}]^{-1} = p_1(g_1g_2^{-1}(h_1h_2)^{-1}),$$

we have

$$\begin{aligned}&(p_1(g_1g_2^{-1}(h_1h_2)^{-1}), p_2(g_1g_2^{-1}(h_1h_2)^{-1})^{-1})\\&= \kappa_2(p_1(h_1h_2p_1(g_2g_1^{-1}))^{-1}, p_2(g_2^{-1}(h_1h_2)^{-1})).\end{aligned}$$

By substituting g_1 for $p_1(g_2g_1^{-1})^{-1}$ and g_2 for g_2^{-1} in the above equation, we have

$$(p_1(\gamma(g_1, g_2; h_1, h_2)), p_2(\gamma(g_1, g_2; h_1, h_2))^{-1}) = \kappa_2(\theta(g_1, g_2; h_1, h_2))).$$

Since we have $p_2(\gamma(g_1, g_2; h_1, h_2)) = (g_2h_2^{-1}) \triangleleft h_1^{-1}$, we have, by Lemma 2.3,

$$\Delta_2(p_2(\gamma(g_1, g_2; h_1, h_2))^{-1}) = \Delta_2(g_2)^{-1}\Delta_2(h_2).$$

(2) Note that we have $(T_1^*\xi)(g_1, g_2) = \Delta_1(g_1)^{1/2}\Delta_2(g_2)^{-1/2}\, \xi \circ \kappa_1(g_1, g_2)$. The equation is an immediate consequence of the formula

$$((\pi_{G_1} \otimes \iota)(a)\xi)(g_1, g_2) = \int_{G_1} a(h_1^{-1})\xi(h_1g_1, g_2)\, d\lambda_{1,r(g_1)}(h_1).$$

(3) Note that we have $(T_2^*\xi)(g_1, g_2) = \Delta_1(g_1)^{1/2}\Delta_2(g_2)^{-1/2}\, \xi \circ \kappa_2(g_1, g_2)$. The equation is an immediate consequence of the formula

$$((\iota \otimes \pi_{G_2})(a)\xi)(g_1, g_2) = \int_{G_2} a(h_2^{-1})\xi(g_1, h_2g_2)\, d\lambda_{2,r(g_2)}(h_2).$$

□

5 C^*-algebras associated with $\mathcal{T}$ For a, $b \in C_c(\mathcal{T})$, define a product $a\sharp b$ and an involution a° as follows:

$$\begin{aligned}&(a\sharp b)(g_1, g_2) = \int_{G_1} a(h_1^{-1}, g_2 \triangleleft (h_1g_1)^{-1})b(h_1g_1, g_2)\, d\lambda_{1,r(g_1)}(h_1),\\&a^\circ = \overline{a} \circ \kappa_1.\end{aligned}$$

For a, $b \in C_c(\mathcal{T})$, define a product $a\flat b$ and an involution $a^\diamondsuit$ as follows:

$$\begin{aligned}&(a\flat b)(g_1, g_2) = \int_{G_2} a(((h_2g_2) \triangleright g_1^{-1})^{-1}, h_2^{-1})b(g_1, h_2g_2)\, d\lambda_{2,r(g_2)}(h_2),\\&a^\diamondsuit = \overline{a} \circ \kappa_2.\end{aligned}$$

Then $(C_c(\mathcal{T}), \sharp, \circ)$ and $(C_c(\mathcal{T}), \flat, \diamondsuit)$ are $*$-algebras.

For $x \in G^{(0)}$, define measures ${}^xm$ and m^x on $\mathcal{T}$ as follows:

$$\begin{aligned}&\int_{\mathcal{T}} f(u)\, d^xm(u) = \iint f(g_1^{-1}, g_2)\, d\lambda_{2,r(g_1)}(g_2)d\lambda_{1,x}(g_1),\\&\int_{\mathcal{T}} f(u)\, dm^x(u) = \iint f(g_1, g_2^{-1})\, d\lambda_{1,r(g_2)}(g_1)d\lambda_{2,x}(g_2)\end{aligned}$$

for $f \in C_c(\mathcal{T})$. The support of ${}^xm$ is ${}^x\mathcal{T} = \{(g_1, g_2) \in \mathcal{T}; r_G(g_1) = x\}$ and the support of m^x is $\mathcal{T}^x = \{(g_1, g_2) \in \mathcal{T}; r_G(g_2) = x\}$. Put ${}^xH = L^2(\mathcal{T}, {}^xm)$ and $H^x = L^2(\mathcal{T}, m^x)$, which are Hilbert spaces whose inner products are linear in the second variables. For a, $\xi \in C_c(\mathcal{T})$, define an element ${}^x\tilde{\rho}(a)\xi$ of $C_c(\mathcal{T})$ by

$$({}^x\tilde{\rho}(a)\xi)(g_1, g_2) = \int a(h_1^{-1}, g_2 \triangleleft h_1^{-1})\xi(g_1h_1^{-1}, g_2 \triangleleft h_1^{-1})\Delta(h_1)^{1/2}\, d\lambda_{1,s(g_1)}(h_1)$$

and define an element $\tilde{\tilde{\rho}}^x(a)\xi$ of $C_c(\mathcal{T})$ by

$$(\tilde{\tilde{\rho}}^x(a)\xi)(g_1, g_2)$$
$$= \int a((h_2 \triangleright g_1^{-1})^{-1}, h_2^{-1})\xi((h_2 \triangleright g_1^{-1})^{-1}, g_2 h_2^{-1})\Delta_2(h_2)^{-1/2}\, d\lambda_{2,s(g_2)}(h_2).$$

We denote by $\mathcal{L}({}^xH)$ the $*$-alegebra of bounded linear operators on ${}^xH$ for each x. Then we have the following theorem.

Proposition 5.1. (1) *For every $a \in C_c(\mathcal{T})$, ${}^x\tilde{\rho}(a)$ is an element of $\mathcal{L}({}^xH)$. The map ${}^x\tilde{\rho}$ is a $*$-represetation of $(C_c(\mathcal{T}), \sharp, \circ)$ on ${}^xH$.*

(2) *For every $a \in C_c(\mathcal{T})$, $\tilde{\tilde{\rho}}^x(a)$ is an element of $\mathcal{L}(H^x)$. The map $\tilde{\tilde{\rho}}^x$ is a $*$-representation of $(C_c(\mathcal{T}), \flat, \diamondsuit)$ on H^x.*

Proof. Let K be a support of a. For $i = 1, 2$, let K_i be the set of $g_i \in G_i$ with $(g_1, g_2) \in K$. Put $M_i(K_i) = \sup\{\lambda_{i,x}(K_i); x \in G^{(0)}\}$. We denote by χ_K the characteristic function of K.

(1) For $(g_1, g_2) \in \mathcal{T}$ with $r_G(g_1) = x$, we have

$$\begin{aligned}
&\|{}^x\tilde{\rho}(a)\xi\|^2_{{}^xH} \\
&\le \|a\|^2_\infty \iint \left\{ \int \chi_K(h_1^{-1}, g_2 \triangleleft h_1^{-1})|\xi(g_1^{-1}h_1^{-1}, g_2 \triangleleft h_1^{-1})|\Delta_1(h_1)^{1/2}\, d\lambda_{1,r(g_1)}(h_1) \right\}^2 \\
&\quad \times d\lambda_{2,r(g_1)}(g_2) d\lambda_{1,x}(g_1) \\
&\le \|a\|^2_\infty \iint \left\{ \int \chi_K(h_1^{-1}, g_2 \triangleleft h_1^{-1})\, d\lambda_{1,r(g_1)}(h_1) \right\} \\
&\quad \times \left\{ \int \chi_K(h_1^{-1}, g_2 \triangleleft h_1^{-1})|\xi(g_1^{-1}h_1^{-1}, g_2 \triangleleft h_1^{-1})|^2 \Delta_1(h_1)\, d\lambda_{1,r(g_1)}(h_1) \right\} \\
&\quad \times d\lambda_{2,r(g_1)}(g_2) d\lambda_{1,x}(g_1) \\
&\le M_1(K_1^{-1})\|a\|^2_\infty \iiint \chi_{K_1}(h_1^{-1})|\xi(g_1^{-1}h_1^{-1}, g_2 \triangleleft h_1^{-1})|^2 \Delta_1(h_1) \\
&\quad \times d\lambda_{1,r(g_1)}(h_1) d\lambda_{2,r(g_1)}(g_2) d\lambda_{1,x}(g_1) \\
&= M_1(K_1^{-1})\|a\|^2_\infty \\
&\quad \times \iiint \chi_{K_1}(h_1^{-1})|\xi(g_1^{-1}h_1^{-1}, g_2)|^2\, d\lambda_{2,r(h_1)}(g_1) d\lambda_{1,r(g_1)}(h_1) d\lambda_{1,x}(g_1) \qquad \text{by (D2)} \\
&= M_1(K_1^{-1})\|a\|^2_\infty \\
&\quad \times \iiint \chi_{K_1}(g_1 h_1^{-1})|\xi(h_1^{-1}, g_2)|^2\, d\lambda_{2,r(h_1)}(g_2) d\lambda_{1,s(g_1)}(h_1) d\lambda_{1,x}(g_1) \\
&= M_1(K_1^{-1})\|a\|^2_\infty \iiint \chi_{K_1}(g_1)|\xi(h_1^{-1}, g_2)|^2\, d\lambda_{1,r(h_1)}(g_1) d\lambda_{2,s(h_1)}(g_2) d\lambda_{1,x}(h_1) \\
&\le M_1(K_1)M_1(K_1^{-1})\|a\|^2_\infty \|\xi\|^2_{{}^xH}.
\end{aligned}$$

Therefore we can extend ${}^x\tilde{\rho}(a)$ to a bounded operator on ${}^xH$, which we denote again by ${}^x\tilde{\rho}(a)$. Note that we have $\|{}^x\tilde{\rho}(a)\| \le M\|a\|_\infty$, where we have $M = (M_1(K_1)M_1(K_1^{-1}))^{1/2}$.

By a straightforward calculation, we can show that ${}^x\tilde{\rho}(a \sharp b)\xi = {}^x\tilde{\rho}(a){}^x\tilde{\rho}(b)\xi$ for a, b, $\xi \in$

$C_c(\mathcal{T})$. We will show that ${}^x\tilde{\rho}(a^\circ) = {}^x\tilde{\rho}(a)^*$. For a, ξ, $\eta \in C_c(\mathcal{T})$, we have

$$\begin{aligned}
&\langle \xi, {}^x\tilde{\rho}(a)\eta\rangle_{{}^xH} \\
&= \iiint \bar{\xi}(g_1^{-1}, (g_2 \triangleleft h_1^{-1}) \triangleleft h_1) a(h_1^{-1}, g_2 \triangleleft h_1^{-1}) \eta(g_1^{-1}h_1^{-1}, g_2 \triangleleft h_1^{-1}) \\
&\times \Delta_1(h_1)^{1/2} d\lambda_{2,s(h_1)}(g_2) d\lambda_{1,r(g_1)}(h_1) d\lambda_{1,x}(g_1) \\
&= \iiint \bar{\xi}(g_1^{-1}, g_2 \triangleleft h_1) a(h_1^{-1}, g_2) \eta(g_1^{-1}h_1^{-1}, g_2) \\
&\times \Delta_1(h_1)^{-1/2} d\lambda_{2,r(h_1)}(g_2) d\lambda_{1,r(g_1)}(h_1) d\lambda_{1,x}(g_1) \qquad \text{by (D2)} \\
&= \iiint \bar{\xi}(g_1^{-1}, g_2 \triangleleft (h_1 g_1^{-1})) a(g_1 h_1^{-1}, g_2) \eta(h_1^{-1}, g_2) \\
&\times \Delta_1(h_1 g_1^{-1})^{-1/2} d\lambda_{2,r(h_1)}(g_2) d\lambda_{1,s(g_1)}(h_1) d\lambda_{1,x}(g_1) \\
&= \iiint \bar{\xi}(h_1^{-1}g_1^{-1}, g_2 \triangleleft g_1^{-1}) a(g_1, g_2) \eta(h_1^{-1}, g_2) \\
&\times \Delta_1(g_1^{-1})^{-1/2} d\lambda_{2,r(h_1)}(g_2) d\lambda_{1,r(h_1)}(g_1) d\lambda_{1,x}(h_1) \\
&= \langle {}^x\tilde{\rho}(a^\circ)\xi, \eta\rangle_{{}^xH},
\end{aligned}$$

where the last equation follows from the fact that $a \circ \kappa_1(g_1^{-1}, g_2 \triangleleft g_1^{-1}) = a \circ \kappa_1^2(g_1, g_2) = a(g_1, g_2)$.

(2) We can prove the statement as in (1). Especially we have $\|\tilde{\tilde{\rho}}(a)\| \le M'\|a\|_\infty$, where $M' = (M_2(K_2)M_2(K_2^{-1}))^{1/2}$. □

Lemma 5.2. (1) *For $a \in C_c(\mathcal{T})$, if ${}^x\tilde{\rho}(a) = 0$ for every $x \in G^{(0)}$, then $a = 0$.*

(2) *For $a \in C_c(\mathcal{T})$, if $\tilde{\tilde{\rho}}^x(a) = 0$ for every $x \in G^{(0)}$, then $a = 0$.*

Proof. (1) We have, for $\xi \in C_c(\mathcal{T})$ and $(g_1, g_2) \in \mathcal{T}$,

$$\begin{aligned}
&({}^x\tilde{\rho}(a)\xi)(g_1, g_2) \\
&\quad = \int a \circ \kappa_1(h_1, g_2) \xi \circ \kappa_1(h_1 g_1^{-1}, g_2 \triangleleft g_1^{-1}) \Delta_1(h_1)^{1/2}\, d\lambda_{1,s(g_1)}(h_1).
\end{aligned}$$

For $\xi_i \in C_c(G_i)$ $(i = 1, 2)$, put $\xi = (\xi_1 \otimes \xi_2) \circ \kappa_1$. Then we have, for $(g_1, g_2) \in {}^x\mathcal{T}$,

$$\begin{aligned}
0 &= ({}^x\tilde{\rho}(a)\xi)(g_1, g_2) \\
&= \xi_2(g_2 \triangleleft g_1^{-1}) \int a \circ \kappa_1(h_1, g_2) \xi_1(h_1 g_1^{-1}) \Delta_1(h_1)^{1/2}\, d\lambda_{1,s(g_1)}(h_1).
\end{aligned}$$

This implies that $a \circ \kappa_1(h_1, g_2) = 0$ for $h_1 \in G_{1,s(g_1)}$. Especially we have $a \circ \kappa_1(g_1, g_2) = 0$ for $(g_1, g_2) \in {}^x\mathcal{T}$. Since x is an arbitrary element of G^0, we have $a = 0$.

(2) We have, for $\xi \in C_c(\mathcal{T})$ and $(g_1, g_2) \in \mathcal{T}$,

$$\begin{aligned}
&(\tilde{\tilde{\rho}}(a)\xi)(g_1, g_2) \\
&= \int a \circ \kappa_2(g_1, h_2) \xi \circ \kappa_2((g_2 \triangleright g_1^{-1})^{-1}, h_2 g_2^{-1}) \Delta_2(h_2)^{-1/2}\, d\lambda_{2,s(g_1)}(h_2).
\end{aligned}$$

We can prove the statement as in (1). □

We introduce a norm on $(C_c(\mathcal{T}), \#, \circ)$ by $\|a\| = \sup\{\|{}^x\tilde{\rho}(a)\|;\, x \in G^{(0)}\}$. We denote by B the completion of $(C_c(\mathcal{T}), \sharp, \circ)$ with respect to this norm. We can extend ${}^x\tilde{\rho}$ to the $*$-representation of B on ${}^xH$, which we denote again by ${}^x\tilde{\rho}$. There exists an injective $*$-homomorphism $\tilde{\rho}_{E_1} : B \longrightarrow \mathcal{L}_{A_0}(E_1)$ such that

$$(\tilde{\rho}_{E_1}(a)\xi)(g_1, g_2) = \int a(h_1^{-1}, g_2 \triangleleft h_1^{-1})\xi(g_1h_1^{-1}, g_2 \triangleleft h_1^{-1})\Delta_1(h_1)^{1/2}\, d\lambda_{1,s(g_1)}(h_1)$$

for $a \in C_c(\mathcal{T})$, $\xi \in C_c(\mathcal{T}) \subset E_1$ and $(g_1, g_2) \in \mathcal{T}$. We introduce a norm on $(C_c(\mathcal{T}), \flat, \diamondsuit)$ by $\|a\| = \sup\{\|\tilde{\tilde{\rho}}^x(a)\|;\, x \in G^{(0)}\}$. We denote by $\hat{B}$ the completion of $(C_c(\mathcal{T}), \flat, \diamondsuit)$ with respect to this norm. We can extend $\tilde{\tilde{\rho}}^x$ to the $*$-representation of $\hat{B}$ on H^x, which we denote again by $\tilde{\tilde{\rho}}^x$. There exists an injective $*$-homomorphism $\tilde{\tilde{\rho}}_{E_2} : \hat{B} \longrightarrow \mathcal{L}_{A_0}(E_2)$ such that

$$\begin{aligned}&(\tilde{\tilde{\rho}}_{E_2}(a)\xi)(g_1, g_2)\\ &= \int a((h_2 \triangleright g_1^{-1})^{-1}, h_2^{-1})\xi((h_2 \triangleright g_1^{-1})^{-1}, g_2h_2^{-1})\Delta_2(h_2)^{-1/2}\, d\lambda_{2,s(g_2)}(h_2)\end{aligned}$$

for $a \in C_c(\mathcal{T})$, $\xi \in C_c(\mathcal{T}) \subset E_2$ and $(g_1, g_2) \in \mathcal{T}$.

6 C^*-algebras arising from a matched pair of groupoids Let $\mathcal{T}_1$ be the fibered product $G_1\,{}_s\times_r G_2 = \{(g_1, g_2) \in G_1 \times G_2; s(g_1) = r(g_2)\}$ and let $\mathcal{T}_2$ be the fibered product $G_2\,{}_s\times_r G_1 = \{(g_2, g_1) \in G_2 \times G_1; s(g_2) = r(g_1)\}$. Define homeomorphisms $\varphi_1 : \mathcal{T} \to \mathcal{T}_1$ and $\varphi_2 : \mathcal{T} \to \mathcal{T}_2$ by $\varphi_1(g_1, g_2) = (g_1, g_2^{-1})$ and $\varphi_2(g_1, g_2) = (g_2, g_1^{-1})$ respectively.

Define a bijection $\Phi_1 : C_c(\mathcal{T}_1) \to C_c(\mathcal{T})$ by

$$\Phi_1(a)(g_1, g_2) = \Delta_1(g_1)^{1/2}\Delta_2(g_2)^{-1/2}(a \circ \varphi_1 \circ \kappa)(g_1, g_2).$$

Since $C_c(\mathcal{T})$ is a dense $*$-subalgebra of the C^*-algebra $C_r^*(\mathcal{T})$, we have a $*$-algebraic structure and a C^*-norm on $C_c(\mathcal{T}_1)$ induced by Φ_1. We denote by $C_r^*(G_1) \bowtie C_r^*(G_2)$ the C^*-algebra that is the completion of $C_c(\mathcal{T}_1)$ with respect to this norm. We can extend Φ_1 to an isomorphism of $C_r^*(G_1) \bowtie C_r^*(G_2)$ onto $C_r^*(\mathcal{T})$, which is denoted again by Φ_1. Define a bijection $\Phi_2 : C_c(\mathcal{T}_2) \to C_c(\mathcal{T})$ by

$$\Phi_2(a)(g_1, g_2) = \Delta_1(g_1)^{-1/2}\Delta_2(g_2)^{1/2}(a \circ \varphi_2)(g_1, g_2).$$

Then we have a $*$-algebraic structure and a C^*-norm on $C_c(\mathcal{T}_2)$ induced by Φ_2. We denote by $C_r^*(G_2) \bowtie C_r^*(G_1)$ the C^*-algebra that is the completion of $C_c(\mathcal{T}_2)$ with respect to this norm. We can extend Φ_2 to an isomorphism of $C_r^*(G_2) \bowtie C_r^*(G_1)$ onto $C_r^*(\mathcal{T})$, which is denoted again by Φ_2. By the construction, we have

$$C_r^*(G_1) \bowtie C_r^*(G_2) \cong C_r^*(G_2) \bowtie C_r^*(G_1) \cong C_r^*(\mathcal{T}) \cong C_r^*(G).$$

Then we have the following injective $*$-homomorphisms:

$$\begin{aligned}\mathrm{Ad}\, \tilde{T}_2^* \circ \pi_{\mathcal{T}} \circ \Phi_1 &: C_r^*(G_1) \bowtie C_r^*(G_2) \longrightarrow \mathcal{L}_{A_0}(\tilde{E}),\\ \mathrm{Ad}\, \tilde{T}_2^* \circ \pi_{\mathcal{T}} \circ \Phi_2 &: C_r^*(G_2) \bowtie C_r^*(G_1) \longrightarrow \mathcal{L}_{A_0}(\tilde{E}),\end{aligned}$$

where $\mathrm{Ad}\, \tilde{T}_2^* \circ \pi_{\mathcal{T}}(a) = \tilde{T}_2^*\pi_{\mathcal{T}}(a)\tilde{T}_2$ for $a \in C_r^*(\mathcal{T})$.

Define a bijection $\varphi_{1*} : C_c(\mathcal{T}_1) \to C_c(\mathcal{T})$ by $\varphi_{1*}(a) = a \circ \varphi_1$. Since $(C_c(\mathcal{T}), \sharp, \circ)$ is a dense $*$-subalgebra of the C^*-algebra B, we have a $*$-algebraic structure and a C^*-norm on $C_c(\mathcal{T}_1)$ induced by φ_{1*}. We denote by $C_r^*(G_1) \ltimes C_0(G_2)$ the C^*-algebra that is the completion of $C_c(\mathcal{T}_1)$ with respect to this norm. We can extend φ_{1*} to an isomorphism of $C_r^*(G_1) \ltimes C_0(G_2)$ onto B, which is denoted again by φ_{1*}. We define a $*$-representation ${}^x\rho$ of

$C_r^*(G_1)\ltimes C_0(G_2)$ on ${}^xH$ by ${}^x\rho(a) = {}^x\tilde{\rho}(\varphi_{1*}(a))$. We also define an injective $*$-homomorphism $\rho_{E_1} : C_r^*(G_1) \ltimes C_0(G_2) \longrightarrow \mathcal{L}_{A_0}(E_1)$ by $\rho_{E_1} = \tilde{\rho}_{E_1} \circ \varphi_{1*}$.

Define a bijection $\varphi_{2*} : C_c(\mathcal{T}_2) \to C_c(\mathcal{T})$ by $\varphi_{2*}(a) = a \circ \varphi_2$. Since $(C_c(\mathcal{T}), \flat, \diamondsuit)$ is a dense $*$-subalgebra of the C^*-algebra $\hat{B}$, we have a $*$-algebraic structure and a C^*-norm on $C_c(\mathcal{T}_2)$ induced by φ_{2*}. We denote by $C_r^*(G_2) \ltimes C_0(G_1)$ the C^*-algebra that is the completion of $C_c(\mathcal{T}_2)$ with respect to this norm. We can extend φ_{2*} to an isomorphism of $C_r^*(G_2) \ltimes C_0(G_1)$ onto $\hat{B}$, which is denoted again by φ_{2*}. We define a $*$-representation $\hat{\rho}^x$ of $C_r^*(G_2)\ltimes C_0(G_1)$ on H^x by $\hat{\rho}^x(a) = \tilde{\hat{\rho}}^x(\varphi_{2*}(a))$. We also define an injective $*$-homomorphism $\hat{\rho}_{E_2} : C_r^*(G_2) \ltimes C_0(G_1) \longrightarrow \mathcal{L}_{A_0}(E_2)$ by $\hat{\rho}_{E_2} = \tilde{\hat{\rho}}_{E_2} \circ \varphi_{2*}$.

7 Representations on Hilbert sapces Let μ be a positive regular Radon measure on $G^{(0)}$. For $i = 1, 2$, we say that μ is G_i-invariant if it satisfies the following equation

$$\int_{G^{(0)}} \int_{G_i} \xi(g_i^{-1})\, d\lambda_{i,x}(g_i) d\mu(x) = \int_{G^{(0)}} \int_{G_i} \xi(g_i)\, d\lambda_{i,x}(g_i) d\mu(x)$$

for $\xi \in C_c(G_i)$. In this section, we assume that there exists a G_1- and G_2-invariant measure μ on $G^{(0)}$ whose support is $G^{(0)}$. Then by equations (C1) and (D2), we have

$$(*)\qquad \begin{aligned} &\int_{G^{(0)}} \int_{G_2} \int_{G_1} \xi \circ \kappa(g_1, g_2)\, d\lambda_{1,x}(g_1) d\lambda_{2,x}(g_2) d\mu(x) \\ &= \int_{G^{(0)}} \int_{G_2} \int_{G_1} \xi(g_1, g_2) \Delta_1(g_1) \Delta_2(g_2)^{-1}\, d\lambda_{1,x}(g_1) d\lambda_{2,x}(g_2) d\mu(x) \end{aligned}$$

for $\xi \in C_c(\mathcal{T})$.

Note that the inner products of the following Hilbert spaces are linear in the second variables. We denote by $H_{\mathcal{T}}$ the completion of the pre-Hilbert space $C_c(\mathcal{T})$ with the following inner product

$$\langle \xi, \eta \rangle = \int_{G^{(0)}} \langle \xi, \eta \rangle_{E_{\mathcal{T}}}(x)\, d\mu(x).$$

The Hilbert space $\tilde{H}_{\mathcal{T}}$, $\tilde{H}$ and H are similarly defined with respect to the A_0-valued inner product $\langle \xi, \eta \rangle_{\tilde{E}_{\mathcal{T}}}$, $\langle \xi, \eta \rangle_{\tilde{E}}$ and $\langle \xi, \eta \rangle_E$ on $C_c(\mathcal{T})$ respectively. Since μ is G_1-invariant, we can define an isomorphism $I_{\mathcal{T}} : H_{\mathcal{T}} \to H$ by $(I_{\mathcal{T}}\xi)(g_1, g_2) = \Delta_2(g_2)^{-1/2} \xi(g_1, g_2)$. Since μ is G_2-invariant, we can define an isomorphism $\tilde{I}_{\mathcal{T}} : \tilde{H}_{\mathcal{T}} \to H$ by $(\tilde{I}_{\mathcal{T}}\xi)(g_1, g_2) = \Delta_1(g_1)^{1/2} \xi(g_1, g_2)$. By the equation $(*)$, we can define an isomorphism $\tilde{I} : \tilde{H} \to H$ by $(\tilde{I}\xi)(g_1, g_2) = \Delta_2(g_2)^{-1/2} \xi(g_1, g_2)$.

For $a \in C_c(\mathcal{T}) \subset C_r^*(\mathcal{T})$ and $\eta \in C_c(\mathcal{T}) \subset E_{\mathcal{T}}$, we have $\pi_{\mathcal{T}}(a)\eta \in C_c(\mathcal{T})$. Then, for $\xi \in C_c(\mathcal{T}) \subset \tilde{E}$, we have $\tilde{T}_2^* \pi_{\mathcal{T}}(a) \tilde{T}_2 \xi \in C_c(\mathcal{T})$. Moreover we have

$$\|\tilde{T}_2^* \pi_{\mathcal{T}}(a) \tilde{T}_2 \xi\|_{\tilde{H}} \leq \|a\|_{C_r^*(\mathcal{T})} \|\xi\|_{\tilde{H}}.$$

Therefore we can extend $\tilde{T}_2^* \pi_{\mathcal{T}}(a) \tilde{T}_2$ to a bounded linear operator on $\tilde{H}$, which we denote by $\mu(\tilde{T}_2^* \pi_{\mathcal{T}}(a) \tilde{T}_2)$. Define $\pi : C_c(\mathcal{T}) \to \mathcal{L}(H)$ by $\pi(a) = \tilde{I} \mu(\tilde{T}_2^* \pi_{\mathcal{T}}(a) \tilde{T}_2) \tilde{I}^*$. Since we have $\|\pi(a)\| \leq \|a\|_{C_r^*(\mathcal{T})}$, we can extend π to $C_r^*(\mathcal{T})$, which we donote again by π. Since $\pi_{\mathcal{T}}$ is injective, the $*$-homomorphism $\pi : C_r^*(\mathcal{T}) \to \mathcal{L}(H)$ is injective. Similarly we can define an injective $*$-homomorphism $\pi_1 : C_r^*(G_1) \to \mathcal{L}(H)$ (resp. $\pi_2 : C_r^*(G_2) \to \mathcal{L}(H)$) by $\pi_1(a) = I_{\mathcal{T}} \mu(T_1(\pi_{G_1} \otimes \iota)(a) T_1^*) I_{\mathcal{T}}^*$ (resp. $\pi_2(a) = \tilde{I}_{\mathcal{T}} \mu(T_2(\iota \otimes \pi_{G_2})(a) T_2^*) \tilde{I}_{\mathcal{T}}^*$). Define an injective $*$-homomorphism $\rho_1 : C_0(G_1) \to \mathcal{L}(H)$ (resp. $\rho_2 : C_0(G_2) \to \mathcal{L}(H)$) by $\rho_1(a) = I_{\mathcal{T}} \mu(T_1(\rho_{G_1} \otimes \iota)(a) T_1^*) I_{\mathcal{T}}^*$ (resp. $\rho_2(a) = \tilde{I}_{\mathcal{T}} \mu(T_2(\iota \otimes \rho_{G_2})(a) T_2^*) \tilde{I}_{\mathcal{T}}^*$). Then we have

$(\rho_1(a)\xi)(g_1, g_2) = a(g_1^{-1})\xi(g_1, g_2)$ and $(\rho_2(a)\xi)(g_1, g_2) = a(g_2^{-1})\xi(g_1, g_2)$. By Theorem 4.1, we have

$$\begin{aligned}
&(\pi(a)\xi)(g_1, g_2)\\
&= \int_{G_2}\int_{G_1} a(h_1, h_2^{-1})\xi(\theta(g_1, g_2; h_1, h_2))\Delta_1(h_1)\, d\lambda_{1,r(h_2)}(h_1)d\lambda_{2,s(g_1)}(h_2),\\
&(\pi_1(a_1)\xi)(g_1, g_2)\\
&= \int_{G_1} a_1(h_1^{-1})\xi(g_1h_1^{-1}, g_2 \triangleleft h_1^{-1})\Delta_1(h_1)^{1/2}\, d\lambda_{1,s(g_1)}(h_1),\\
&(\pi_2(a_2)\xi)(g_1, g_2)\\
&= \int_{G_2} a_2(h_2^{-1})\xi((h_2 \triangleright g_1^{-1})^{-1}, g_2h_2^{-1})\Delta_2(h_2)^{-1/2}\, d\lambda_{2,s(g_2)}(h_2)
\end{aligned}$$

for $a \in C_c(\mathcal{T}), a_1 \in C_c(G_1), a_2 \in C_c(G_2)$ and $\xi \in C_c(\mathcal{T})$.

Proposition 7.1. *The following equations hold:*

$$\begin{aligned}
\pi_2(a_2)\pi_1(a_1) &= \pi((\Delta_1^{1/2}a_1)\check{}\otimes(\Delta_2^{1/2}a_2))\\
\pi_1(a_1)\pi_2(a_2) &= \pi(((\Delta_1^{-1/2}a_1)\otimes(\Delta_2^{-1/2}a_2)\check{})\circ\kappa)
\end{aligned}$$

for $a_i \in C_c(G_i) \subset C_r^*(G_i)$ $(i = 1, 2)$, where $\check{a}_i(g_i) = a_i(g_i^{-1})$.

Proof. For $\xi \in C_c(\mathcal{T})$ and $(g_1, g_2) \in \mathcal{T}$, we have

$$\begin{aligned}
&(\pi_2(a_2)\pi_1(a_1)\xi)(g_1, g_2)\\
&= \iint \check{a}_1(h_1)a_2(h_2^{-1})\xi((h_2 \triangleright g_1^{-1})^{-1}h_1^{-1}, (g_2h_2^{-1}) \triangleleft h_1^{-1})\\
&\quad \times \Delta_2(h_2)^{-1/2}\Delta_1(h_1)^{1/2}\, d\lambda_{1,r(h_2)}(h_1)d\lambda_{2,s(g_2)}(h_2)\\
&= \iint \check{a}_1(h_1)a_2(h_2^{-1})\xi(\theta(g_1, g_2; h_1, h_2))\\
&\quad \times \Delta_1(h_1)^{1/2}\Delta_2(h_2)^{-1/2}\, d\lambda_{1,r(h_2)}(h_1)d\lambda_{2,s(g_2)}(h_2)\\
&= \iint (\Delta_1^{1/2}a_1)\check{}(h_1)(\Delta_2^{1/2}a_2)(h_2^{-1})\xi(\theta(g_1, g_2; h_1, h_2))\\
&\quad \times \Delta_1(h_1)\, d\lambda_{1,r(h_2)}(h_1)d\lambda_{2,s(g_1)}(h_2)\\
&= (\pi((\Delta_1^{1/2}a_1)\check{}\otimes(\Delta_2^{1/2}a_2))\xi)(g_1, g_2).
\end{aligned}$$

We also have

$$\begin{aligned}
&(\pi_1(a_1)\pi_2(a_2)\xi)(g_1, g_2)\\
&= \iint a_1(h_1^{-1})a_2(h_2^{-1})\xi((h_2 \triangleright (g_1h_1^{-1})^{-1})^{-1}, (g_2 \triangleleft h_1^{-1})h_2^{-1})\\
&\quad \times \Delta_1(h_1)^{1/2}\Delta_2(h_2)^{-1/2}\, d\lambda_{2,r(h_1)}(h_2)d\lambda_{1,s(g_1)}(h_1).
\end{aligned}$$

Since we have $h_2h_1 = (h_2 \triangleright h_1)(h_2 \triangleleft h_1)$, $h_2 \triangleright (g_1h_1^{-1})^{-1} = p_1(h_2h_1g_1^{-1})$ and $(g_2 \triangleleft h_1^{-1})h_2^{-1} =$

$p_2(g_2(h_2h_1)^{-1})$, the last integral equals to

$$\begin{aligned}
&\iint a_1(h_1^{-1})a_2(h_1(h_2 \triangleleft h_1)^{-1}(h_2 \triangleright h_1)^{-1}) \\
&\quad \times \xi(p_1((h_2 \triangleright h_1)(h_2 \triangleleft h_1)g_1^{-1})^{-1}, p_2(g_2(h_2 \triangleleft h_1)^{-1}(h_2 \triangleright h_1)^{-1})) \\
&\quad \times \Delta_1(h_1)^{1/2}\Delta_2((h_2 \triangleright h_1)(h_2 \triangleleft h_1)h_1^{-1})^{-1/2}\, d\lambda_{2,r(h_1)}(h_2) d\lambda_{1,s(g_1)}(h_1) \\
&= \iint a_1(h_1^{-1})a_2(h_1h_2^{-1}(h_2 \triangleright h_1^{-1})) \\
&\quad \times \xi(p_1((h_2 \triangleright h_1^{-1})^{-1}h_2g_1^{-1})^{-1}, p_2(g_2h_2^{-1}(h_2 \triangleright h_1^{-1}))) \\
&\quad \times \Delta_1(h_1)^{3/2}\Delta_2(h_1h_2^{-1}(h_2 \triangleright h_1^{-1}))^{1/2} d\lambda_{1,s(g_1)}(h_1) d\lambda_{2,s(g_1)}(h_2)
\end{aligned}$$

by (C2). Since we have $h_1^{-1} = h_2^{-1} \triangleright (h_2 \triangleright h_1^{-1})$, the last integral equals to

$$\begin{aligned}
&\iint a_1((h_2^{-1} \triangleright (h_2 \triangleright h_1^{-1}))a_2(\{h_2^{-1} \triangleright (h_2 \triangleright h_1^{-1})\}^{-1}h_2^{-1}(h_2 \triangleright h_1^{-1})) \\
&\quad \times \xi(p_1((h_2 \triangleright h_1^{-1})^{-1}h_2g_1^{-1})^{-1}, p_2(g_2h_2^{-1}(h_2 \triangleright h_1^{-1}))) \\
&\quad \times \Delta_1(h_2^{-1} \triangleright (h_2 \triangleright h_1^{-1}))^{-3/2}\Delta_2(\{h_2^{-1} \triangleright (h_2 \triangleright h_1^{-1})\}^{-1}h_2^{-1}(h_2 \triangleright h_1^{-1}))^{1/2} \\
&\quad \times d\lambda_{1,s(h_2)}(h_1) d\lambda_{2,s(g_1)}(h_2) \\
&= \iint a_1(h_2^{-1} \triangleright h_1^{-1})a_2(\{h_2^{-1} \triangleright h_1^{-1}\}^{-1}h_2^{-1}h_1^{-1}) \\
&\quad \times \xi(p_1(h_1h_2g_1^{-1})^{-1}, p_2(g_2h_2^{-1}h_1^{-1})) \\
&\quad \times \Delta_1(h_2^{-1} \triangleright h_1^{-1})^{-3/2}\Delta_2(\{h_2^{-1} \triangleright h_1^{-1}\}^{-1}h_2^{-1}h_1^{-1})^{1/2}\Delta_2(h_2) \\
&\quad \times d\lambda_{1,r(h_2)}(h_1) d\lambda_{2,s(g_1)}(h_2)
\end{aligned}$$

by (D1). Since we have $\{h_2^{-1} \triangleright h_1^{-1}\}^{-1}h_2^{-1}h_1^{-1} = h_2^{-1} \triangleleft h_1^{-1}$, the last integral equals to

$$\begin{aligned}
&\iint a_1(h_2^{-1} \triangleright h_1^{-1})a_2(h_2^{-1} \triangleleft h_1^{-1})\xi(p_1(h_1h_2g_1^{-1})^{-1}, p_2(g_2h_2^{-1}h_1^{-1})) \\
&\quad \times \Delta_1(h_1)^{3/2}\Delta_2(h_2)^{1/2}\, d\lambda_{1,r(h_2)}(h_1) d\lambda_{2,s(g_1)}(h_2)
\end{aligned}$$

by Lemma 2.3. Therefore we have

$$\begin{aligned}
&(\pi_1(a_1)\pi_2(a_2)\xi)(g_1, g_2) \\
&= \iint (\Delta_1^{-1/2}a_1)(h_2^{-1} \triangleright h_1^{-1})(\Delta_2^{-1/2}a_2)\check{}((h_2^{-1} \triangleleft h_1^{-1})^{-1})\xi(\theta(g_1, g_2; h_1, h_2)) \\
&\quad \times \Delta_1(h_1)\, d\lambda_{1,r(h_2)}(h_1) d\lambda_{2,s(g_1)}(h_2) \\
&= (\pi(((\Delta_1^{-1/2}a_1) \otimes (\Delta_2^{-1/2}a_2)\check{}) \circ \kappa)\xi)(g_1, g_2).
\end{aligned}$$

□

From the above arguments, we have injective $*$-homomorphisms $\pi \circ \Phi_1 : C_r^*(G_1) \bowtie C_r^*(G_2) \to \mathcal{L}(H)$ and $\pi \circ \Phi_2 : C_r^*(G_2) \bowtie C_r^*(G_1) \to \mathcal{L}(H)$. The invariance of μ implies that $H = \int^{\oplus}{}^{x}H\, d\mu(x) = \int^{\oplus} H^x\, d\mu(x)$. Then we can define injective $*$-homomorphisms $\rho : C_r^*(G_1) \ltimes C_0(G_2) \to \mathcal{L}(H)$ and $\hat{\rho} : C_r^*(G_2) \ltimes C_0(G_1) \to \mathcal{L}(H)$ by $\rho = \int^{\oplus}{}^{x}\rho\, d\mu(x)$ and $\hat{\rho} = \int^{\oplus} \hat{\rho}^x\, d\mu(x)$ respectively.

Theorem 7.2. *The following equations hold:*

(1) $\pi \circ \Phi_1(a \otimes b) = \pi_1(a)\pi_2(b) \qquad (a \in C_c(G_1),\ b \in C_c(G_2)).$

(2) $\pi \circ \Phi_2(b \otimes a) = \pi_2(b)\pi_1(a) \qquad (a \in C_c(G_1),\ b \in C_c(G_2)).$

(3) $\rho(a \otimes b) = \pi_1(a)\rho_2(b) \qquad (a \in C_c(G_1),\ b \in C_0(G_2)).$

(4) $\hat{\rho}(b \otimes a) = \pi_2(b)\rho_1(a) \qquad (a \in C_0(G_1),\ b \in C_c(G_2)).$

Proof. The statements (1) and (2) are immediate consequences of the definitions of $C_r^*(G_1) \bowtie C_r^*(G_2)$ and $C_r^*(G_2) \bowtie C_r^*(G_1)$ and Proposition 7.1. The statements of (3) and (4) are immediate consequences of the definitions of representations involved. □

8 Examples

8.1 Actions of semidirect product groups Let Γ_1 and Γ_2 be a locally compact second countable Hausdorff groups. Let $\sigma : \Gamma_2 \to \mathrm{Aut}(\Gamma_1)$ be a continuous homomorphism. Let $\Gamma = \Gamma_1 \times_\sigma \Gamma_2$ be a seimidirect product group. Suppose that Γ acts on a topological space X. We have groupoids $G = \Gamma \times X$ and $G_i = \Gamma_i \times X$ $(i = 1, 2)$. For every $(\gamma, x) \in G$, we have $(\gamma, x) = (\gamma_1, \gamma_2 \cdot x)(\gamma_2, x)$ for $\gamma_i \in G_i$ with $\gamma = \gamma_1\gamma_2$. That is, $p_1(\gamma, x) = (\gamma_1, \gamma_2 \cdot x)$ and $p_2(\gamma, x) = (\gamma_2, x)$. Then (G_1, G_2) is a matched pair. Let λ_i be a right Haar measure of Γ_i. Defin $\Delta_2 : G_2 \longrightarrow \mathbb{R}_{>0}$ by $\Delta_2(\gamma_2, x) = d(\sigma_{\gamma_2} \cdot \lambda_1)/d\lambda_1$, which is constant on X since λ_1 is a Haar measure. We set $\Delta_1 = 1$. Then the equation (Ci) is satisfied for the right Haar system $\{\lambda_i \times \delta_x\}$ $(i = 1, 2)$, where δ_x is the Dirac measure. Let δ_i be the modular function of Γ_i. Then a positive measure μ on X is G_i-invariant if and only if $(d(\gamma_i \cdot \mu)/d\mu)(x) = \delta_i(\gamma_i)$ for $\lambda_i \times \mu$-a.a. $(\gamma_i, x) \in G_i$.

8.2 An action of a matche pair given by S. Majid Using a matched pair given by S. Majid [5], Example 6.2.16, we describe an action of a matched pair on a two torus. Let Γ_1 be the group of 3×3 lower triangular matrices with 1 on the diagonal and Γ_2 the the group of 3×3 upper triangular matrices with 1 on the diagonal. We take the entries in the integers $\mathbb{Z}$. That is

$$\Gamma_1 = \left\{ \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix} \middle| \, a, b, c \in \mathbb{Z} \right\}, \qquad \Gamma_2 = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \middle| \, a, b, c \in \mathbb{Z} \right\}.$$

Define a bijection $\sigma : \Gamma_1 \cup \Gamma_2 \to \Gamma_1 \cup \Gamma_2$ by $\sigma(\gamma) = {}^t\gamma^{-1}$, where ${}^t\gamma$ is the transpose of γ. For $\gamma_i \in \Gamma_i$ $(i = 1, 2)$, define

$$\gamma_2 \triangleleft \gamma_1 = I + (\gamma_2 - I)\sigma(\gamma_1) \in \Gamma_2,$$
$$\gamma_2 \triangleright \gamma_1 = I + \sigma(\gamma_2)(\gamma_1 - I) \in \Gamma_1.$$

Then (Γ_1, Γ_2) is a matched pair of groups. Note that we have $\gamma_2\sigma(\gamma_1) = \sigma(\gamma_2 \triangleright \gamma_1)(\gamma_2 \triangleleft \gamma_1)$. We can form the bicrossed product group $\Gamma = \Gamma_1 \bowtie \Gamma_2$. Let $X = \mathbb{T}^2$. Define an action of Γ on X by

$$(\gamma_1, \gamma_2) \cdot (u_1, u_2) = \sigma(\gamma_1)\gamma_2 \begin{pmatrix} u_1 \\ u_2 \\ 1 \end{pmatrix}$$

for $(\gamma_1, \gamma_2) \in \Gamma$ and $(u_1, u_2) \in X$, where we identify (u_1, u_2) with ${}^t(u_1, u_2, 1)$. Define r-dicrete groupoids G, G_1 and G_2 by $G = \Gamma \times X$ and $G_i = \Gamma_i \times X$ $(i = 1, 2)$. Then (G_1, G_2) is a matched pair of groupoids.

Remark. The groups Γ, Γ_1 and Γ_2 are amenable. In fact, they are semidirect product groups of amenable groups: $\Gamma_1 \simeq \Gamma_2 \simeq \mathbb{Z}^2 \times_s \mathbb{Z}$ and $\Gamma \simeq \mathbb{Z}^3 \times_s \Gamma_2$.

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