Option Pricing and Hedging under Stochastic Verhulst-Gompertz Equation

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Abstract. Emphasis is placed on the valuation of plain vanilla option when the price process of underlying asset is described by the stochastic Verhulst-Gompertz Equation with network externality effects in a complete market. The method is based on the change of measure, Girsanov theorem and martingale valuation techinique. The application to an exchange option is made attempt and the valuation formula for this option like the Black-Scholes one is derived. A simple relationship similar to the put-call-parity between the exchange call option and the put option is provided. Our results will be useful to analyze and to hedge the price evolution at a sudden rise or crash of stock and commodity markets.

1 Introduction. The prices of some assets traded in the financial market have the tendency raised more when the prices begin to rise, it is because most traders expect that a price rises more and the purchase orders of other traders are induced as a result. Conversely, when the prices begin to fall, selling orders of other traders are induced. In the actual market, we often observe such a phenomenon called network externality price effects. These effects have been recognized as very significant concept in the field of marketing. Nerlove and Arrow[8] , Bass[2], Vidale and Wolfe[11] and Gould[4] made use of this deterministic version to study the dynamic behavior of new product and advertising policy. Their models were stimulated by a classical theory of logistic curve by Verhulst[10]. Gompertz extended Verhulst's model to so-called a deterministic Verhulst-Gompertz model in order to investigate the growth of population. In the fields of finance, Schwartz[9] gave a stochastic version of Verhulst-Gompertz model and developed the random behavior of commodity price. His model was based on the biological problem studied by Goel and Richer-Dyn[3].

The main purpose of this paper is devoted to describe the network externality effects by a generalized stochastic Verhulst-Gompertz equation and to derive pricing formulae for a plain vanilla and an exchange options when the price of underlying asset shows the network externality price effects. The organization of this paper is as follows. The next section briefly introduces the generalized stochastic Verhulst-Gompertz equation to express the network externality effects and in Section 3 we develope the martingale measure to evaluate the value of option. In Section 4 we derive a hedging strategy and a pricing formula of an exchange option that gives the holder the right which exchanges the underlying asset with the stochastic Verhulst-Gompertz equation for the asset with the geometric Brownian motion (Black-Scholes model) by means of martingale pricing method[7]. If we regard the former as the spot price of energy commodity and the latter as the stock price of business firm that trades the energy commodity, such an exchang option will be very useful to hedge the sudden rise or heavy fall in price of energy and make a contribution to the stabilized price of energy. The paper concludes with a brief summary.

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2 Undelying Price Evolution Consider a market consisting of one bond (riskless asset S^0) and two risky assets (S^1 and S^2) . $(S_t^1)_{0\leq t\leq T}$ denotes the price of the energy commodity such as crude oil and $(S_t^2)_{0\leq t\leq T}$ the stock price of energy business company at time t. The evolution of the economy is described by the dynamics of S^0, S^1, S^2, which satisfy the stochastic differential equations:

$$(1)\qquad \begin{cases} dS_t^0 = rS_t^0 dt, \\ dS_t^1 = \mu_1 S_t^1 \Big[1-(\frac{S_t^1}{\beta})^\alpha\Big]dt + \sigma_1 S_t^1 \Big(\sqrt{1-\rho_{12}^2}dB_t^1 + \rho_{12}dB_t^2\Big), \\ dS_t^2 = \mu_2 S_t^2 dt + \sigma_2 S_t^2 dB_t^2, \end{cases}$$

with $S_0^0 = 1, S_0^1 = s_1 > 0, S_0^2 = s_2 > 0$. The processes $B^1 = (B_t^1)$ and $B^2 = (B_t^2)$ are two independent standard Brownian motions defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Parameters $\mu_1, \mu_2, \sigma_1, \sigma_2, \alpha > 0$ and $\beta > 0$ are real numbers, with $\sigma_1 > 0$ and $\sigma_2 > 0$. We denote $\mathcal{F}_s$ the σ-algebra generated by the random variables B_s^1 and B_s^2 for $s \leq t$. Then the vector $(B_t^1 - B_s^1, B_t^2 - B_s^2)$ is independent of $\mathcal{F}_s$. Note that $\mathbb{P}$ is the physical probability measure, that captures the underlying uncertainty in this market. Tradings in these assets are unrestriced, i.e., no taxes, transactions costs, constraints, or other frictions. Likewise, investors can invest without restrictions, at the constant risk-free rate r. The cross variation of S^1 and S^2 is given by

$$d\langle S^1_\cdot, S^2_\cdot\rangle_t = \rho_{12}\sigma_2\sigma_2 S_t^1 S_t^2 dt.$$

This fact shows that our model deals with the case of correlated rate of returns S_t^1 and S_t^2.

The solutions to the first and the third equation in (1) are easily shown to be

$$S_t^0 = e^{rt}$$

and

$$(2)\qquad S_t^2 = s_2 \exp\left[\Big(\mu_2 - \frac{\sigma_2^2}{2}\Big)t + \sigma_2 B_t^2\right] = s_2 e^{\mu_2 t}\mathcal{E}_t,$$

where $\mathcal{E}_t = \exp(\sigma_2 B_t^2 - (\frac{\sigma_2^2}{2})t)$ is an exponential martingale for $t \geq 0$.

Note that the second equation is the extension of the deterministic Verhulst-Gompertz equation

$$\frac{dx_t}{dt} = \mu x_t\big[1-(\frac{x_t}{\beta})^\alpha\big]$$

to the stochastic version. The Verhulst-Gompertz equation has been extensively studied in biology[3] and characterized the transition of the number of individuals and the growth of populations in some species[10].

It is clear that the process defined by

$$\hat{B}_t = \sqrt{1-\rho_{12}^2}B_t^1 + \rho_{12}B_t^2$$

is a $\mathbb{P}$-standard Brownian motion on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, since $\mathbb{E}_{\mathbb{P}}[\hat{B}_t] = 0$ and the quadratic variation of $\hat{B}_t$ is given by

$$d\langle \hat{B}_\cdot, \hat{B}_\cdot\rangle_t = dt.$$

Then, the second equation in (1) is rewritten as follows :

$$(3)\qquad dS_t^1 = \mu_1 S_t^1\Big[1-(\frac{S_t^1}{\beta})^\alpha\Big]dt + \sigma_1 S_t^1 d\hat{B}_t.$$

We introduce $\mathbb{P}$-independent processes u_t^1 and u_t^2 by

$$u_t^1 \equiv \frac{1}{\sqrt{1-\rho_{12}^2}}\Big[\frac{\mu_1 - r}{\sigma_1} - \rho_{12}\frac{\mu_2 - r}{\sigma_2} - \frac{\mu_1}{\sigma_1}(\frac{S_t^1}{\beta})^\alpha\Big], \quad u_t^2 = \frac{\mu_2 - r}{\sigma_2}, \tag{4}$$

which satisfy the Novikov condition

$$\mathbb{E}_{\mathbb{P}}\Big[\exp(\frac{1}{2}\int_0^t (u_s^k)^2 ds)\Big] < \infty, \ (k=1,2).$$

If we define

$$\begin{aligned} \hat{W}_t &= \hat{B}_t + \int_0^t [\sqrt{1-\rho_{12}^2}u_s^1 + \rho_{12}u_s^2]ds \\ &= \sqrt{1-\rho_{12}^2}\big(B_t^1 + \int_0^t u_s^1 ds\big) + \rho_{12}\big(B_t^2 + \int_0^t u_s^2 ds\big) \\ &= \sqrt{1-\rho_{12}^2}W_t^1 + \rho_{12}W_t^2, \end{aligned}$$

where

$$W_t^k = B_t^k + \int_0^t u_s^k ds \ (k=1,2), \tag{5}$$

then equation (3) can be written as

$$dS_t^1 = rS_t^1 dt + \sigma_1 S_t^1 d\hat{W}_t, \tag{6}$$

by substituting equation (4) to equation (5). Clearly the process $(\hat{W}_t)_{0\le t\le T}$ is not a $\mathbb{P}$-standard Brownian motion. Here we can define the processes $(M_t^k)_{0\le t\le T}$, $k=1,2$ by

$$M_t^k = \exp\Big(-\int_0^t u_s^k dB_s^k - \frac{1}{2}\int_0^t (u_s^k)^2 ds\Big),$$

which are positive $\mathbb{P}$-martingales. Since $\mathbb{E}[M_T^k]=1$, then by Girsanov theorem, the measure $\tilde{P}_k$ defined by

$$\frac{d\tilde{\mathbb{P}}_k}{d\mathbb{P}} = M_T^k$$

is a probability measure, equivalent to $\mathbb{P}$, such that under $\tilde{\mathbb{P}}_k$ the process $(W_t^k)_{0\le t\le T}$ is a Brownian motion.

By the application of Itô's formula, the process $(e^{\xi W_t^k - \frac{\xi^2}{2}t}M_t^k)$ is a martingale relative to $(\mathcal{F}_t)_{0\le t\le T}$ under $\mathbb{P}$, where ξ is a real number. It follows that for $0\le s\le t\le T$,

$$\frac{1}{M_s^k}\mathbb{E}_{\mathbb{P}}(e^{\xi(W_t^k - W_s^k)}M_t^k \mid \mathcal{F}_s) = e^{\xi^2(t-s)/2}, \quad (k=1,2). \tag{7}$$

Lemma 1 The process $(M_t) = M_t^1 M_t^2$ is a $\mathbb{P}$-maritingale and $\mathbb{E}[M_T]=1$. Under the probability measure $\tilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ defined by $d\tilde{\mathbb{P}} = M_T d\mathbb{P}$, the processes W_t^1 and W_t^2 are independent $\tilde{\mathbb{P}}$-standard Brownian motions.

Proof. Using the fact that the processes M^1 and M^2 are independent martigales, we have for $0\le s\le t$

$$\mathbb{E}_{\mathbb{P}}(M_t|\mathcal{F}_s) = \mathbb{E}_{\mathbb{P}}(M_t^1 M_t^2|\mathcal{F}_s) = \mathbb{E}_{\mathbb{P}}(M_t^1|\mathcal{F}_t)\mathbb{E}_{\mathbb{P}}(M_t^2|\mathcal{F}_t) = M_s^1 M_s^2 = M_s$$

This shows that $(M_t)_{0\leq t\leq T}$ is a $\mathbb{P}$-martingale. Hence $\mathbb{E}_{\mathbb{P}}[M_T]=\mathbb{E}_{\mathbb{P}}[M_0]=1$. Then by Girsanov theorem, the measure $\tilde{\mathbb{P}}$ defined by

$$\frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}=M_T$$

is a probability measure, equivalent to $\mathbb{P}$, such that under $\tilde{\mathbb{P}}$ the process W_t^k is a standard Brownian motion. In detail, we may use Bayes's rule and equation (7) to convert the conditional expectation with respect to $\tilde{\mathbb{P}}_k$ to a conditional expectation with respect to the original probability measure $\mathbb{P}$, namely for $k=1,2$

$$\begin{aligned}\mathbb{E}_{\tilde{\mathbb{P}}_k}(e^{\xi(W_t^k-W_s^k)}|\mathcal{F}_s) &= \frac{\mathbb{E}_{\mathbb{P}}[e^{\xi(W_t^k-W_s^k)}M_t^k|\mathcal{F}_s]}{\mathbb{E}_{\mathbb{P}}[M_t^k|\mathcal{F}_s]}\\ &= \frac{1}{M_s^k}\mathbb{E}_{\mathbb{P}}(e^{\xi(W_t^k-W_s^k)}M_t^k|\mathcal{F}_s)=e^{\xi^2(t-s)/2}\end{aligned}$$

The independence of W_t^1 and W_t^2 under $\tilde{\mathbb{P}}_k$ is obvious. This implies that $(W_t^k-W_s^k)$ is independent of $\mathcal{F}_s$ and has a normal distribution as $N(0,t-s)$ under the measure $\tilde{\mathbb{P}}_k$. Since the martigales M_t^1 and M_t^2 are independent, we have

$$\begin{aligned}\mathbb{E}_{\tilde{\mathbb{P}}}(e^{\xi(W_t^1-W_s^1)}|\mathcal{F}_s) &= \frac{1}{M_s}\mathbb{E}_{\mathbb{P}}(e^{\xi(W_t^1-W_s^1)}M_t|\mathcal{F}_s)\\ &= \frac{1}{M_s^1M_s^2}\mathbb{E}_{\mathbb{P}}(e^{\xi(W_t^1-W_s^1)}M_t^1M_t^2|\mathcal{F}_s)\\ &= \frac{1}{M_s^1}\mathbb{E}_{\mathbb{P}}(e^{\xi(W_t^1-W_s^1)}M_t^1|\mathcal{F}_s)\frac{1}{M_t^2}\mathbb{E}_{\mathbb{P}}(M_t^2|\mathcal{F}_s)\\ &= e^{\xi^2(t-s)/2}.\end{aligned}$$

This shows that the process $W_t^1-W_s^1$ is independent of $\mathcal{F}_s$ under $\tilde{\mathbb{P}}$ and the final expression corresponds to the generating function of a normal distribution $N(0,t-s)$. The same argument can be applied to $W_t^2-W_s^2$. Therefore, $(W_t^1)_{0\leq t\leq T}$ and $(W_t^2)_{0\leq t\leq T}$ are $\tilde{\mathbb{P}}$-standard Brownian motions. □

From this lemma

$$\mathbb{E}_{\tilde{\mathbb{P}}}(W_t^k)=0,\quad \mathrm{Var}_{\tilde{\mathbb{P}}}(W_t^k)=t.$$

So the process $\hat{W}_t$ defined by $\sqrt{1-\rho_{12}^2}W_t^1+\rho_{12}W_t^2$ is a $\tilde{P}$-standard Brownian motion.

It should be noted that this result implies that the stochastic Verhulst-Gompertz equation (1) is reduced to the Black-Scholes-Merton equation or geometric Brownian motion. Consequently, the solution of the geometric equation (6) under $\tilde{\mathbb{P}}$ is given by

$$\begin{aligned}S_t^1 &= s_1\exp\Big[\Big(r-\frac{\sigma_1^2}{2}\Big)t+\sigma_1\hat{W}_t\Big]\\ &= \frac{s_1\exp\Big[\Big(\mu_1-\frac{\sigma_1^2}{2}\Big)t+\sigma_1\hat{B}_t\Big]}{\exp\Big[\mu_1\Big(\int_0^t(\frac{S_u^1}{\beta})^\alpha du\Big]}=\frac{f(t)}{R_t^{1/\alpha}},\end{aligned}$$

where

$$f(t)=s_1\exp\Big[\Big(\mu_1-\frac{\sigma_1^2}{2}\Big)t+\sigma_1\hat{B}_t\Big],\quad R_t^{1/\alpha}=\exp\Big[\mu_1\int_0^t(\frac{S_u^1}{\beta})^\alpha du\Big].$$

Then $(f(t))^\alpha = R_t(S_t^1)^\alpha$ and

$$dR_t = R_t\alpha\mu_1(\frac{S_t^1}{\beta})^\alpha dt = \alpha\mu_1(\frac{f(t)}{\beta})^\alpha.$$

Thus,

$$R_t = 1 + \int_0^t \alpha\mu_1(\frac{f(u)}{\beta})^\alpha du.$$

Notice that in terms of the process $\hat{B}_t$ the price process S_t^1 can be expressed as

$$S_t^1 = \frac{s_1 \exp\Big[\Big(\mu_1 - \frac{\sigma_1^2}{2}\Big)t + \sigma_1\hat{B}_t\Big]}{\Big(1 + \alpha\mu_1\frac{s_1^\alpha}{\beta^\alpha}\int_0^t \exp\Big[\alpha\Big(\mu_1 - \frac{\sigma_1^2}{2}\Big)s + \alpha\sigma_1\hat{B}_s\Big]ds\Big)^{1/\alpha}}. \tag{8}$$

3 Option Pricing by Martingale Measure In this section we develop a valuation method for the market price of European style call (plain vanilla) option with exercise price K and maturity T on the underlying asset (S_t^1) described by the stochastic Verhulst-Gompertz equation mentioned above. To this end, we can use the unified martingale measure $\tilde{\mathbb{P}}(= M_T d\mathbb{P})$ common to two risky assets S^1 and S^2.

It can be shown by standard method that, under the unified measure $\tilde{\mathbb{P}}$ derived in the last section, the discounted price process $(\tilde{S}_t^1)_{0\le t\le T}$ can be expressed as

$$\begin{aligned} d\tilde{S}_t^1 &= d(e^{-rt}S_t^1) \\ &= -r\tilde{S}_t^1 dt + \tilde{S}_t^1\big\{\mu_1\big[1 - (\frac{S_t^1}{\beta})^\alpha\big]dt + \sigma_1\big(\sqrt{1-\rho_{12}^2}dB_t^1 + \rho_{12}dB_t^2\big)\big\} \\ &= -r\tilde{S}_t^1 dt + \tilde{S}_t^1[rdt + \sigma_1(\sqrt{1-\rho_{12}^2}\sigma_1 S_t^1 dW_t^1 + \rho_{12}\sigma_1 S_t^1 dW_t^2)] \\ &= \sigma_1\tilde{S}_t^1(\sqrt{1-\rho_{12}^2}\sigma_1 S_t^1 dW_t^1 + \rho_{12}\sigma_1 S_t^1 dW_t^2) = \sigma_1\tilde{S}_t^1 d\hat{W}_t, \end{aligned}$$

where $(\hat{W}_t)_{0\le t\le T}$ is a $\tilde{\mathbb{P}}$-standard Brownian motion. This implies that $(\tilde{S}_t^1)$ forms a $\tilde{\mathbb{P}}$-martingale. Therefore, our arguments establish that the option price $C(t, S_t^1)$ with underlying asset S_t^1 satisfies the same type of Black-Scholes valuation formula $C_{\mathrm{BS}}(t, S_t^2)$, that is, an application of martingale (risk-neutral) valuation gives the option price representation as

$$C(t, S_t^1) = \mathbb{E}_{\tilde{\mathbb{P}}}(e^{-r(T-t)}h \mid \mathcal{F}_t) = S_t^1 N(d_1) + e^{-r(T-t)}KN(d_2), \tag{9}$$

where $h = (S_T^1 - K)_+$, $d_1 = \frac{\log(S_t^1/K) + (r+\sigma_1^2/2)(T-t)}{\sigma_1\sqrt{T-t}}$, $d_2 = d_1 - \sigma_1\sqrt{T-t}$. And $N(\cdot)$ denotes a standard normal distribution function.

This expression shows that the option's price is simply the expected payoff discounted at the risk-free rate, as should be the case with risk neutorality. In the case of $\sigma_1 = \sigma_2$ and $S_s^1 = S_s^2$ for some $s \in [0, T]$, we have $C(s, S_s^1) = C_{\mathrm{BS}}(s, S_s^2)$.

It is also important to note that the discounted values $\tilde{C}(t, S_t^1)$ and $\tilde{C}(t, S_t^1)$ of option values are martingales under the measure $\tilde{\mathbb{P}}$. This can be easily seen by taking conditional expectations and using the tower property in the following manner : For $s \le t$ and $k = 1, 2$,

$$\begin{aligned} \mathbb{E}_{\tilde{\mathbb{P}}}[\tilde{C}(t, S_t^k) \mid \mathcal{F}_s] &= \mathbb{E}_{\tilde{\mathbb{P}}}[e^{-rt}C(t, S_t^k) \mid \mathcal{F}_s] = \mathbb{E}_{\tilde{\mathbb{P}}}[\mathbb{E}_{\tilde{\mathbb{P}}}(e^{-rT}h \mid \mathcal{F}_t)\big|\mathcal{F}_s] \\ &= \mathbb{E}_{\tilde{\mathbb{P}}}[e^{-rs}\mathbb{E}_{\tilde{\mathbb{P}}}(e^{-r(T-s)}h \mid \mathcal{F}_t)\big|\mathcal{F}_s] = e^{-rs}\mathbb{E}_{\tilde{\mathbb{P}}}[e^{-r(T-s)}h\big|\mathcal{F}_s] \\ &= e^{-rs}C(s, S_s^k) = \tilde{C}(s, S_s^k). \end{aligned}$$

The martingale property implies that for any t $(0 \le t \le T)$,

$$\mathbb{E}_{\tilde{\mathbb{P}}}[\tilde{C}(t, S_t^1)] = \mathbb{E}_{\tilde{\mathbb{P}}}[\tilde{C}_{\mathrm{BS}}(t, S_t^2)]$$

and we obtain the following proposition about the option pricing, immediately.

Proposition 1 Assume that the price processes (S_t^1) and (S_t^2) of asset 1 and asset 2 are given by the stochastic Verhulst-Gompertz equation and the Black-Scholes-Merton equation with same volatility $\sigma_1 = \sigma_2$. If $S_s^1 = S_s^2$, ($\tilde{\mathbb{P}}$-a.s.) for some $s \in [0, T]$, then $\mathbb{E}_{\tilde{\mathbb{P}}}[C(t, S_t^1)] = \mathbb{E}_{\tilde{\mathbb{P}}}[C_{\mathrm{BS}}(t, S_t^2)]$ for any $t \in [0, T]$.

In place of the price process S_t^1 given by the second equation in (1), Schwartz[9] dealt with the equation as

$$dS_t^3 = \mu_3 S_t^3 \Big(1 - \frac{1}{\beta} \log S_t^3\Big) dt + \sigma_3 S_t^3 \Big(\rho_{23} dB_t^2 + \sqrt{1 - \rho_{23}^2} dB_t^3\Big), \; S_0^3 = s_3 > 0. \tag{10}$$

We can appeal to Girsanov change of measure to construct a new measure $\hat{\mathbb{P}}$ under which $W_t^k = (k = 2, 3)$ has the standard Browinian motion property.

Proposition 2 Defining

$$u_t^2 = \frac{\mu_2 - r}{\sigma_2}, \; u_t^3 = \frac{\mu_3 - r}{\sigma_3} - \rho_{23} \frac{\mu_2 - r}{\sigma_2} - \frac{\mu_3}{\beta \sigma_3} \log S_t^3 \tag{11}$$

and

$$M_t^k = \exp\Big(- \int_0^t u_s^k dw_s - \frac{1}{2} \int_0^t (u_s^k)^2 ds \Big), \quad k = 2, 3$$

then the stochastic process $(M_t)_{0 \le t \le T}$ defined by $M_t = M_t^2 M_t^3$ is a non-negative $\hat{\mathbb{P}}$-martingale. The random variable M_T represents the Radon-Nikodym derivative of $\hat{\mathbb{P}}$ with respect to $\mathbb{P}$ (i.e., $M_T = d\hat{\mathbb{P}}/d\mathbb{P}$). Girsanov's transformation

$$W_t^k \equiv B_t^k + \int_0^t u_s^k ds, \quad k = 2, 3$$

can then be invoked to assert that $(W_t^k)_{0 \le t \le T}$ is a $\hat{\mathbb{P}}$-standard Brownian motion and the stochastic differential equation (10) is reduced to the Black-Scholes-Merton equation

$$dS_t^3 = r S_t^3 dt + \sigma_3 S_t^3 d\tilde{W}_t, \tag{12}$$

where $\tilde{W}_t$ defined by

$$\tilde{W}_t = \rho_{23} W_t^2 + \sqrt{1 - \rho_{23}^2} W_t^3$$

is a $\hat{\mathbb{P}}$-standard Brownian motion.

4 Exchange Option and Hedging Strategy This section is devoted to pricing an exchange option which plays a prominant weapon in risk hedging alternatives of energy company. An exchange option gives the holder the right to exchange one asset S_1 for another S_2, in some rates q_1 and q_2. The payoff for this contract at maturity T is

$$\max(q_1 S_1 - q_2 S_2, 0),$$

where q_1 and q_2 are specified constants. For simplicity from now on, it is assumed that $q_1 = q_2 = 1$.

First of all, we present two preliminary lemmas to obtain our main purpose. The proof of Lemma 2 is found in the published paper[1] as the answer to exercise 27 in the book [6].

Lemma 2 Suppose that X_1 and X_2 are independent random variables with standard normal distributions. Then for any real numbers $a, b, \lambda_0, \lambda_1$ and λ_2, the following relation holds:

$$\begin{aligned}\mathbb{E}\{[e^{a+\lambda_1 X_1+\lambda_2 X_2} - e^{b+\lambda_2 X_2}]_+\} &= e^{a+(\lambda_0^2+\lambda_1^2)/2} N\Big(\frac{a-b+(\lambda_1^2+\lambda_0^2)-\lambda_0\lambda_2}{\sqrt{\lambda_1^2+(\lambda_0-\lambda_2)^2}}\Big) \\ &\quad - e^{b+\lambda_2^2/2} N\Big(\frac{a-b-\lambda_2^2}{\sqrt{\lambda_1^2+(\lambda_0-\lambda_2)^2}}\Big).\end{aligned}$$

Consider an investor with initial endowment $V_0(\phi) \geq 0$ and investing in these three kinds of assets described in the last section. Let H_t^0 be the number of riskless assets S^0, and H_t^1, H_t^2 be the number of risky assets S^1 and S^2, respectively, owned by the investor at time t. The triplet $\phi(t) = (H_t^0, H_t^1, H_t^2)_{0\leq t\leq T}$ is called a trading strategy or a portfolio. We assume that H_t^0, H_t^1 and H_t^2 are $\mathcal{F}_t$-measurable and adapted processes such that

$$\int_0^T |H_t^0| dt + \int_0^T [(H_t^1)^2 + (H_t^2)^2] dt < \infty, \quad \mathbb{P}\text{-a.s.}$$

Then $V_0(\phi) = H_0^0 + H_0^1 S_0^1 + H_0^2 S_0^2$ and the investor's wealth at time t (the value of the strategy) is represented by

$$V_t(\phi) = H_t^0 S_t^0 + H_t^1 S_t^1 + H_t^2 S_t^2.$$

We say that the strategy ϕ is self-financing if there is no fresh investment and consumption. This means that the investor's wealth equals to the initial investment plus the gain :

$$V_t(\phi) = V_0(\phi) + \int_0^t H_s^0 dS_s^0 + \int_0^t H_s^1 dS_s^1 + \int_0^t H_s^2 dS_s^2,$$

that is,

$$dV_t(\phi) = H_t^0 dS_t^0 + H_t^1 dS_t^1 + H_t^2 dS_t^2.$$

From now on we will consider only self-financing strategies and let $\tilde{S}$ and $\tilde{V}$ be the discounted processes defined by

$$\tilde{S}_t^k = (S_t^0)^{-1} S_t^k = e^{-rt} S_t^k, \ (k = 1, 2)$$

and

$$\tilde{V}(\phi) = (S_t^0)^{-1} V_t(\phi) = H_t^0 + H_t^1 \tilde{S}_t^2 + H_t^2 \tilde{S}_t^2.$$

Then, the self-financing strategy ϕ is written by

$$\tilde{V}_t(\phi) = V_0(\phi) + \int_0^t H_s^1 d\tilde{S}_s^1 + \int_0^t H_s^2 d\tilde{S}_s^2,$$

that is, if the equation

$$\begin{aligned} d\tilde{V}_t &= H_t^1 d\tilde{S}_t^1 + H_t^2 d\tilde{S}_t^2 \\ &= H_t^1 \tilde{S}_t^1 \sigma_1 (\sqrt{1-\rho_{12}^2} dW_t^1 + \rho_{12} dW_t^2) + H_t^2 \sigma_2 \tilde{S}_t^2 dW_t^2 \\ &= e^{-rt}[H_t^1 \sigma_1 S_t^1 (\sqrt{1-\rho_{12}^2} dW_t^1 + \rho_{12} dW_t^2) + H_t^2 \sigma_2 S_t^2 dW_t^2] \\ &= e^{-rt} H_t^1 \sigma_1 S_t^1 d\hat{W}_t + e^{-rt} H_t^2 \sigma_2 S_t^2 dW_t^2 \end{aligned}$$

holds, then the strategy ϕ is self-financing. Moreover, assume that the processes of the self-financing strategy H_t^1 and H_t^2 are uniformly bounded, that is, for any $(t,\omega) \in [0,T]\times\Omega$ and some $K > 0$, $|H_t^k(\omega)| \le K$, $k = 1,2$. Then from equations (2) and (6), we have $(\tilde{S}_t^1)^2 = (S_0^1)^2 e^{-\sigma_1^2 t + 2\sigma_1 \hat{W}_t}$ and

$$\mathbb{E}_{\tilde{\mathbb{P}}}[(S_t^k)^2] = (S_0^k)^2 e^{\sigma_k^2 t} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-(y-2\sigma_k\sqrt{t})^2/2} dy \le (S_0^k)^2 e^{\sigma_k^2 t}, \quad \text{for } k = 1,2.$$

Thus,

$$\mathbb{E}_{\tilde{\mathbb{P}}}\Big(\int_0^T (H_t^k)^2 e^{-2rt} (S_t^1)^2 dt\Big) \le \int_0^T K^2 \mathbb{E}_{\tilde{\mathbb{P}}}[(\tilde{S}_t^k)^2 dt \le \int_0^T K^2 (S_0^k)^2 e^{\sigma_k^2 t} dt < \infty.$$

Hence $\int_0^T e^{-rt} H_t^1 \sigma_1 S_t^1 d\hat{W}_t$ and $\int_0^T e^{-rt} H_t^2 \sigma_2 S_t^2 dW_t^2$ are $\tilde{\mathbb{P}}$-martingales since $\hat{W}_t$ and W_t^2 are $\tilde{\mathbb{P}}$-standard Brownian motions. So, the discounted value process $\tilde{V}_t$ is represented by the sum of these two $\tilde{\mathbb{P}}$-martingales. Consequently,

$$\tilde{V}_t = V_0 + \int_0^t e^{-rs} H_s^1 \sigma_1 S_s^1 d\hat{W}_s + \int_0^t e^{-rt} H_s^2 \sigma_2 S_s^2 dW_s^2$$

is also a $\tilde{\mathbb{P}}$-martingale.

The argument described above leads to the following lemma :

Lemma 3 Assume that the self-financing strategy ϕ satisfies the uniformly bounded and the integrable conditions

$$\int_0^T |H_t^0| dt + \int_0^T (H_t^1)^2 dt + \int_0^T (H_t^2)^2 dt + < \infty \text{ (a.s.)}.$$

If the terminal value V_T is expressed by

$$V_T = (S_T^1 - S_T^2)_+,$$

then the value of the strategy ϕ is given by

$$V_t = F(t, S_t^1, S_t^2), \quad t \le T,$$

where the function F is repesented by

$$F(t,x_1,x_2) = \mathbb{E}_{\tilde{\mathbb{P}}}\Big[\big(x_1 e^{\sigma_1(\hat{W}_T - \hat{W}_t) - \sigma_1^2(T-t)/2} - x_2 e^{\sigma_2(W_T^2 - W_t^2) - \sigma_2^2(T-t)/2}\big)_+\Big]. \tag{13}$$

Proof. Since the discounted value $(\tilde{V}_t)_{0\le t\le T}$ of the strategy ϕ is a $\tilde{\mathbb{P}}$-martingale, the following relation holds :

$$\begin{aligned} \tilde{V}_t &= \mathbb{E}_{\tilde{\mathbb{P}}}(\tilde{V}_T|\mathcal{F}_t) = \mathbb{E}_{\tilde{\mathbb{P}}}(e^{-rT} V_T|\mathcal{F}_t) \\ &= \mathbb{E}_{\tilde{\mathbb{P}}}(e^{-rT}(S_T^1 - S_T^2)_+|\mathcal{F}_t) = \mathbb{E}_{\tilde{\mathbb{P}}}((\tilde{S}_T^1 - \tilde{S}_T^2)_+|\mathcal{F}_t). \end{aligned}$$

Using the relations $d\tilde{S}_t^1 = \tilde{S}_t^1 \sigma_1 d\hat{W}_t$ and $d\tilde{S}_t^2 = \tilde{S}_t^2 \sigma_2 dW_t^2$, we have

$$\tilde{S}_T^1 = e^{-rt} S_t^1 \exp\big(\sigma_1(\hat{W}_T - \hat{W}_t) - \frac{\sigma_1^2}{2}(T-t)\big)$$

and

$$\tilde{S}_T^2 = e^{-rt} S_t^2 \exp\big(\sigma_2(W_T^2 - W_t^2) - \frac{\sigma_1^2}{2}(T-t)\big).$$

Since $\hat{W}_T - \hat{W}_t$ and $W_T^2 - W_t^2$ are independent of $\mathcal{F}_t$,

$$\begin{aligned} \tilde{V}_t &= e^{-rt}\mathbb{E}_{\tilde{\mathbb{P}}}\Big\{\Big[S_t^1 \exp\Big(\sigma_1(\hat{W}_T - \hat{W}_t) - \frac{\sigma_1^2}{2}(T-t)\Big) \\ &\quad -S_t^2 \exp\Big(\sigma_2(W_T^2 - W_t^2) - \frac{\sigma_2^2}{2}(T-t)\Big)\Big]_+\Big\} = e^{-rt}F(t, S_t^1, S_t^2). \end{aligned}$$

Then we have $V_t = F(t, S_t^1, S_t^2)$. From the basic property of Brownian motion, two processes $\hat{W}_{T-t} = \hat{W}_T - \hat{W}_t$ and $W_{T-t}^2 = W_T^2 - W_t^2$ are independent $\tilde{\mathbb{P}}$-standard Brownian motions. Hence, substituting

$$a = \ln x_1 - \frac{\sigma_1^2}{2}(T-t), \quad \lambda_1 = \sigma_1\sqrt{1-\rho_{12}^2}\sqrt{T-t}, \quad \lambda_0 = \sigma_1\rho_{12}\sqrt{T-t},$$

$$b = \ln x_2 - \frac{\sigma_2^2}{2}(T-t), \quad \lambda_2 = \sigma_2\sqrt{T-t}$$

into Lemma 2, we get

$$F(t, x_1, x_2) = x_1 N\Big(\frac{\ln(x_1/x_2) + \frac{1}{2}D^2(T-t)}{D\sqrt{T-t}}\Big) - x_2 N\Big(\frac{\ln(x_1/x_2) - \frac{1}{2}D^2(T-t)}{D\sqrt{T-t}}\Big),$$

where $D = \sqrt{\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2}$. This expression shows the explicit form of F. □

From this lemma, we have main result as

Theorem 1 The value V_t of the trading strategy ϕ for any point in time $t \le T$ is given by

$$V_t = F(t, S_t^1, S_t^2),$$

where

$$F(t, x_1, x_2) = \mathbb{E}_{\tilde{\mathbb{P}}}\big[\big(x_1 e^{\sigma_1(\hat{W}_T - \hat{W}_t) - \sigma_1^2(T-t)/2} - x_2 e^{\sigma_2(W_T^2 - W_t^2) - \sigma_2^2(T-t)/2}\big)_+\big]. \tag{14}$$

Proof. Since the discounted value $(\tilde{V}_t)_{0\le t\le T}$ of the trading strategy ϕ is a $\tilde{\mathbb{P}}$-martingale,

$$\begin{aligned} \tilde{V}_t &= \mathbb{E}_{\tilde{\mathbb{P}}}(\tilde{V}_T|\mathcal{F}_t) = \mathbb{E}_{\tilde{\mathbb{P}}}(e^{-rT}V_T|\mathcal{F}_t) \\ &= \mathbb{E}_{\tilde{\mathbb{P}}}(e^{-rt}(S_T^1 - S_T^2)_+|\mathcal{F}_t) = \mathbb{E}_{\tilde{\mathbb{P}}}((\tilde{S}_T^1 - \tilde{S}_T^2)_+|\mathcal{F}_t) \end{aligned}$$

By the way, $d\tilde{S}_t^k = \tilde{S}_t^k\sigma_i d\tilde{W}_t^k, (k = 1, 2)$ gives

$$\tilde{S}_T^k = e^{-rt} S_t^k \exp\Big(\sigma_k(\hat{W}_T^k - \hat{W}_t^k) - \frac{\sigma_i^2}{2}(T-t)\Big)$$

Thus,

$$\begin{aligned} \tilde{V}_t &= e^{-rt}\mathbb{E}_{\tilde{\mathbb{P}}}\Big\{\big[S_t^1 \exp\big(\sigma_1(\hat{W}_T^1 - \hat{W}_t^1) - \frac{\sigma_1^2}{2}(T-t)\big) \\ &\quad -S_t^2 \exp\big(\sigma_2(\hat{W}_T^2 - \hat{W}_t^2) - \frac{\sigma_2^2}{2}(T-t)\big)\big]_+\Big\} \\ &= e^{-rt}F(t, S_t^1, S_t^2). \end{aligned}$$

Then we obtain

$$V_t = F(t, S_t^1, S_t^2). \qquad \square$$

Next, we derive the explicit form of the hedging strategy. Let $\tilde{C}_t = F(t, \tilde{S}_t^1, \tilde{S}_t^2)$ be the discounted value of exchange option. By Itô's formula with

$$d\tilde{S}_t^1 = \sigma_1^2 \tilde{S}_t^1 d\hat{W}_t, \quad d\tilde{S}_t^2 = \sigma_2^2 \tilde{S}_t^2 dW_t^2,$$

we obtain

$$\begin{aligned} d\tilde{C}_t &= \frac{\partial F}{\partial x_1}(t, \tilde{S}_t^1, \tilde{S}_t^2) d\tilde{S}_t^1 + \frac{\partial F}{\partial x_2}(t, \tilde{S}_t^1, \tilde{S}_t^2) d\tilde{S}_t^2 \\ &= \frac{\partial F}{\partial x_1}(t, \tilde{S}_t^1, \tilde{S}_t^2)\sigma_1^2 \tilde{S}_t^1 d\hat{W}_t + \frac{\partial F}{\partial x_2}(t, \tilde{S}_t^1, \tilde{S}_t^2)\sigma_2^2 \tilde{S}_t^2 dW_t^2. \end{aligned} \tag{15}$$

Thus,

$$\tilde{C}_t (= \tilde{V}_t) = F(0, \tilde{S}_0^1, \tilde{S}_0^2) + \int_0^t \frac{\partial F}{\partial x_1}(u, \tilde{S}_u^1, \tilde{S}_u^2) d\tilde{S}_u^1 + \int_0^t \frac{\partial F}{\partial x_2}(u, \tilde{S}_u^1, \tilde{S}_u^2) d\tilde{S}_u^2$$

Therefore, the trading strategy $(H^0, H^1, H^2)_{0 \le t \le T}$ which satisfies

$$H_t^0 = F(t, \tilde{S}_t^1, \tilde{S}_t^2) - \frac{\partial F}{\partial x_1}(t, \tilde{S}_t^1, \tilde{S}_t^2)\tilde{S}_t^1 - \frac{\partial F}{\partial x_2}(t, \tilde{S}_t^1, \tilde{S}_t^2)\tilde{S}_t^2$$

and

$$H_t^1 = \frac{\partial F}{\partial x_1}(t, \tilde{S}_t^1, \tilde{S}_t^2), \quad H_t^2 = \frac{\partial F}{\partial x_1}(t, \tilde{S}_t^1, \tilde{S}_t^2)$$

is a self-financing strategy, since the discounted value of this strategy satisfies the relation

$$H_t^0 + H_t^1 \tilde{S}_t^1 + H_t^2 \tilde{S}_t^2 = F(t, \tilde{S}_t^1, \tilde{S}_t^2)$$

and the self-financing condition

$$\tilde{V}_t = V_0 + \int_0^t H_u^1 d\tilde{S}_u^1 + \int_0^t H_u^2 d\tilde{S}_u^2.$$

Using $D = \sqrt{\sigma_1^2 - 2\rho_{12}\sigma_1\sigma_2 + \sigma_2^2}$ and equation (15), we have

$$\begin{aligned} \frac{\partial F}{\partial x_1} = N\Big(\frac{\ln(x_1/x_2) + \frac{1}{2}D^2(T-t)}{D\sqrt{T-t}}\Big) + \frac{1}{D\sqrt{T-t}}\Big\{N'\Big(\frac{\ln(x_1/x_2) + \frac{1}{2}D^2(T-t)}{D\sqrt{T-t}}\Big) \\ -\frac{x_2}{x_1}N'\Big(\frac{\ln(x_1/x_2) - \frac{1}{2}(D^2 + 2\rho_{12}\sigma_1\sigma_2)(T-t)}{D\sqrt{T-t}}\Big)\Big\}. \end{aligned}$$

Also, we obtain $\partial F/\partial x_2$. Hence, we find that

$$\partial F(t, \tilde{S}_t^1, \tilde{S}_t^2)/\partial x_k = \partial F(t, S_t^1, S_t^2)/\partial x_k, \quad k = 1, 2.$$

It should be noted that for any t, there exists some $K > 0$ that satisfies

$$\big|H_t^1(\omega)\big| = \big|\frac{\partial F}{\partial x_1}(\omega)\big| < K$$

since $0 < N'(x) \le 1/\sqrt{2\pi}$. The same is true of H_t^2. Then the trading strategy $\phi = (H_t^0, H_t^1, H_t^2)$ presented above satisfies the uniformly bounded and the discounted value

$\tilde{V}_t$ is a $\tilde{\mathbb{P}}$-martingale. The argument in consideration of the above is summarized as the following thoerem.

Theorem 2 Assume that the price S_t^1 of asset 1 with network externality price effect and the price S_t^2 of asset 2 with the geometric Brownian motion are given by equation (1). The valuation formula of exchange (call) option C_t that exchanges the 1 unit of asset 1 to the 1 unit of asset 2 at expiration date T is expressed as

$$C_t = F(t, S_t^1, S_t^2),$$

where the function F is given by

$$F(t, S_t^1, S_t^2) = x_1 N(d_1) - x_2 N(d_2), \tag{16}$$

and $d_1 = \frac{\ln(x_1/x_2)+\frac{1}{2}D^2(T-t)}{D\sqrt{T-t}}$, $d_2 = d_1 - \sqrt{T-t}(D + \frac{\rho_{12}\sigma_1\sigma_2}{D})$. Moreover, this exchange option is relicated by the self-financing strategy $\phi = (H^0, H^1, H^2)$, where H^0, H^1 and H^2 are given by

$$H_t^0 = e^{-rt}\Big(F(t, S_t^1, S_t^2) - \frac{\partial F}{\partial x_1}(t, S_t^1, S_t^2)S_t^1 - \frac{\partial F}{\partial x_2}(t, S_t^1, S_t^2)S_t^2\Big)$$

$$H_t^1 = \frac{\partial F}{\partial x_1}(t, S_t^1, S_t^2), \quad H_t^2 = \frac{\partial F}{\partial x_2}(t, S_t^1, S_t^2).$$

The hedging of a derivative security is the problem faced by a financial institution that sells to a client some contract designed to reduce the client's risk. This theorem shows that for risk hedging, the writer of this option should hold the portfolio V_t composed of H_t^0 units of riskfree asset, H_t^1 units of energy S_t^1, and H_t^2 units of stocks of business company S_t^2. Finally, we present an interesting relationship between the value of the (call) option with payoff $(S_T^1 - S_T^2)_+$ and the symmetrical (put) option with payoff $(S_T^2 - S_T^1)_+$, similar to the put-call parity relationship. Let C_t be the price of exchange (call) option and P_t be the price of exchange (put) option which is symmetric to C_t, that is, the holder of the option P_t has the right, but not the obligation, to exchange an underlying security at a specified date T for a contractually. Using the identity

$$(A)_+ - (-A)_+ = A$$

and the $\tilde{\mathbb{P}}$-martingale relation for $(\tilde{S}_t^1 - \tilde{S}_t^2)$

$$\mathbb{E}_{\tilde{\mathbb{P}}}[\tilde{S}_T^1 - \tilde{S}_T^2 | \mathcal{F}_t] = \tilde{S}_t^1 - \tilde{S}_t^2$$

deduces an important link between the process of call and put options which corresponds to the put-call-parity for European options.

Proposition 3 For any t,

$$C_t - P_t = S_t^1 - S_t^2, \quad (0 \leq t \leq T).$$

This proposition and the theory of arbitrage suggest that if $S_s^1 = S_s^2$ for some s, $0 \leq s \leq T$, then $C_t = P_t$ for $t \in [0, T]$. Furthermore, if the trading unit is adjusted as $S_0^1 = S_0^2$ at time 0, the option premiums of these two symmetric exchange options are reduced to $C_0 = P_0$ and $C_t = P_t$ for any $t \in [0, T]$. Therefore, we cannot get any profit by such a option trading only.

5 Conclusion In this paper, we proposed the generalized stochastic Verhulst-Gompertz equation in order to describe the commodity price evolution with network price externality effects. It was shown that the equation was reduced to the geometric Brownian motion (Black-Scholes-Merton Equation) under the unified martingale measure. Explicit pricing expressions for the European call option and the exchange option were derived by means of the martingale valuation method and the self-financing strategies replicating these options were established. We presented a parity relationship between the value of the call option and the value of the put option, similar to the put-call-parity relationship. These results obtained in this paper will be useful to hedge the sudden price changes in the spot market and to stabilize the real economy.

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