ON THREE CLASSICAL RESULTS ABOUT COMPACT GROUPS

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Abstract. In this paper we reprove three classical results on compact groups from Hofmann - Mostert book [8] . Notably we get a very short proof of Borel one point theorem [8, p.310].

In this paper we obtain three classical results from the book "Elements of compact semigroups" by Hofmann - Mostert [8]:
1. [8, p.302] If G is a compact connected n-dimensional topological group such that H^n(G) = Z (Ćech cohomology), then G is a Lie group.
2. [8, p.303] If G is a compact connected group and H closed connected subgroup of G such that dim G/H =1, then H is a normal subgroup of G.
3. Borel one point theorem [8,p.310]. If G is a compact group and H a closed subgroup of G such that G/H is **Q**-acyclic and **Z**/2**Z**-acyclic (Ćech cohomology), then H=G.

Throughout this paper we let G be a compact group. Then G=$\varprojlim$ G$_j$, G$_j$'s, finite dimentional Lie groups, $j \in J$. Let p_j : G $\to$ G$_j$ be the canonical map for all $j \in J$. Hence G=$\varprojlim$ G/ker p_j. We let G_0 be the connected component of 1 $\in$G.

The proof of the first result depends on Capel-Gordh - Mardešic characterization of local connectedness in inverse limits [4,6].

Proposition 1: [8,p.302]

Let G be a compact connected n-dimensional topological group such that $H^n(G)$ = Z(Ćech cohomology) .Then G is a Lie group.

Proof:

We may assume that G/ker p_j is an n-dimensional Lie group for all $j \in J$. By the continuity property of Ćech cohomology [13, p.319] we have $H^n(G) = \varinjlim$ H^n(G/ker p_j). The hypothesis and the universal property of direct limits show that we may further assume that for $k \geq j$, the canonical map G/ker $p_k \to G$/ker p_j has degree 1, hence it induces a surjection on fundamental groups[10] . The homotopy exact sequence of this fibration shows that π_0(ker p_j/ker p_k) = 0 so that G is locally connected [4 or 6] (note that in [6,Theorem 1] one must add the hypothesis that the bonding maps are surjective). Since G is locally connected, Mostert's Theorem [12, Theorem 13] applied to the fiber bundle G $\to$ G/ker p_j shows that the zero dimensional group ker p_j is discrete hence finite and G is a Lie group. //

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The proof of the second result depends on Scheffer's results on maps between topological groups that are homotopic to homomorphisms[11].

Proposition 2: [8,p.303]

Let G be a compact connected group and H a closed connected subgroup of G such that dim G/H =1. Then H is a normal subgroup of G.

Proof:

We may assume that dim G/H ker $p_j = 1$ for all $j \in J$. Fix $j \in J$ and let φ_j :(G/H ker p_j ,H ker $p_j) \to (S^1, 1)$ be a homeomorphism and q_j :G/ker $p_j \to$ G/H ker p_j the canonical map, so that by [11,Theorem 1], there exists f: G/ker $p_j \to \mathbf{R}$ and $g \in$ Hom(G/ker p_j, S^1) such that $\varphi_j \circ q_j = e^{2\pi i f} g$.
Note that H ker p_j/ker $p_j \subseteq$ ker g [11,Corollary 2] and g is onto [9,Corollary 1.9] so that H ker p_j/ker p_j is an open subgroup of (ker $g)_0$. Hence H ker p_j/ker p_j = (ker $g)_0$, a normal subgroup of G/ker p_j , and Hker p_j is a normal subgroup of G. It follows that H=$\cap${H ker $p_j : j \in J$} is a normal subgroup of G[2,TGIII.61]. //

Our new proof of Borel one-point theorem depends on observing that the **Q**-acyclic assumption shows that the decomposition of G as inverse limit as above gives rise to coset spaces that have non-zero Euler characteristic. The $\mathbf{Z}/2\mathbf{Z}$ -acyclic assumption then shows by virtue of the transfer theorem that these successive coset spaces are also $\mathbf{Z}/2\mathbf{Z}$ -acyclic. Therefore they are all trivial. The known proofs of that theorem are very complicated [8].

Theorem 3 : Borel one point theorem [8,p.310]

Let G be a compact group and H a closed subgroup of G such that G/H is **Q-**acyclic and $\mathbf{Z}/2\mathbf{Z}$-acyclic (Ćech cohomology). Then H=G.

Proof:

Since G/H is **Q-**acyclic, it is connected. By [3,TGIII.36] G/H= G_0H/H . Since G_0 H/H $\cong G_0/G_0 \cap H$ we may assume that G is connected . We have G/H=$\varprojlim$ G/H ker p_j .
Claim: χ (G/H ker $p_j) \neq 0$ for all $j \in J$ (χ=Euler Characteristic)

Proof of Claim: Let $j \in J$ and let T_j be a maximal torus subgroup of G/ker p_j . By [8,p.299] there exists M compact connected normal subgroup of G ,M $\cap$ker p_j totally disconnected and M ker p_j =G. Then for N=M $\cap p_j^{-1}(T_j)$ we have $p_j|_{N_0} : N_0 \to T_j$ is onto [3,TGIII.36] . Since ker$(p_j|_{N_0}) \leq$N $\cap$ ker p_j we have dim N_0=dim T_j . Note also that the commutator subgroup (N_0, N_0) is a connected subgroup of ker $(p_j|_{N_0}) \leq$ N$\cap$ ker p_j [3,TGIII.8], hence N_0 is abelian. Since G/H is **Q-**acyclic, the left action of N_0 on G/H has a fixed point[8,p.332] so H $\supseteq$ some conjugate ofN_0 and H ker p_j /ker $p_j \supseteq$ some conjugate of T_j . Therefore χ(G/H ker $p_j) \neq 0$ [14]. //

Let $j \in J$, the continuity property of Ćech cohomology shows that there exists $k \geq j$ such that the canonical map G/H ker $p_k \to$ G/H ker p_j induces the zero map on reduced $\mathbf{Z}/2\,\mathbf{Z}$ cohomology. Since G/H ker p_j are all homotopic to finite CW complexes [15] ,the transfer theorem [5] and the above claim show that G/H ker p_j must be $\mathbf{Z}/2\mathbf{Z}$-acyclic by virtue of [2,Lemma in Theorem 7] . Hence G=H kerp_j .Therefore G=$\bigcap_{j\in J}$ H ker p_j =H [2,TGIII.61]. //

Corollary 4: [1 and 7]

Let G be a compact group, H closed $\leq$ G such that G/H is contractible. Then G=H.

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