MONOTONIC PROPERTIES FOR A SEQUENTIAL DECISION PROBLEM WITH PARTIAL MAINTENANCE ON A MARKOV PROCESS

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Abstract. In the present paper, a sequential decision problem on a Markov process is set up which takes into account a partial maintenance, and observe some monotonic properties of an optimal policy. We develop an optimal maintenance policy for products. During their life cycle, a condition of this item deteriorates, and a state of an item goes from state to state according to a Markovian transition rule based on the stochastic convexity. The decision-maker decides a level of repair with cost which varies with this level. This problem is how much to expend to maintain this item to minimize the total expected cost. A dynamic programming formulation implies a recursive equation about expected cost obtainable under optimal policy.

1 Introduction A sequential decision problem on a Markov process in which states are closely related to outcome is treated in Nakai [6]. In [6], expending an additional amount within a range of the budget improves a state, and the process goes from this state to new state according to a Markovian transition rule based on the total positivity of order two (TP_2). In the present paper, a sequential decision problem on a Markov process is set up which takes into account a partial maintenance to minimize the total expected cost. Especially, the decision-maker selects a level of repair to maintain a condition of this item.

We develop an optimal maintenance policy for products such as electrical devices, cars and so on. During their life cycle, a condition of this item deteriorates, and this condition is represented as an element of a state space $(0, \infty)$. The process goes from a state to a new state according to a Markovian transition rule based on stochastic convexity. For $s \in (0, \infty)$, as s becomes larger, this item complied with user, and it is not sufficiently complied with their demands as s approaches to 0. On the other hand, the decision-maker decides to select a level of repair with cost which varies with this level. This problem is how much to expend to maintain this item to minimize the total expected cost. In this paper, a selection of a level improves a state as a multiplicative manner which is a difference to [6]. A dynamic programming formulation implies a recursive equation about total expected cost obtainable under optimal policy. The purpose of this paper is to observe some monotonic properties of an optimal policy. This is one of a partially observable Markov decision processes such as Monahan[5], Grosfeld-Nir[2], Albright[1], White[10], Itoh and Nakamura[3], Ohnishi, Kawai and Mine[8] for example.

As for a total expected cost obtainable under optimal policy, some monotonic properties are obtained in Nakai [7]. Monotonic properties about an optimal policy are treated in this paper. In Section 2, some essential properties are considered for a case when a decision only makes a transition among states as a preparation. By using this result, monotonic properties concerning an optimal policy are considered in Section 3 under assumptions based on stochastic convexity.

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2 Sequential Decision Problem with Partial Maintenance Consider a product such as electrical devices, cars, etc. During their life cycle, a condition of this item deteriorates. Let $(0, \infty)$ be a state space which represents this condition. A condition becomes better as s increases, and this condition becomes worse as s approaches to 0. Let $u(s)$ be a terminal cost when a problem is in state s, and assume $u(s)$ to be a decreasing function of s. If the process is in state s and a level α is selected to maintain this item ($1 \leq \alpha$), this decision makes a transition from state s to a new state αs with cost $C(\alpha)$. $C(\alpha)$ is assumed to be a non-decreasing and non-negative function of α with $C(1) = 0$. This is a problem to select a level α to maintain this item to minimize the total expected cost.

Initially, we will consider a problem where a decision α only makes a transition from a state in order to prepare for analyzing this problem as a Markov decision process. When the process is in state s at time n, let $w_n(s)$ be a total expected cost obtainable under optimal policy. The principle of the optimality implies recursive equation (1).

$$w_n(s) = \min_{1 \leq \alpha} \{C(\alpha) + w_{n-1}(\alpha s)\}, \tag{1}$$

where $w_1(s) = \min_{1 \leq \alpha}\{C(\alpha) + u(\alpha s)\}$.

Consider a function $u(s)$ defined on $s \in (0, \infty)$ which satisfies an inequality

$$u(s^{\lambda} t^{1-\lambda}) \geq \lambda u(s) + (1 - \lambda) u(t) \tag{2}$$

for any $s < t$ and λ where $0 < \lambda < 1$. Throughout this paper, this function $u(s)$ is termed as P-concave function. In order to observe some monotonic properties concerning an optimal policy, two properties (Lemmas 1 and 2) are obtained concerning P-concave function.

In subsequent discussions, a terminal cost $u(s)$ is assumed to be a P-concave function. It is also assumed that $C(\alpha)$ is an increasing and P-concave function of α, and $C(\alpha) = \log \alpha$ and $u(s) = -s$ satisfy these properties. Since $\lambda x + (1 - \lambda) y \geq x^{\lambda} y^{1-\lambda}$ for $x < y$ and $0 \leq \lambda \leq 1$, if a function $u(s)$ defined on $s \in (0, \infty)$ is a decreasing concave function, then this $u(s)$ is a P-concave function.

Lemma 1 *Let $u(s)$ be a P-concave function. If $s < t, s' < t'$ where $s/t = s'/t'$, then*

$$u(t') - u(s') \leq u(t) - u(s). \tag{3}$$

Proof: Assume $s < t < s' < t'$ $(s, t, s', t' \geq 0)$. Let $s \leq s = s^{\lambda} t^{1-\lambda} \leq t$ for any $0 < \lambda < 1$, then $\lambda u(s) + (1 - \lambda) u(t) \leq u(s^{\lambda} t^{1-\lambda}) = \lambda u(s) + (1 - \lambda) u(s)$, and, therefore, $(u(s) - u(s))/(1 - \lambda) \geq (u(t) - u(s))/\lambda$. Since $\log t - \log s > 0$, $s/s = (t/s)^{1-\lambda}$ and $t/s = (t/s)^{\lambda}$ yield $(u(s) - u(s))/(\log s - \log s) \geq (u(t) - u(s))/(\log t - \log s)$. This inequality implies

$$\frac{u(t) - u(s)}{\log t - \log s} \geq \frac{u(s') - u(t)}{\log s' - \log t} \geq \frac{u(t') - u(s')}{\log t' - \log s'}$$

for any s, t, s', t' where $s < t < s' < t'$ by simple calculations.

Since $s/t = s'/t'$ for $s < t < s' < t'$, an inequality $(u(t) - u(s))/(\log t - \log s) \geq (u(t') - u(s'))/(\log t' - \log s')$ implies Equation (3) since $\log t - \log s = \log t' - \log s' > 0$.

On the other hand, when $s < s' < t < t'$ $(s, t, s', t' \geq 0)$, it is also possible to show inequalities $u(t') - u(t) \leq u(s') - u(s)$ since $s/s' = t/t'$ by a method similar to one used above, i.e. Equation (3), and this completes the proof. □

Lemma 2 *Let $v(s)$ be a function defined by $v(s) = \min_{\alpha \geq 1}\{C(\alpha) + u(\alpha s)\}$. If $u(s)$ is a P-concave function, then $v(s)$ is also a P-concave function.*

Proof: Let $v(s) = C(\widehat{\alpha}) + u(\widehat{\alpha}s)$ and $v(t) = C(\overline{\alpha}) + u(\widehat{\alpha}t)$. Since $\widehat{\alpha}^{\lambda}\overline{\alpha}^{1-\lambda} \geq 1$ and $u(s^{\lambda}t^{1-\lambda}) \geq \lambda u(s) + (1-\lambda)u(t)$ for any λ $(0 < \lambda < 1)$ and $s < t$,

$$\begin{aligned} v(s^{\lambda}t^{1-\lambda}) &= \min_{\alpha \geq 1}\{C(\alpha) + u(\alpha s^{\lambda}t^{1-\lambda})\} \\ &= C(\widehat{\alpha}^{\lambda}\overline{\alpha}^{1-\lambda}) + u(\widehat{\alpha}^{\lambda}\overline{\alpha}^{1-\lambda}s^{\lambda}t^{1-\lambda}) \\ &\geq -(\lambda C(\widehat{\alpha}) + (1-\lambda)C(\overline{\alpha})) + \lambda u(\widehat{\alpha}s) + (1-\lambda)u(\overline{\alpha}t) \\ &= \lambda v(s) + (1-\lambda)v(t) \end{aligned}$$

by an assumption for $C(\alpha)$. This inequality implies $v(s^{\lambda}t^{1-\lambda}) \geq \lambda v(s) + (1-\lambda)v(t)$ for any $0 < \lambda < 1$ and $s < t$. □

If $w_{n-1}(s)$ is a decreasing and P-concave function of s as an induction assumption, then Lemma 2 yields that $w_n(s)$ is also a P-concave function of s. By employing an induction principle on n, $w_n(s)$ is a decreasing function of s as Lemma 3.

Lemma 3 *$w_n(s)$ is a decreasing and P-concave function of s.*

When this problem is in state s at time n, let $\alpha_n(s)$ be an optimal decision of this problem. An optimal decision $\alpha_n(s)$ has following monotonic properties related to a state s and time n as Properties 1 and 2.

Property 1 *$\alpha_n(s)$ increases as s increases.*

Proof: For $s < t$, put $\alpha^* = \alpha_n(s)$ for $n \geq 1$. Since α^* is an optimal solution for s, $C(\alpha^*) + w_{n-1}(\alpha^* s) \leq C(\alpha) + w_{n-1}(\alpha s)$ for any $\alpha \geq 1$. For any $\alpha < \alpha^*$, if

$$C(\alpha^*) + w_{n-1}(\alpha^* t) \leq C(\alpha) + w_{n-1}(\alpha t), \tag{4}$$

then $\alpha_n(t) \geq \alpha^*$.

For any α (≥ 1), an inequality $C(\alpha^*) + w_{n-1}(\alpha^* s) \leq C(\alpha) + w_{n-1}(\alpha s)$ implies $C(\alpha^*) - C(\alpha) \leq w_{n-1}(\alpha s) - w_{n-1}(\alpha^* s)$. If $\alpha^* > \alpha$, Lemma 1 yields $w_{n-1}(\alpha t) - w_{n-1}(\alpha^* t) \geq w_{n-1}(\alpha s) - w_{n-1}(\alpha^* s)$ since $s < t$ and $\alpha^* s/\alpha s = \alpha^* t/\alpha t$. Combining these inequalities implies $C(\alpha^*) - C(\alpha) \leq w_{n-1}(\alpha t) - w_{n-1}(\alpha^* t)$ for any $\alpha < \alpha^*$, and this yields Equation (4).□

Property 2 *$\alpha_n(s)$ increases as n increases.*

Proof: Let $s < t$ and put $\alpha^* = \alpha_n(s)$ for $n \geq 1$, then $w_n(s) = C(\alpha^*) + w_{n-1}(\alpha^* s)$ and $w_n(t) \leq C(\alpha^*) + w_{n-1}(\alpha^* t)$. These equations yield $w_n(t) - w_n(s) \leq w_{n-1}(\alpha^* t) - w_{n-1}(\alpha^* s)$. Since $w_{n-1}(s)$ is a P-concave function of s and $\frac{\alpha^* t}{\alpha^* s} = \frac{t}{s}$ $(s < t, \alpha^* > 1)$, Equation (3) implies $w_{n-1}(\alpha^* t) - w_{n-1}(\alpha^* s) \leq w_{n-1}(t) - w_{n-1}(s)$. Combining these inequalities yields an inequality

$$w_n(t) - w_n(s) \leq w_{n-1}(t) - w_{n-1}(s). \tag{5}$$

Since $\alpha_n(s) = \alpha^*$, if $\alpha^* > \alpha \geq 1$, then $C(\alpha) + w_{n-1}(\alpha s) \geq C(\alpha^*) + w_{n-1}(\alpha^* s)$. Inequality (5) yields $w_n(\alpha s) - w_n(\alpha^* s) \geq w_{n-1}(\alpha s) - w_{n-1}(\alpha^* s)$ since $\alpha^* > \alpha \geq 1$ and $s > 0$, and, therefore, $C(\alpha) + w_n(\alpha s) \geq C(\alpha^*) + w_n(\alpha^* s)$. This yields $\alpha^* \leq \alpha_{n+1}(s)$, and $\alpha_n(s) \leq \alpha_{n+1}(s)$ for any $n \geq 1$ and $s > 0$. □

When the process is in state s at time n, an optimal decision $\alpha_n(s)$ becomes large as s decreases by Property 1, i.e. it is necessary to repair adequately when the condition is good. On the other hand, $\alpha_n(s)$ becomes large as n increases by Property 2, i.e. it is optimal to repair adequately when the residual time is long.

3 Monotonic Properties of Markov Decision Process with Partial Maintenance

3.1 Stochastic Convexity and Concavity Initially, stochastic convexity and concavity are introduced according to Shaked and Shanthikumar [9]. Let $\{X(s)\}_{s\in(-\infty,\infty)}$ be a set of random variables with parameter s, and SICX(stochastically increasing and convex) and SICV(stochastically increasing and concave) are defined as follows.

Definition 1 *If $E[u(X(s))]$ is increasing convex (concave) function of s for any increasing convex (concave) function $u(s)$, then $\{X(s)\}_{s\in(-\infty,\infty)}$ contains to* SICX (SICV).

Consider a set of $\{X(s)\}_{s\in(-\infty,\infty)}$ of random variables with parameter s. Let s_1, s_2, s_3, s_4 be any four values with $s_1 \leq s_2 \leq s_3 \leq s_4$ and $s_1 + s_4 = s_3 + s_2$. Let $X_i = X(s_i)$ be four random variable defined on a common probability space ($i = 1, 2, 3, 4$). Define sample path convexity and concavity as following definitions.

Definition 2 *If $\min\{X_2, X_3\} \leq X_4$ ($X_1 \leq \min\{X_2, X_3\}$) a.s. and $X_2 + X_3 \leq (\geq) X_1 + X_4$ a.s., then $\{X(s)\}_{s\in(-\infty,\infty)}$ contains to* SICX(sp) (SICV(sp)).

Shaked and Shanthikumar [9] show Lemma 4 concerning SICX (SICV) and SICX(sp) (SICV(sp)).

Lemma 4 *If $\{X(s)\}_{s\in(-\infty,\infty)}$ contains to* SICX(sp) (SICV(sp))*, then $\{X(s)\}$ contains to* SICX (SICV)*. If $\{X(s)\}_{s\in(-\infty,\infty)}$ contains to* SICX(sp) (SICV(sp)) *and $u(\cdot)$ is an increasing and convex (concave) function, then $\{u(X(s))\}_{s\in(-\infty,\infty)}$ contains to* SICX(sp) (SICV(sp)).

These Lemmas implies Example 1 which is useful for a problem treated in this paper.

Example 1 *Let $Y(\mu)$ be a normal random variable $N(\mu, \sigma^2)$ with common variance σ^2, then $\{Y(\mu)\}_{\mu\in(-\infty,\infty)}$ contains to* SICX(sp) *and* SICV(sp).

When $X(\mu) = e^{Y(\mu)}$, set $\{X(\mu)\}_{\mu\in(-\infty,\infty)}$ of random variables contains to SICX(sp) *since $u(x) = e^x$ is an increasing and convex function and, therefore, set $\{X(\mu)\}_{\mu\in(-\infty,\infty)}$ of a log-normal random variable $X(\mu)$ contains to* SICX(sp)*, and also* SICX.

Let $u(x)$ be a P-convex function of x, then $w(y) \equiv u(e^y)$ is a convex function of y since $w(\lambda \log a + (1-\lambda)\log b) = u(e^{\lambda \log a + (1-\lambda)\log b}) \leq \lambda u(e^{\log a}) + (1-\lambda)u(e^{\log b}) = \lambda w(a) + (1-\lambda)w(b)$. On the other hand, let $X(s)$ be a log-normal random variable with a density function $f_s(t) = \frac{1}{\sqrt{2\pi}\sigma t} e^{-\frac{(\log t - \log s)^2}{2\sigma^2}} = \frac{\phi_{\log s, \sigma^2}(\log t)}{t}$, where $\phi_{\mu,\sigma^2}(x)$ is a density function of a normal distribution $N(\mu, \sigma^2)$, and $Y(s)$ be a set of random variables $N(s, \sigma^2)$ with common σ^2. It is easy to show

$$\begin{aligned} E[u(X(a^\lambda b^{1-\lambda}))] &= \int_0^\infty f_{a^\lambda b^{1-\lambda}}(t)u(t)dt \\ &= \int_0^\infty \frac{\phi_{\lambda \log a - (1-\lambda)\log b, \sigma^2}(\log t)}{t} u(e^{\log t})dt \\ &= \int_{-\infty}^\infty \phi_{\lambda \log a - (1-\lambda)\log b, \sigma^2}(x)w(x)dx. \end{aligned}$$

By Example 1, since $\{Y(s)\}$ contains to SICX, $E[u(X(a^\lambda b^{1-\lambda}))] = E[w(Y(\lambda \log a - (1-\lambda)\log b))] \leq \lambda E[w(Y(\log a))] + (1-\lambda)E[w(Y(\log b))]$ because $w(y) = u(e^y)$ is a convex function of y. Since $E[w(Y(\log a))] = E[u(X(a))]$ and $E[w(Y(\log b))] = E[u(X(b))]$, $E[u(X(a^\lambda b^{1-\lambda}))] \leq \lambda E[u(X(a))] + (1-\lambda)E[u(X(b))]$, and, therefore, $E[u(X(s))]$ be a P-convex function of x.

Definition 3 *Let $u(s)$ be an increasing and P-convex function. If $E[u(X(s))]$ is an increasing and P-convex function of s, then $\{X(s)\}_{s\in(-\infty,\infty)}$ contains to* SIPCX (*stochastically increasing and P-convex*).

3.2 Markov Decision Process with Partial Maintenance In this section, we treat this sequential decision problem as a Markov decision process, i.e. a decision makes a transition from current state to a new state, and after that, the process goes from this state to a state at the next instant according to a Markovian transition rule $\boldsymbol{P} = (p_s(t))_{s,t\in(0,\infty)}$. Whenever a process is in state s, let $T(s)$ be a random variable which represents a new state after making a transition according to $\boldsymbol{P} = (p_s(t))_{s,t\in(0,\infty)}$.

When a process is in state s, the decision-maker selects a level α to maintain this item ($\alpha \geq 1$), which makes a transition from state s to a state αs with cost $C(\alpha)$. After that, process goes from this state to a next state according to the Markovian transition rule $\boldsymbol{P}$. When the process is in state s, $u(s)$ is a terminal cost at this state, which is decreasing and P-concave function of s. This is a problem to select a level α to maintain this item to minimize the total expected cost. A similar problem is treated in Nakai [6], in which states are closely related to outcome and expending an additional amount makes a transition from a current state.

If a process is in state s at time n, let $v_n(s)$ be a total expected cost obtainable by employing an optimal policy. Since the decision-maker initially selects a level α to maintain this item ($\alpha \geq 1$), this decision makes a transition from current state to a state αs, and after that, this process goes from this state to new state according to $\boldsymbol{P}$. A random variable $T(\alpha s)$ represents this new state of the process. After making a transition from a state, if a process is in state t at the next instant, a total expected cost obtainable under optimal policy is $v_{n-1}(t)$, and, therefore, a total expected cost is $\int_0^\infty p_s(t)v_{n-1}(t)dt = E[v_{n-1}(T(s))]$, when a process is in state s. The principle of the optimality implies the optimality equation

$$v_n(s) = \min_{\alpha\geq 1}\{C(\alpha) + E[v_{n-1}(T(\alpha s))]\}, \tag{6}$$

where $v_1(s) = \min_{\alpha\geq 1}\{C(\alpha) + E[u(T(\alpha s))]\}$.

Assumption 1 *$E[u(T(s))]$ is an increasing and P-concave function of s for any increasing and P-convex function $u(t)$ of t, i.e. $\{T(s)|s \in (0,\infty)\}$ contains to* SIPCX.

If $\{T(s)|s \in (0,\infty)\}$ contains to SIPCX and $u(s)$ be a decreasing and P-concave function, then $E[-u(T(s))]$ is an increasing and P-convex function of s since $-u(s)$ is an increasing and P-convex function, and, therefore, $E[u(T(s))]$ is a decreasing and P-concave function of s. From this fact, for any decreasing and P-concave function $u(t)$ of t, $E[u(T(s))]$ is a decreasing and P-concave function of s under Assumption 1.

Example 2 *Let $p_s(t) = \frac{1}{\sqrt{2\pi}\sigma t}e^{-\frac{(\log t-\log s)^2}{2\sigma^2}} = \frac{\phi_{\log s,\sigma^2}(\log t)}{t}$, where $\phi(x)$ is a density function of a normal distribution $N(\mu,\sigma^2)$, then $p_s(t)$ is a density function of a log-normal distribution, and, therefore, $\{T(s)|s \in (0,\infty)\}$ contains to* SIPCX *for these $p_s(t)$.*

Lemma 5 *$v_n(s)$ is a decreasing and P-concave function.*

Proof: We employ an induction principle on n. Since $v_0(s) = u(s)$, $v_0(s)$ is a decreasing and P-concave function. If we assume $v_{n-1}(s)$ to be a decreasing and P-concave function, then $E[v_{n-1}(T(\alpha s))]$ is also a decreasing and P-concave function of s by Assumption 1. Because $E[v_{n-1}(T(\alpha s))]$ is a decreasing function of s, $v_n(s) = \min_{\alpha\geq 1}\{C(\alpha) + E[v_{n-1}(T(\alpha s))]\}$ is also a decreasing function of s. On the other hand, Lemma 2 yields that $v_n(s)$ is a P-concave function, and this completes the proof. □

Lemma 5 yields that $v_n(s)$ is a decreasing and P-concave function of s. By Assumption 1, $E[v_n(T(s))]$ is a decreasing and P-concave function of s. By Lemma 1 this function is P-concave function.

When the process is in state s at time n, let $\alpha_n^*(s)$ be an optimal decision for this problem, then it has monotonic properties by Lemma 5.

Proposition 1 *If $s \le t$, then $\alpha_n^*(s) \le \alpha_n^*(t)$, i.e. $\alpha_n^*(s)$ increases as s increases.*

Proof: We employ an induction principle on n. A proof of a case for $n = 1$ is derived by a method similar to one used in the general case. For $n(> 1)$, let $\alpha_n^*(s) = \alpha^*$, then Equation (6) yields

$$v_n(s) = \min_{\alpha \ge 1}\{C(\alpha) + E[v_{n-1}(T(\alpha s))]\} = C(\alpha^*) + E[v_{n-1}(T(\alpha^* s))], \tag{7}$$

For any α where $\alpha^* \ge \alpha \ge 1$, if an inequality

$$C(\alpha) + E[v_{n-1}(T(\alpha t))] \ge C(\alpha^*) + E[v_{n-1}(T(\alpha^* t))] \tag{8}$$

is obtained, then $\alpha_n^*(s) = \alpha^* \le \alpha_n^*(t)$, and this completes a proof.

By Equation (7),

$$C(\alpha) + E[v_{n-1}(T(\alpha s))] \ge C(\alpha^*) + E[v_{n-1}(T(\alpha^* s))]$$

for any $\alpha \ge 1$, and, therefore,

$$C(\alpha) - C(\alpha^*) \ge E[v_{n-1}(T(\alpha^* s))] - E[v_{n-1}(T(\alpha s))]. \tag{9}$$

On the other hand, $E[v_{n-1}(T(\alpha s))]$ is a decreasing and P-concave function of s by Assumption 1. Since $\alpha^* t/\alpha^* s = \alpha t/\alpha s$ and $\alpha^* \ge \alpha \ge 1$,

$$E[v_{n-1}(T(\alpha t))] - E[v_{n-1}(T(\alpha s)) \ge E[v_{n-1}(T(\alpha^* t))] - E[v_{n-1}(T(\alpha^* s))]$$

as Equation (3). Combining Inequality (9) and this inequality implies Inequality (8), and this completes the proof. □

In order to consider a monotonic property for an optimal policy concerning n, Assumption 2 is prepared.

Assumption 2 *If $s \le t$, then $E[u(T(t))] - E[u(T(s))] \le u(t) - u(s)$ for any decreasing and P-concave function $u(s)$ of s.*

Since $v_n(s)$ is a decreasing and P-concave function, Assumption 2 yields $E[v_n(T(t))] - E[v_n(T(s))] \le v_n(t) - v_n(s)$ for $s < t$, and if $E[u(T(s))] - u(s)$ decreases as s increases for a increasing and P-concave function $u(t)$, then Assumption 2 is satisfied.

First observe a relationship between $E[v_n(T(t))] - E[v_n(T(s))]$ and $E[v_{n-1}(T(t))] - E[v_{n-1}(T(s))]$ for $s < t$ and $n \ge 1$. Put $\alpha^* = \alpha_n^*(s)$, then

$$\begin{aligned} v_n(t) - v_n(s) &= \min_{\alpha \ge 1}\{C(\alpha) + E[v_{n-1}(T(\alpha t))]\} - \{C(\alpha^*) + E[v_{n-1}(T(\alpha s^*))]\} \\ &\le \quad E[v_{n-1}(T(\alpha^* t))] - E[v_{n-1}(T(\alpha^* s))]. \end{aligned}$$

On the other hand, Lemma 5 yields that $E[v_{n-1}(T(s))]$ is a decreasing and P-concave function of s by Assumption 1. Since $\alpha^* t/\alpha^* s = t/s$ and $s < t, 1 \le \alpha^*$, Equation (3) implies

$$E[v_{n-1}(T(\alpha^* t))] - E[v_{n-1}(T(\alpha^* s))] \le E[v_{n-1}(T(t))] - E[v_{n-1}(T(s))].$$

Combining these inequalities yields an inequality

$$v_n(t) - v_n(s) \leq E[v_{n-1}(T(t))] - E[v_{n-1}(T(s))].$$

By Lemma 5, $v_n(s)$ is a P-concave function, and Assumption 2 yields $E[v_n(T(t))] - E[v_n(T(s))] \leq v_n(t) - v_n(s)$. Combining these two inequalities implies

$$E[v_n(T(t))] - E[v_n(T(s))] \leq E[v_{n-1}(T(t))] - E[v_{n-1}(T(s))], \tag{10}$$

for any $n \geq 1$.

Proposition 2 *If $s \leq t$, then $\alpha_n^*(s) \leq \alpha_{n+1}^*(s)$ for any $n \geq 1$.*

Proof: We employ an induction principle on n. Since it is possible to obtain a proof for a case where $n = 1$ by employing a method similar to one used in the general case, consider for $n(> 2)$.

When $s \leq t$, let $\alpha_n^*(s) = \alpha^*$ for $n > 1$, then

$$C(\alpha) + E[v_{n-1}(T(\alpha s))] \geq C(\alpha^*) + E[v_{n-1}(T(\alpha^* s))]$$

for any $\alpha^* \geq \alpha \geq 1$. On the other hand, Equation (10) implies

$$E[v_{n-1}(T(\alpha^* s))] - E[v_{n-1}(T(\alpha s))] \geq E[v_n(T(\alpha^* s))] - E[v_n(T(\alpha s))],$$

and, therefore,

$$C(\alpha) + E[v_n(T(\alpha s))] \geq C(\alpha^*) + E[v_n(T(\alpha^* s))]. \tag{11}$$

This implies $\alpha^* \leq \alpha_{n+1}^*(s)$ by an induction principle on n, and $\alpha_n^*(s) \leq \alpha_{n+1}^*(s)$ for any $n \geq 1$. □

By this proposition, a monotonic property of $\alpha_n^*(s)$ concerning n is obtained under Assumption 2. When a process is in state s at time n, an optimal decision $\alpha_n^*(s)$ becomes large as s increases, i.e. it is necessary to repair adequately when a condition is good. On the other hand, $\alpha_n^*(s)$ becomes large as n increases, i.e. it is optimal to repair adequately when a residual time becomes long. Section 3 concerns monotonic properties for an optimal policy $\alpha_n^*(s)$ under a stochastic convexity when the process goes from a state to a new state according to a Markovian transition rule. The stochastic convexity is defined for a set of random variable with parameter as $\{T(s)|s \in (0, \infty)\}$, which is different to a stochastic convex order as Shaked and Shanthikumar [9], Kijima and Ohnishi [4] etc.

In this problem, optimal decision varies with a state, which makes a transition from a state, and after that, the process goes from this state to new state according to a Markov transition rule. This implies that an order of decision affects future states and decisions. Let α and α' be two different decisions ($\alpha \neq \alpha' \geq 1$), then $T(\alpha' T(\alpha s))$ is a random variable representing a state after taking decisions initially α and secondary α' when a process is in state s. Whenever $T(s)$ is a log-normal random variable with a density function $f_s(t) = \frac{\phi_{\log s, \sigma^2}(\log t)}{t}$ as Example 2, it is easy to show two random variables $T(\alpha' T(\alpha s))$ and $T(\alpha t(\alpha' s))$ are equivalent for any two decisions α and α', but it is not true in general. Moreover, an optimal decision $\alpha_{n-1}^*(t)$ at the next stage depends on a state t, and an expected cost by this decision is $E[C(\alpha_{n-1}^*(T(s))]$.

When a terminal cost $u(s)$ is assumed to be a decreasing and P-convex function, it is also possible to obtain similar monotonic properties by a method similar to one used in this paper. For this problem, $\{T(s)|s \in (0, \infty)\}$ is assumed to be contained to SIPCV instead of Assumption 1, which is defined by a manner similar to one used in Definition 3, but it is not easy to find a simple example of a set of random variables with a property SIPCV.

By Lemma 5, $v_n(s)$ is a decreasing and P-concave function of s. Finally, consider a property of $v_n(s)$ for n. If $E[u(T(s))] \leq u(s)$, then $v_1(s) \leq u(s)$, since $v_1(s) = \min_{\alpha \geq 1}\{C(\alpha) + E[u(T(\alpha s))]\}$. By employing an induction principle on n, it is easy to show a property that $v_n(s)$ increases as n increases for this case.

Nakai [7] treats a similar problem for a partially observable Markov decision process, and some monotonic properties are observed concerning a total expected cost obtainable under optimal policy. In this paper, monotonic properties concerning optimal policy are obtained as Propositions 1 and 2. But, for a problem on a partially observable Markov process, similar monotonic properties are rest for future observations, since a property similar to Equation (2) is not obtained for this case.

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