Corona rank for C^*-algebras

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Abstract. We introduce a notion of rank for C^*-algebras (or Banach algebras), which is viewed as a replacement of the topological stable rank of Rieffel. We study its basic properties and close relation with the stable rank. We also consider a replacement of the connected stable rank of Rieffel.

1 Introduction

The (topological) stable rank for C^*-algebras (or Banach algebras) was introduced by Rieffel [6], and also the real rank for those was done by Brown and Pedersen [1]. It seems to be known that the definitions of the stable and real ranks are subtly different in a technical sense that the stable rank is defined from viewing an algebra as a (left) module in a sense, while the real rank is done from invertivility of operators in a direct sense. Furthermore, there seems to be a non-trivial step if we view the stable rank as in the sense of Brown and Pedersen.

We then introduce a notion of rank for C^*-algebras (or Banach algebras), that we call the corona rank, which is viewed as a replacement of the topological stable rank of Rieffel by using invertibility of operators. Naming it comes from that we found that this notion is closely related to the corona theorem in function algebra theory ([2]).

In this paper we study the basic properties of the corona rank and its close relation with the stable rank, more precisely, we develop a parallel theory by using the corona rank, as that of the stable rank (or instead of). We also consider a replacement of the connected stable rank of Rieffel, as in that sense of the corona rank. In particular, we show that the corona theorem for C^*-algebras in our sense does hold.

2 Corona rank

Definition 2.1. Let $\mathfrak{A}$ be a unital Banach or C^*-algebra and $\mathfrak{A}^n$ its n-direct sum. Denote by $C_n(\mathfrak{A})$ the set of all elements $(a_j) \in \mathfrak{A}^n$ such that $\sum_{j=1}^n |a_j|$ is invertible in $\mathfrak{A}$.

Remark. It is easily seen that for a unital C^*-algebra $\mathfrak{A}$, $C_1(\mathfrak{A})$ is just the set $\mathfrak{A}_l^{-1}$ of all left invertible elements of $\mathfrak{A}$. Indeed, if $|a|$ is invertible in $\mathfrak{A}$, then $|a|^2 = a^*a$ is also invertible. Conversely, if $a \in \mathfrak{A}_l^{-1}$, use the polar decomposition $a = u|a|$ for a partial isometry u. Hence $|a|$ is invertible in $\mathbb{B}(H)$ and also in $\mathfrak{A}$, where $\mathbb{B}(H)$ is the C^*-algebra of all bounded operators on a Hilbert space H on which $\mathfrak{A}$ may act. Thus $C_1(\mathfrak{A}) = \mathfrak{A}_l^{-1}$.

Definition 2.2. Let $\mathfrak{A}$ be a unital Banach or C^*-algebra. We define the corona rank of $\mathfrak{A}$ to be the least positive integer n such that $C_n(\mathfrak{A})$ is dense in $\mathfrak{A}^n$. Denote $n = \mathrm{cor}(\mathfrak{A})$. If there is no such n, then set $\mathrm{cor}(\mathfrak{A}) = \infty$. For a non-unital Banach or C^*-algebra $\mathfrak{A}$, define its corona rank to be $\mathrm{cor}(\mathfrak{A}^+)$, where $\mathfrak{A}^+$ is the unitization of $\mathfrak{A}$.

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Remark. Note that if $C_n(\mathfrak{A})$ is dense in $\mathfrak{A}^n$, so is $C_{n+1}(\mathfrak{A})$ in $\mathfrak{A}^{n+1}$. Note also that $C_n(\mathfrak{A})$ is an open subset of $\mathfrak{A}^n$. Indeed, the function $\mathfrak{A} \ni x \mapsto |x|$ is continuous because $|x|$ is a uniform limit of polynomials with variables 1 and x (and x^*).

Proposition 2.3. *Let X be a compact Hausdorff space and $C_{\mathbb{R}}(X)$ the Banach algebra of all continuous, real-valued functions on X. Then*

$$\mathrm{cor}(C_{\mathbb{R}}(X)) = \dim X + 1,$$

where $\dim X$ *means the covering dimension of X.*

Proof. It is known in dimension theory for spaces that the covering dimension of X is equal to the smallest non-negative integer n such that every continuous function f from X into $\mathbb{R}^{n+1}$ can be approximated closely by another continuous function g such that $g(X)$ does not contain zero in $\mathbb{R}^{n+1}$, i.e. $0 \notin g(X)$. Note that $0 \notin g(X)$ if and only if $\sum_{j=1}^{n+1} |g_j(x)| > 0$ for every $x \in X$, i.e. $\sum_{j=1}^{n+1} |g_j|$ is invertible in $C(X)$. □

Proposition 2.4. *Let X be a compact Hausdorff space and $C(X)$ the C^*-algebra of all continuous, complex-valued functions on X. Then*

$$\mathrm{cor}(C(X)) = \left[\frac{\dim X}{2}\right] + 1.$$

Proof. Let $(a_j) \in C(X)^n$. The element corresponds to a continuous map from X into $\mathbb{R}^{2n}$. If $\mathrm{cor}(C(X)) \le n$, then $\dim X \le 2n - 1$. Thus,

$$\left[\frac{\dim X}{2}\right] + 1 \le \mathrm{cor}(C(X)),$$

where $[x]$ means the maximum integer $\le x$. Conversely, if $\dim X \le k$ even, then $\mathrm{cor}(C(X)) \le \frac{k}{2} + 1$, and if $\dim X \le k$ odd, then $\mathrm{cor}(C(X)) \le \frac{k+1}{2}$. Thus,

$$\mathrm{cor}(C(X)) \le \left[\frac{\dim X}{2}\right] + 1.$$

□

Denote by $\mathrm{sr}(\mathfrak{A})$ the topological (or Bass) stable rank of a C^*-algebra $\mathfrak{A}$ (see [6] and [4]).

Proposition 2.5. *Let $\mathfrak{A}$ be a C^*-algebra. Then $\mathfrak{A}$ has stable rank one if and only if it has corona rank one.*

Proof. By the definition of $\mathrm{sr}(\mathfrak{A})$, $\mathfrak{A}$ has stable rank one if $\mathfrak{A}_l^{-1}$ is dense in $\mathfrak{A}$. The proof completes by the remark above. □

Corollary 2.6. *Let $\mathfrak{A}$ be a C^*-algebra. Then* $\mathrm{cor}(\mathfrak{A}) \ge 2$ *if and only if* $\mathrm{sr}(\mathfrak{A}) \ge 2$.

As a variation of corona rank,

Definition 2.7. For $p > 0$, define the corona p-rank of a unital Banach or C^*-algebra $\mathfrak{A}$ by replacing $\sum |a_j|$ with $\sum |a_j|^p$. Denote it by $\mathrm{cor}^p(\mathfrak{A})$.

Denote by $C_n^p(\mathfrak{A})$ the set of all $(a_j) \in \mathfrak{A}^n$ such that $\sum_{j=1}^n |a_j|^p$ is invertible in $\mathfrak{A}$. Then $\mathrm{cor}^p(\mathfrak{A}) \le n$ if and only if $C_n^p(\mathfrak{A})$ is dense in $\mathfrak{A}^n$.

Remark. The propositions above also hold for $\mathrm{cor}^p(\cdot)$.

It is shown by Brown and Pedersen [1, Proposition 1.2] that

$$\mathrm{RR}(\mathfrak{A}) \leq 2\,\mathrm{cor}^2(\mathfrak{A}) - 1,$$

for the real rank $\mathrm{RR}(\mathfrak{A})$ of a C^*-algebra $\mathfrak{A}$. Moreover, it seems to be known that $\mathrm{cor}^2(\mathfrak{A}) = \mathrm{sr}(\mathfrak{A})$ for a C^*-algebra $\mathfrak{A}$, by definition. However, showing this exactly seems to involve a non-trivial step. Clear is $C_n^2(\mathfrak{A}) \subset L_n(\mathfrak{A})$ of all elements $(a_j) \in \mathfrak{A}^n$ such that $\sum_j b_j a_j \in \mathfrak{A}^{-1}$ for some $(b_j) \in \mathfrak{A}^n$. In particular, $L_1(\mathfrak{A}) = \mathfrak{A}_l^{-1} = C_1^p(\mathfrak{A})$, but its reverse inclusion seems to be non-trivial.

Corollary 2.8. *Let $\mathfrak{A}$ be a C^*-algebra and $\mathbb{K}$ the C^*-algebra of all compact operators on a separable Hilbert space. Then $\mathrm{cor}^p(\mathfrak{A} \otimes \mathbb{K}) = 1$ if and only if $\mathrm{cor}^p(\mathfrak{A}) = 1$.*

Proof. Use [6, Theorem 3.6] and the remark above. □

3 Corona rank of ideals and quotients

Theorem 3.1. *Let $\mathfrak{A}$ be a C^*-algebra and $\mathfrak{I}$ be a closed ideal of $\mathfrak{A}$. Then*

$$\mathrm{cor}^p(\mathfrak{A}/\mathfrak{I}) \leq \mathrm{cor}^p(\mathfrak{A}).$$

Proof. Since $(\mathfrak{A}/\mathfrak{I})^+ \cong \mathfrak{A}^+/\mathfrak{I}$, it suffices to consider the case where $\mathfrak{A}$ is unital. Let $\pi : \mathfrak{A} \to \mathfrak{A}/\mathfrak{I}$ be the quotient map. Note that for $(a_j) \in \mathfrak{A}^n$,

$$\pi(\sum_j |a_j|^p) = \sum_j \pi((a_j^* a_j)^{p/2}) = \sum_j |\pi(a_j)|^p.$$

Therefore, $C_n^p(\mathfrak{A})$ is mapped into $C_n^p(\mathfrak{A}/\mathfrak{I})$. Thus if $C_n^p(\mathfrak{A})$ is dense in $\mathfrak{A}^n$, then $C_n^p(\mathfrak{A}/\mathfrak{I})$ is also dense in $(\mathfrak{A}/\mathfrak{I})^n$. □

Theorem 3.2. *Let $\mathfrak{A}$ be a C^*-algebra and $\mathfrak{I}$ a closed ideal of $\mathfrak{A}$. Then*

$$\mathrm{cor}^p(\mathfrak{I}) \leq \mathrm{cor}^p(\mathfrak{A}).$$

Proof. We may assume that $\mathfrak{A}$ is unital, otherwise we consider the unitization of $\mathfrak{A}$. Let $\mathrm{cor}^p(\mathfrak{A}) \leq n$. Let $(b_j + \mu_j 1) \in (\mathfrak{I}^+)^n$, where 1 is the unit of $\mathfrak{A}$ and $\mu_j \in \mathbb{C}$. Let $\{i_\alpha\}$ be an approximate identity of $\mathfrak{I}$ with $\|i_\alpha\| \leq 1$. Since $(b_j + \mu_j 1) \in \mathfrak{A}^n$, it can be approximated closely by $(a_j + \mu_j 1) \in \mathfrak{A}^n$ such that $\|b_j - a_j\| < \varepsilon/3$ and $\sum_j |a_j + \mu_j 1|^p$ is invertible in $\mathfrak{A}$. If for $\varepsilon > 0$ (small enough) we take α to have $\|b_j - b_j i_\alpha\| < \varepsilon/3$ for every j, then

$$\begin{aligned}\|a_j - a_j i_\alpha\| &= \|a_j - b_j + b_j - b_j i_\alpha + b_j i_\alpha - a_j i_\alpha\| \\ &\leq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon,\end{aligned}$$

so that $\sum_j |a_j i_\alpha + \mu_j 1|^p$ can be invertible in $\mathfrak{A}$ by continuity of the sum, and $(a_j i_\alpha + \mu_j 1) \in C_n^p(\mathfrak{I}^+)$ as desired. □

Theorem 3.3. *Let X be a compact Hausdorff space. Let $\Gamma(X, \{\mathfrak{A}_t\}_{t\in X})$ be the continuous field C^*-algebra over X with fibers C^*-algebras $\mathfrak{A}_t$. Then*

$$\mathrm{cor}^p(\Gamma(X, \{\mathfrak{A}_t\}_{t\in X})) = \sup_{t\in X} \mathrm{cor}^p(\mathfrak{A}_t).$$

Proof. Let $\mathfrak{A} = \Gamma(X, \{\mathfrak{A}_t\}_{t\in X})$ and let $(f_j) \in \mathfrak{A}^n$. Note that $\sum_j |f_j|^p$ is invertible in $\mathfrak{A}$ if and only if for each $t \in X$, $\sum_j |f_j(t)|^p$ is invertible in $\mathfrak{A}_t$. Thus $(f_j) \in C_n^p(\mathfrak{A})$ if and only if for every $t \in X$, $(f_j(t)) \in C_n^p(\mathfrak{A}_t)$. It follows that $\mathrm{cor}^p(\mathfrak{A}) \leq n$ if and only if $\sup_{t\in X} \mathrm{cor}^p(\mathfrak{A}_t) \leq n$, by using Theorem 3.1 above. See [3] for the definition of a continuous field C^*-algebra. □

Remark. This can be a benefit by using our notion, but not many.

Definition 3.4. Let $\mathfrak{A}$ be a unital C^*-algebra. Define the connected corona rank of $\mathfrak{A}$ to be the least positive integer n such that for any $m \geq n$, the identity matrix connected component $GL_m(\mathfrak{A})_0$ of $GL_m(\mathfrak{A})$ of all $m \times m$ invertible matrices over $\mathfrak{A}$ acts transitively on $C_m(\mathfrak{A})$ by multiplication from the left. Denote $n = \mathrm{ccor}(\mathfrak{A})$.

For a non-unital C^*-algebra $\mathfrak{A}$, its connected corona rank is defined by that of the unitization $\mathfrak{A}^+$.

Also, the connected corona p-rank of $\mathfrak{A}$ is defined by replacing $C_n(\mathfrak{A})$ with $C_n^p(\mathfrak{A})$ and is denoted by $\mathrm{ccor}^p(\mathfrak{A})$.

Remark. Probably it is quite hard to know such a number in general.

Anyway,

Theorem 3.5. *Let $\mathfrak{A}$ be a C^*-algebra and $\mathfrak{I}$ a closed ideal of $\mathfrak{A}$. Then*

$$\mathrm{cor}^p(\mathfrak{A}) \leq \max\{\mathrm{cor}^p(\mathfrak{I}), \mathrm{cor}^p(\mathfrak{A}/\mathfrak{I}), \mathrm{ccor}^p(\mathfrak{A}/\mathfrak{I})\}.$$

Proof. This can be proved by using the argument of [6, Theorem 4.11], exactly the same way. □

4 Inductive limits

Lemma 4.1. *Let $\mathfrak{A} \oplus \mathfrak{B}$ be the direct sum of unital C^*-algebras $\mathfrak{A}$ and $\mathfrak{B}$. Then*

$$\mathrm{cor}^p(\mathfrak{A} \oplus \mathfrak{B}) = \max\{\mathrm{cor}^p(\mathfrak{A}), \mathrm{cor}^p(\mathfrak{B})\}.$$

This also holds when $\mathfrak{A}$ or $\mathfrak{B}$ are non-unital if we replace each with its unitization.

Proof. Let $x = (a, b) \in \mathfrak{A} \oplus \mathfrak{B}$. Since $x^*x = (a^*a, b^*b)$, we have $|x|^p = (|a|^p, |b|^p)$. Thus, $(x_j) = (a_j, b_j) \in C_n^p(\mathfrak{A} \oplus \mathfrak{B})$ if and only if $(a_j) \in C_n^p(\mathfrak{A})$ and $(b_j) \in C_n^p(\mathfrak{B})$. □

Theorem 4.2. *Let $\mathfrak{A}$ be an inductive limit C^*-algebra of C^*-algebras $\mathfrak{A}_k$. Then*

$$\mathrm{cor}^p(\mathfrak{A}) \leq \liminf \mathrm{cor}^p(\mathfrak{A}_k).$$

Proof. We may assume that $\mathfrak{A}$ is unital and each $\mathfrak{A}_n$ has the unit of $\mathfrak{A}$ if necessary by considering the unitization, so that $*$-homomorphisms in the inductive system are also unital. Let $n = \liminf \mathrm{cor}^p(\mathfrak{A}_k)$. Any element (a_j) of $\mathfrak{A}^n$ can be approximated closely by an element $(a_{j,k})_j$ of $\mathfrak{A}_k^n$, where k is large enough and $\mathrm{cor}^p(\mathfrak{A}_k) = n$. Then there is $(b_{j,k})_j \in C_n^p(\mathfrak{A}_k)$ that approximates closely $(a_{j,k})_j$. Its image in $\mathfrak{A}^n$ is in $C_n^p(\mathfrak{A})$ because $C_n^p(\cdot)$ are preserved under unital $*$-homomorphisms. □

Theorem 4.3. *Let $\mathfrak{A}$ be the direct sum of C^*-algebras $\mathfrak{A}_k$, i.e. $\mathfrak{A} = \oplus_{k=1}^\infty \mathfrak{A}_k$ an inductive limit of finite direct sums. Then*

$$\mathrm{cor}^p(\mathfrak{A}) = \sup_k \mathrm{cor}^p(\mathfrak{A}_k).$$

Corollary 4.4. *If* $\mathfrak{A}$ *is an AF* C^**-algebra, i.e. an inductive limit of finite dimensional* C^**-algebras, then* $\mathrm{cor}^p(\mathfrak{A}) = 1$ *for any* $p > 0$.

Proof. Note that a finite dimensional C^*-algebra $\mathfrak{A}_k$ is a finite direct sum of matrix algebras over $\mathbb{C}$, so that $\mathrm{cor}^p(\mathfrak{A}_k) = 1 = \mathrm{sr}(\mathfrak{A}_k)$. Thus the equalities also hold for $\mathfrak{A}$. Or use [6, Proposition 3.5]. □

Corollary 4.5. *If* $\mathfrak{A}$ *is an AT* C^**-algebra, i.e. an inductive limit of finite direct sums of matrix algebras over* $C(\mathbb{T})$ *the* C^**-algebra of all continuous functions on the* 1*-torus, then* $\mathrm{cor}^p(\mathfrak{A}) = 1$ *for any* $p > 0$.

Proof. Note that a finite direct sum of matrix algebras over $C(\mathbb{T})$ has both corona and stable ranks one by using [6, Theorem 6.1]. Note also that the stable rank of $\mathfrak{A}$ is one, so is the corona p-rank. □

5 The corona theorem for C^*-algebras In function algebra theory it is well known that the corona theorem for the unit open disc D holds, i.e., that D is dense in the maximal ideal space of $H^\infty(D)$ the Banach algebra of all bounded holomorphic functions on D with the supremum norm, where each point of D is identified with the point evaluation at the point and its kernel as well, and that the corona theorem is equivalent to say that if any $(f_j) \in H^\infty(D)^n$ satisfies that $\sum_j |f_j| \geq \varepsilon > 0$ on D, then there is $(g_j) \in H^\infty(D)^n$ such that $\sum_j g_j f_j = 1$ (see [2]).

Definition 5.1. We say that the corona theorem holds for a unital (Banach or) C^*-algebra $\mathfrak{A}$ if $C_n(\mathfrak{A})$ is contained in $L_n(\mathfrak{A})$ for any $n \geq 1$, where $(a_j) \in L_n(\mathfrak{A}) \subset \mathfrak{A}^n$ if there is $(b_j) \in \mathfrak{A}^n$ such that $\sum b_j a_j = 1$.

We say that the corona theorem for p holds for a unital C^*-algebra $\mathfrak{A}$ if $C_n^p(\mathfrak{A})$ is contained in $L_n(\mathfrak{A})$ for any $n \geq 1$.

For a non-unital C^*-algebra $\mathfrak{A}$, we say that the corona theorem for p holds for $\mathfrak{A}$ if it holds for the unitization $\mathfrak{A}^+$.

Proposition 5.2. *Suppose that the corona theorem holds for a* C^**-algebra* $\mathfrak{A}$*. Then*

$$\mathrm{sr}(\mathfrak{A}) \leq \mathrm{cor}(\mathfrak{A}).$$

If the corona theorem for p *holds for a* C^**-algebra* $\mathfrak{A}$*. Then*

$$\mathrm{sr}(\mathfrak{A}) \leq \mathrm{cor}^p(\mathfrak{A}).$$

Proof. It follows from assumption that if $C_n(\mathfrak{A})$ is dense in $\mathfrak{A}^n$, then $L_n(\mathfrak{A})$ is dense in $\mathfrak{A}^n$, i.e. $\mathrm{sr}(\mathfrak{A}) \leq n$ by definition. □

Proposition 5.3. *If the corona theorem for* p *holds for a* C^**-algebra* $\mathfrak{A}$*, then*

$$\mathrm{ccor}^p(\mathfrak{A}) \leq \mathrm{csr}(\mathfrak{A}),$$

where $\mathrm{csr}(\mathfrak{A})$ *is the connected stable rank of* $\mathfrak{A}$.

Proof. By definition, if $\mathrm{csr}(\mathfrak{A}) \leq n$, then $GL_m(\mathfrak{A})_0$ acts transitively on $L_m(\mathfrak{A})$ for any $m \geq n$. By assumption, $C_m^p(\mathfrak{A})$ is contained in $L_m(\mathfrak{A})$, so that $GL_m(\mathfrak{A})_0$ acts transitively on $C_m^p(\mathfrak{A})$ as well. □

Corollary 5.4. *If the corona theorem for* p *holds for a* C^**-algebra* $\mathfrak{A}$*, then*

$$\mathrm{ccor}^p(\mathfrak{A}) \leq \mathrm{cor}^p(\mathfrak{A}) + 1.$$

Proof. In addition, use the estimate $\mathrm{csr}(\mathfrak{A}) \leq \mathrm{sr}(\mathfrak{A}) + 1$ by [6, Corollary 4.10]. □

Theorem 5.5. *The corona theorem for p holds for a commutative C^* algebra $\mathfrak{A}$.*

Proof. Indeed, $\mathfrak{A} \cong C(X)$ for a compact Hausdorff space X, where we may assume that $\mathfrak{A}$ is unital. For any element $(f_j) \in C(X)^n$ such that $\sum |f_j|^p \geq \delta 1 > 0$ for some $\delta > 0$ on X, but suppose that $(f_j) \notin L_n(C(X))$, which implies that the (two-sided) closed ideal of $C(X)$ generated by $f_j \in C(X)$ $(1 \leq j \leq n)$ in $C(X)$ is contained in a maximal closed ideal of $C(X)$ that is the kernel of the evaluation map at a point $x \in X$ by Gel'fand theory, which contradicts to that inequality since the sum $\sum |f_j|^p$ belongs to the closed ideal generated by f_j $(1 \leq j \leq n)$. □

Corollary 5.6. *For a compact Hausdorff space X, we have*

$$\mathrm{sr}(C(X)) \leq \mathrm{cor}^p(C(X)) \quad \text{and} \quad \mathrm{ccor}^p(C(X)) \leq \mathrm{csr}(C(X)).$$

More generally, we do have

Theorem 5.7. *The corona theorem for p holds for a C^* algebra $\mathfrak{A}$.*

Proof. We may assume that $\mathfrak{A}$ is unital. For any $(a_j) \in \mathfrak{A}^n$ such that $\sum_j |a_j|^p \geq \delta 1 > 0$ for some $\delta > 0$, but suppose that $(a_j) \notin L_n(A)$, which implies that the closed left ideal $\mathfrak{L}$ generated by a_j $(1 \leq j \leq n)$ does not contain 1. Then $\mathfrak{B} = \mathfrak{L} \cap \mathfrak{L}^*$ is a hereditary C^*-subalgebra of $\mathfrak{A}$ by C^*-algebra theory (see [5]) and $1 \notin \mathfrak{B}$. Since $|a_j|^2 = a_j^* a_j \in \mathfrak{B}$, so that $\sum_j |a_j|^p \in \mathfrak{B}$. Since $\sum_j |a_j|^p$ is positive and invertible, we have its inverse belonging to $\mathfrak{B}$ by spectral theory, so that $1 \in \mathfrak{B}$. This is a contradiction. □

Corollary 5.8. *Let $\mathfrak{A}$ be a C^*-algebra. Then*

$$\mathrm{sr}(\mathfrak{A}) \leq \mathrm{cor}^p(\mathfrak{A}) \quad \text{and} \quad \mathrm{ccor}^p(\mathfrak{A}) \leq \mathrm{csr}(\mathfrak{A}).$$

Also, we obtain

$$\mathrm{ccor}^p(\mathfrak{A}) \leq \mathrm{cor}^p(\mathfrak{A}) + 1.$$

Corollary 5.9. *For a C^*-algebra $\mathfrak{A}$,*

$$\mathrm{cor}^p(\mathfrak{A} \otimes \mathbb{K}) \geq \min\{2, \mathrm{cor}^2(\mathfrak{A})\}, \quad \mathrm{ccor}^p(\mathfrak{A} \otimes \mathbb{K}) \leq 2.$$

Proof. Use [6, Theorem 6.4] and [8, Theorem 3.10] respectively. □

Corollary 5.10. *For a C^*-algebra $\mathfrak{A}$,*

$$\mathrm{cor}^p(\mathfrak{A} \otimes M_n(\mathbb{C})) \geq \lceil \frac{\mathrm{sr}(\mathfrak{A}) - 1}{n} \rceil + 1, \quad \mathrm{ccor}^p(\mathfrak{A} \otimes M_n(\mathbb{C})) \leq \lceil \frac{\mathrm{csr}(\mathfrak{A}) - 1}{n} \rceil + 1$$

where $\lceil x \rceil$ means the maximum integer $\geq x$.

Proof. Use [6, Theorem 6.1] and [7, Theorem 4.7] respectively. □

Example 5.11. Let $\mathfrak{T}$ be the Toeplitz algebra, which is the C^*-algebra generated by the unilateral shift S (or a proper isometry). It is well known that $\mathfrak{T}$ is decomposed into the following exact sequence:

$$0 \to \mathbb{K} \to \mathfrak{T} \to C(\mathbb{T}) \to 0,$$

where $\mathbb{T}$ is the 1-torus (see [5]). Using the results obtained above we have

$$\begin{aligned}\mathrm{cor}^p(\mathfrak{T}) &\le \max\{\mathrm{cor}^p(\mathbb{K}), \mathrm{cor}^p(C(\mathbb{T})), \mathrm{ccor}^p(C(\mathbb{T}))\} \\ &\le \max\{1, 1, \mathrm{csr}(C(\mathbb{T}))\} = 2.\end{aligned}$$

On the other hand, we have $\mathrm{sr}(\mathfrak{T}) \ge 2$ because the Fredholm index theory implies that $\mathfrak{T}^{-1}$ is not dense in $\mathfrak{T}$, equivalently, $L_1(\mathfrak{A}) = \mathfrak{A}_l^{-1}$ is not dense in $\mathfrak{A}$, since the Fredholm index of S is nonzero. Thus we get $\mathrm{cor}^p(\mathfrak{T}) = 2$.

Example 5.12. Let $\mathbb{B}(H)$ be the C^*-algebra of all bounded operators on a Hilbert space H. It is shown by [6, Proposition 6.5] that $\mathrm{sr}(\mathbb{B}(H)) = \infty$ since $\mathbb{B}(H)$ has two orthogonal isometries. Thus $\mathrm{cor}^p(\mathbb{B}(H)) = \infty$.

The same reason implies that the Cuntz algebras O_n $(2 \le n < \infty)$ (and O_∞) generated by n orthogonal isometries with the sum of their range projections equal to 1 (and equal to a subprojection of 1 respectively) have $\mathrm{cor}^p(\cdot)$ infinite.

Corollary 5.13. *Let $\mathfrak{A} \rtimes_\alpha \mathbb{Z}$ be the crossed product of a C^*-algebra $\mathfrak{A}$ by an action α of $\mathbb{Z}$ by automorphisms. Then*

$$\mathrm{ccor}^p(\mathfrak{A} \rtimes_\alpha \mathbb{Z}) \le \mathrm{sr}(\mathfrak{A}) + 1.$$

Proof. Use [6, Corollary 8.6]. □

Corollary 5.14. *Let $\mathfrak{A}$ be a unital C^*-algebra. For all $n \ge \mathrm{cor}^p(\mathfrak{A})$, the map from $GL_n(\mathfrak{A})/GL_n(\mathfrak{A})_0$ to $GL_{n+1}(\mathfrak{A})/GL_{n+1}(\mathfrak{A})_0$ is an isomorphism, so that the K_1-group $K_1(\mathfrak{A})$ of $\mathfrak{A}$ is isomorphic to $GL_n(\mathfrak{A})/GL_n(\mathfrak{A})_0$.*

Proof. Use [7, Theorem 2.10]. □

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