Wavelet estimation for hidden periodic components in spatial series.

Junichi Hirukawa

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Abstract. An overlapped wavelet method is proposed to detect the number and locations of the hidden periodicities in two dimensionally indexed random fields, by checking if the empirical wavelet coefficients of periodogram have significantly large absolute values across fine scale levels. The magnitudes of the amplitudes are also estimated using wavelet coefficients of smaller scale levels than that for the detection of periodicities. The strong consistency of the estimators is established. Some numerical examples are given to test the performance of our method.

1 Introduction. Two dimensional hidden periodic model is an important model in random fields. In this model, the observation consists of two parts; $Y_{n,m} = G_{n,m} + X_{n,m}$, $(n,m) \in \mathbb{N}^2$. The signal part

$$G_{n,m} = \sum_{r=1}^{q} A_r \exp(in\lambda_r + im\mu_r) \tag{1}$$

is a sum of the q sinusoidal signals where $\lambda_r, \mu_r \in (-\pi, \pi]$, $r = 1, \ldots, q$ are the horizontal and vertical frequencies, respectively and A_r, $r = 1, \ldots, q$ are the amplitudes. The noise part $\{X_{n,m}\}$ is a two dimensionally indexed stationary random field with absolutely continuous spectral function $F_X(\lambda, \mu)$ and whose spectral density $f_X(\lambda, \mu)$. Our statistical problem is how to estimate the parameters q, (λ_r, μ_r) and A_r of (1) based on an observation $\{Y_{n,m} : 1 \le n \le N, 1 \le m \le M\}$. Suppose that A_r, $r = 1, \ldots, q$ are zero mean complex random variables with $\mathcal{P}(A_r \neq 0) = 1$, $\sigma_r = E|A_r|^2 < \infty$ and uncorrelated with each other. If A_r, $r = 1, \ldots, q$ are also uncorrelated with noise $\{X_{n,m}\}$, then $\{Y_{n,m}\}$ is a weakly stationary random field with spectral function $F_Y(\lambda, \mu) = \sum_{r=1}^{q} \sigma_r \mathcal{I}[\lambda_r, \mu_r] + F_X(\lambda, \mu)$, where $\mathcal{I}[\lambda_r, \mu_r]$ denotes the indicator function of the interval $[\lambda_r, \infty) \times [\mu_r, \infty)$. Since $F_X(\lambda, \mu)$ is absolutely continuous, $F_Y(\lambda, \mu)$ has precisely q jump points with jump heights σ_r at (λ_r, μ_r). Hidden periodic model has been considered by many authors including

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Hannan (1970). He (1987) gave the strong consistent estimators of the parameters in one dimensional hidden periodic model by checking the magnitudes of the periodogram $J_{Y_N}(\lambda) := (2\pi N)^{-1} \left|\sum_{n=1}^{N} Y_n \exp(in\lambda)\right|^2$ and extended his results to two dimensional case (He (1999)).

Wavelets have been successfully applied to many fields such as data compression, signal analysis and image processing. The recently developed mathematical theory of wavelets has drawn much attention from both statisticians and engineers. In the context of time series, Wang (1995) considered the detection of jumps and sharp cusps by wavelets in a function which is observed contaminated with noise. Luan and Xie (2001) proposed a method to detect the number, locations and heights of jump points of the derivative in the regression model by wavelets. In regard to one dimensional hidden periodic model, Li and Xie (1997) gave the strong consistent estimators of the number and locations of hidden periodicities by following Wang's wavelet method for the identification of jumps and cusps. However, it will be seen that their method does not work in many cases. They proposed to use the empirical wavelet coefficients

$$\alpha_{J_{Y_N}}(j,k) = \int_{-\pi}^{\pi} \psi_{j,k}^{per}(\lambda) J_{Y_N}(\lambda) d\lambda$$

for detection of hidden periodicities, where $\psi(x)$ is so-called mother wavelets (See e.g., Daubechies (1992), Meyer (1992)) and

$$\psi_{j,k}^{per}(x) = \sum_{n\in\mathbb{N}} \sqrt{2\pi}\psi_{j,k}\left(\frac{x+\pi}{2\pi}+n\right) = \sum_{n\in\mathbb{N}} \frac{\sqrt{2\pi}}{2^{\frac{j}{2}}}\psi\left\{2^j\left(\frac{x+\pi}{2\pi}+n\right)-k\right\}$$

is the 2π-extension of wavelets (See Wojtaszczyk (1997)). This coefficient has significantly large absolute value if k is in the set $I(\lambda_r, 2^{-2j}) = \{k : |\widetilde{\lambda}_k - \lambda_r| < 2^{-2j}$, or $|\widetilde{\lambda}_k - \lambda_r| > 2\pi - 2^{-2j},\ k \in I_j\}$, where $\widetilde{\lambda}_k = (k/2^j)2\pi - \pi$ and $I_j = \{0, 1, \ldots, 2^j - 1\}$. But this set will be empty set as $N \to \infty$ (so $j \to \infty$) in many cases, since the sampling interval of k is too wide. For instance, when $\lambda_r = -\frac{\pi}{3}$ this set is an empty set for sufficiently large j, hence their method does not work. So, we have to modify their method.

Let $j, \bar{j}, J$ be natural numbers which tend to infinity with the order $j \ll \bar{j} \ll J$ as $N \to \infty$. We take a nonnegative integer $\tau = 2^{J-j}\zeta_j + \eta_j$ from the set $I_J = \{0, 1, \ldots, 2^J - 1\}$, where ζ_j and η_j are the quotient and the remainder of τ divided by 2^{J-j}. We replace $\widetilde{\lambda}_k$, $k \in I_j$ and $I(\lambda_r, 2^{-2j})$ by $\widetilde{\lambda}_\tau = (\tau/2^J)2\pi - \pi$, $\tau \in I_J$ and $I(\lambda_r, 2^{-\bar{j}}) = \{\tau : |\widetilde{\lambda}_\tau - \lambda_r| < 2^{-\bar{j}}$, or $|\widetilde{\lambda}_\tau - \lambda_r| > 2\pi - 2^{-\bar{j}},\ \tau \in I_J\}$, respectively, then $I(\lambda_r, 2^{-\bar{j}})$ is not an empty set even for large j. This suggests that we employ the following empirical wavelet coefficient

$$\alpha_{J_{Y_N}}^{(\eta_j)}(j,k) = \int_{-\pi}^{\pi} \psi_{j,k}^{(\eta_j)per}(\lambda) J_{Y_N}(\lambda) d\lambda, \quad \psi_{j,k}^{(\eta_j)per}(\lambda) \equiv \psi_{j,k}^{per}\left(\lambda - \frac{2\pi}{2^J}\eta_j\right),$$

as a tool for detection of the hidden periodicities. That is, we take an alternative overlapped wavelet method to detect the hidden periodicities. This method is essentially the same as the so-called maximal overlap discrete wavelet transform (MODWT) (See e.g. Percival and Walden (2000), Nason and Silverman (1995)).

In the first of this paper, we give the always working modification of Li and Xie's method and extend it to two dimensionally indexed random fields. We give the estimators of the number and locations of periodicities. Next, we propose the consistent estimators of the amplitudes. We employ different scale parameters of wavelets for detection of hidden periodicities and for estimation of amplitudes. This method is motivated by Wu and Chu (1993) which employed the kernel-type estimators with different bandwidths for locations of jump points and for corresponding jump heights.

This paper is organised as follows. Section 2 interprets our modified overlap wavelet methods. In Section 3 we describe the model and list several assumptions on two dimensionally indexed noise random fields. Section 4 gives main theoretical results and Section 5 gives numerical examples. All the proofs of the theorems are arranged in Section 6.

2 The overlapped wavelets on $L_2[-\pi,\pi)^2$. Li and Xie (1997) proposed an (usual) wavelet method to detect the hidden periodicities, but their procedure does not work in many cases because sampling interval of k is too wide. Therefore, in this section we introduce the overlapped wavelets on $L_2[-\pi,\pi)^2$ which enable us to always detect the hidden periodicities. In the context of this paper, the following results in Theorem 1 are true if the MODWT is replaced by the standard DWT as done in Li and Xie (1997), but those in Theorem 2 and 3 are not true, so the modifications are essential.

2.1 The ϵ-decimated wavelet transform. Suppose the data $x_0, \ldots, x_{T-1}$ is given. To simplify, let $T = 2^J$ for some integer J. The standard discrete wavelet transform is based on scaling and wavelets filters $\mathcal{G}$ and $\mathcal{H}$, and on 'binary decimation' operator $\mathcal{D}_0$.

The filter $\mathcal{G}$ ($\mathcal{H}$) is a low (high) pass filter defined by a sequence conventionally denoted as $\{g_t\}$ ($\{h_t\}$). The action of the low pass filter is defined by $(\mathcal{G}x)_k = \sum_{t=0}^{T-1} g^\circ_{k-t} x_t$, where $g^\circ_t = \sum_{m\in\mathbb{Z}} g_{t+mT}$, $t = 0, \dots, T-1$. The scaling filter should satisfy the internal orthogonality $\sum_t g_t \overline{g}_{t+2k} = 0$ for all integers $k \neq 0$ and $\sum_t g_t^2 = 1$. The wavelet filter $\mathcal{H}$ is defined by $h_t = (-1)^t \overline{g}_{1-t}$, for all t, where $\overline{\{\cdot\}}$ denotes complex conjugate of $\{\cdot\}$. The binary decimation operator $\mathcal{D}_0$ is simply defined as choosing every even member of a sequence, so $(\mathcal{D}_0 x)_k = x_{2k}$. Then, the mapping of a sequence x to the pair of sequences $(\mathcal{D}_0\mathcal{H}x, \mathcal{D}_0\mathcal{G}x)$ is an orthogonal transformation (See Nason and Silverman (1995)). Let x^J be the original data at level J, that is, $v^J_t = x_t,\ t = 0, \dots, T-1$. Now, for $j = J-1, \dots, 0$, recursively define the smooth v^j at level j and the detail w^j at level j by $v^j = \mathcal{D}_0\mathcal{G}v^{j+1}$ and $w^j = \mathcal{D}_0\mathcal{H}v^{j+1}$, then we have $v^j = (\mathcal{D}_0\mathcal{G})^{J-j}v^J$ and $w^j = \mathcal{D}_0\mathcal{H}(\mathcal{D}_0\mathcal{G})^{J-j-1}v^J$.

Define other binary decimation operator $\mathcal{D}_1$ by $(\mathcal{D}_1 x)_k = x_{2k+1}$, then it is not difficult to show that the mapping $(\mathcal{D}_1\mathcal{H}, \mathcal{D}_1\mathcal{G})$ is still an orthogonal transform. Suppose that $\epsilon_{J-1}, \epsilon_{J-2}, \dots, \epsilon_0$ is a binary sequence of $0, 1$, then we can apply operator $\mathcal{D}_{\epsilon_j}$ at level j. For each choice of the sequence $\epsilon = (\epsilon_{J-1}, \epsilon_{J-2}, \dots, \epsilon_0)$, this will give a different orthogonal transformation of the original sequence. We shall refer to this transformation as the ϵ-decimated discrete wavelet transform. Let $\mathcal{S}$ be the shift operator $(\mathcal{S}x)_t = x_{t+1}$ and τ be the integer which have binary representation $\epsilon_0\epsilon_1 \dots \epsilon_{J-1}$, namely for example $31 = \{11111\}$. For any fixed j, let ζ_j and η_j be the integers with binary representations $\epsilon_0\epsilon_1 \dots \epsilon_{j-1}$ and $\epsilon_j\epsilon_{j+1} \dots \epsilon_{J-1}$, therefore ζ_j and η_j are the quotient and the remainder of τ divided by 2^{J-j}, respectively, that is. $\tau = 2^{J-j}\zeta_j + \eta_j$. Then, in the ϵ-decimated case, we have the smooth $v^j_\epsilon = (\mathcal{D}_0\mathcal{G})^{J-j}\mathcal{S}^{\eta_j}v^J$ and the detail $w^j_\epsilon = \mathcal{D}_0\mathcal{H}(\mathcal{D}_0\mathcal{G})^{J-j-1}\mathcal{S}^{\eta_j}v^J$.

2.2 The continuous form of overlapped discrete wavelet transform. Wavelets are based on so-called scaling functions ϕ which have two key properties. Firstly, $\phi(t)$ and all its integer translates $\phi(t+k)$ form an orthonormal set in L_2, so that $\int \phi(t)\overline{\phi(t+k)}dt = 0$ for all integers $k \neq 0$, and $\int \phi(t)^2 dt = 1$. Secondly, ϕ satisfy the two scale relationship $\phi(t) = \sqrt{2}\sum_k \tilde{g}_k \phi(2t-k)$, where $\tilde{g}_k \equiv \overline{g}_{-k}$. Then, the mother wavelet ψ is defined by $\psi(t) = \sqrt{2}\sum_k \tilde{h}_k \phi(2t-k)$, where $\tilde{h}_k \equiv (-1)^k g_{k+1} = \overline{h}_{-k}$ (See e.g. Daubechies (1992), Meyer (1992)). Now, let $\phi_{j,k}^{per}(\cdot)$ and $\psi_{j,k}^{per}(\cdot)$ be the periodic scaling functions and wavelets

$$\phi_{j,k}^{per}(t) = \sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{2\pi}}\phi_{j,k}\left(\frac{t+\pi}{2\pi}+n\right), \quad \psi_{j,k}^{per}(t) = \sum_{n\in\mathbb{Z}} \frac{1}{\sqrt{2\pi}}\psi_{j,k}\left(\frac{t+\pi}{2\pi}+n\right)$$

(See Wojtaszczyk (1997)) and associate with a sequence v^J to the function $f(t) \in L_2[-\pi,\pi)$ by $\langle f(t), \phi_{J,l}^{per}(t)\rangle_{L_2[-\pi,\pi)} = v_l^J$ and $f(t) \equiv \sum_k v_k^J \phi_{J,k}^{per}(t)$. Then, it is seen that

$$\begin{aligned} v_k^j &= \left((\mathcal{D}_0\mathcal{G})^{J-j}v^J\right)_k = \langle f(t), \phi_{j,k}^{per}(t)\rangle_{L_2[-\pi,\pi)} \\ w_k^j &= \left(\mathcal{D}_0\mathcal{H}(\mathcal{D}_0\mathcal{G})^{J-j-1}v^J\right)_k = \langle f(t), \psi_{j,k}^{per}(t)\rangle_{L_2[-\pi,\pi)}. \end{aligned}$$

Similarly, we have in the ϵ-decimated case

$$\begin{aligned} (v_\epsilon^j)_k &= \left((\mathcal{D}_0\mathcal{G})^{J-j}\mathcal{S}^{\eta_j}v^J\right)_k = \left\langle f(t), \phi_{j,k}^{per}\left(t-\frac{2\pi}{2^J}\eta_j\right)\right\rangle_{L_2[-\pi,\pi)} \\ (w_\epsilon^j)_k &= \left(\mathcal{D}_0\mathcal{H}(\mathcal{D}_0\mathcal{G})^{J-j-1}\mathcal{S}^{\eta_j}v^J\right)_k = \left\langle f(t), \psi_{j,k}^{per}\left(t-\frac{2\pi}{2^J}\eta_j\right)\right\rangle_{L_2[-\pi,\pi)}. \end{aligned}$$

Motivated by the above fact, for wavelets and corresponding scaling functions $\psi, \phi \in L_1(\mathbb{R})\cap L_2(\mathbb{R})$, we define

$$\begin{aligned} \phi_{j,k}^{(\epsilon)per}(t) &\equiv \phi_{j,k}^{per}\left(t-\frac{2\pi}{2^J}\eta_j\right) = \sum_{n\in\mathbb{Z}}\frac{1}{\sqrt{2\pi}}\phi_{j,k}\left(\frac{t+\pi}{2\pi}+n-\frac{\eta_j}{2^J}\right) \\ \psi_{j,k}^{(\epsilon)per}(t) &\equiv \psi_{j,k}^{per}\left(t-\frac{2\pi}{2^J}\eta_j\right) = \sum_{n\in\mathbb{Z}}\frac{1}{\sqrt{2\pi}}\psi_{j,k}\left(\frac{t+\pi}{2\pi}+n-\frac{\eta_j}{2^J}\right), \end{aligned}$$

then for each $\tau \in \{0,1,\ldots,2^J-1\}$, $\{1, \psi_{j,k}^{(\epsilon)per}\}_{j\geq 0, k=0,\ldots,2^j-1}$ forms orthonormal basis (ONB) in $L_2[-\pi,\pi)$. Since $\phi_{j,k}^{(\epsilon)per}(t)$ and $\psi_{j,k}^{(\epsilon)per}(t)$ depends on τ only through η_j at level j, we can represent them as $\phi_{j,k}^{(\eta_j)per}(t)$ and $\psi_{j,k}^{(\eta_j)per}(t)$. Then, we have for $f(t) \in L_2[-\pi,\pi)$,

$$f(t) = \sum_{k=0}^{2^{j_0}-1}(v_\epsilon^{j_0})_k\phi_{j_0,k}^{(\eta_{j_0})per}(t) + \sum_{j\geq j_0}\sum_{k=0}^{2^j-1}(w_\epsilon^j)_k\psi_{j,k}^{(\eta_j)per}(t).$$

If we retake $\tau_1 = 2^{J-j}k+\eta_j$, then τ_1 takes the value in the set I_J and we have the maximal overlap wavelet coefficients

$$\begin{aligned} (w_\epsilon^j)_k &= \frac{\sqrt{2\pi}}{2^{\frac{j}{2}}}\int_{-\infty}^{\infty} f\left(\frac{2\pi}{2^j}(x+k)+\frac{2\pi}{2^J}\eta_j-\pi\right)\psi(x)dx \\ &= \frac{\sqrt{2\pi}}{2^{\frac{j}{2}}}\int_{-\infty}^{\infty} f\left(\frac{2\pi}{2^j}x+\frac{2\pi}{2^J}\tau_1-\pi\right)\psi(x)dx. \end{aligned}$$

Usually, one needs boundary corrections at $-\pi$ and π. However, since our objects are 2π-period functions, we need no boundary corrections.

2.3 The extension of maximal overlap discrete wavelet transform to spatial series. Let $\tau^{(i)}$, $i = 1, 2$ be the integers whose binary representations are $\epsilon_0^{(i)}\epsilon_1^{(i)}\dots\epsilon_{J-1}^{(i)}$. Consider any fixed j and let $\zeta_j^{(i)}$ and $\eta_j^{(i)}$ be the integers whose binary representations $\epsilon_0^{(i)}\epsilon_1^{(i)}\dots\epsilon_{j-1}^{(i)}$ and $\epsilon_j^{(i)}\epsilon_{j+1}^{(i)}\dots\epsilon_{J-1}^{(i)}$, namely $\tau^{(i)} = 2^{J-j}\zeta_j^{(i)} + \eta_j^{(i)}$. For wavelets and corresponding scaling functions $\psi^{(i)}, \phi^{(i)} \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$, $i = 1, 2$, define

$$\phi_{j,k}^{(i,\eta_j^{(i)})per}(t) \equiv \phi_{j,k}^{(i)per}\left(t - \frac{2\pi}{2^J}\eta_j^{(i)}\right), \quad \psi_{j,k}^{(i,\eta_j^{(i)})per}(t) \equiv \psi_{j,k}^{(i)per}\left(t - \frac{2\pi}{2^J}\eta_j^{(i)}\right)$$

and

$$\begin{aligned}
f_{1\,(j,k,l)}^{(\eta_j)per}(t_1,t_2) &\equiv \psi_{j,k}^{(1,\eta_j^{(1)})per}(t_1)\psi_{j,l}^{(2,\eta_j^{(2)})per}(t_2),\\
f_{2\,(j,k,l)}^{(\eta_j)per}(t_1,t_2) &\equiv \psi_{j,k}^{(1,\eta_j^{(1)})per}(t_1)\phi_{j,l}^{(2,\eta_j^{(2)})per}(t_2),\\
f_{3\,(j,k,l)}^{(\eta_j)per}(t_1,t_2) &\equiv \phi_{j,k}^{(1,\eta_j^{(1)})per}(t_1)\psi_{j,l}^{(2,\eta_j^{(2)})per}(t_2),\\
\phi_{(j,k,l)}^{(\eta_j)per}(t_1,t_2) &\equiv \phi_{j,k}^{(1,\eta_j^{(1)})per}(t_1)\phi_{j,l}^{(2,\eta_j^{(2)})per}(t_2),
\end{aligned}$$

where $(\eta_j) \equiv (\eta_j^{(1)}, \eta_j^{(2)})$. Then, the system $\{1, f_{i\,(j,k,l)}^{(\eta_j)per}(t_1,t_2)\}_{j\ge 0, k,l=0,\dots,2^j-1, i=1,2,3}$ forms ONB in $L_2[-\pi,\pi)^2$ for each $(\eta_j^{(1)}, \eta_j^{(2)})$. Therefore, for any $g(x_1,x_2) \in L_2[-\pi,\pi)^2$, we have

$$\begin{aligned}
g(x_1,x_2) &= \sum_{k=0}^{2^{j_0}-1}\sum_{l=0}^{2^{j_0}-1} \beta^{(\eta_{j_0})}(j_0,k,l)\phi_{(j_0,k,l)}^{(\eta_{j_0})per}(x_1,x_2)\\
&\quad + \sum_{j\ge j_0}\left\{\sum_{i=1,2,3}\sum_{k=0}^{2^j-1}\sum_{l=0}^{2^j-1}\alpha_i^{(\eta_j)}(j,k,l)f_{i\,(j,k,l)}^{(\eta_j)per}(x_1,x_2)\right\}
\end{aligned}$$

with

$$\begin{aligned}
\alpha_i^{(\eta_j)}(j,k,l) &= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f_{i\,(j,k,l)}^{(\eta_j)per}(t_1,t_2)g(t_1,t_2)dt_1dt_2,\\
\beta^{(\eta_j)}(j,k,l) &= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} \phi_{(j,k,l)}^{(\eta_j)per}(t_1,t_2)g(t_1,t_2)dt_1dt_2.
\end{aligned}$$

Let $\tau_1 = 2^{J-j}k + \eta_j^{(1)}$, $\tau_2 = 2^{J-j}l + \eta_j^{(2)}$, so $\tau_1, \tau_2 \in I_J$ and write $\widetilde{\lambda}_{\tau_1} = \frac{2\pi}{2^J}\tau_1 - \pi$, $\widetilde{\mu}_{\tau_2} = \frac{2\pi}{2^J}\tau_2 - \pi$. Then, it is easily shown that the maximal overlap wavelet coefficients become

$$\begin{aligned}
(2)\qquad \gamma_g^{(\eta_j)}(j,k,l) &\equiv \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\varphi_{(j,k,l)}^{(\eta_j)per}(t_1,t_2)g(t_1,t_2)dt_1dt_2\\
&= \frac{2\pi}{2^j}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi(x_1,x_2)g\left(\frac{2\pi}{2^j}x_1 + \frac{2\pi}{2^J}\tau_1 - \pi, \frac{2\pi}{2^j}x_2 + \frac{2\pi}{2^J}\tau_2 - \pi\right)dx_1dx_2\\
&= \frac{2\pi}{2^j}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\varphi(x_1,x_2)g\left(\frac{2\pi}{2^j}x_1 + \widetilde{\lambda}_{\tau_1}, \frac{2\pi}{2^j}x_2 + \widetilde{\mu}_{\tau_2}\right)dx_1dx_2\\
&\equiv \gamma_g(j, \widetilde{\lambda}_{\tau_1}, \widetilde{\mu}_{\tau_2}),
\end{aligned}$$

where $\varphi = \phi$ or f_i, $i = 1, 2, 3$, and $\gamma = \beta$ or α_i, $i = 1, 2, 3$.

In the following, we use the mother wavelet and the scaling function introduced by Y. Meyer (See e.g. Meyer (1992), Wojtaszczyk (1997)), which satisfies the following Assumptions 1 and 2:

Assumption 1. *Let* $\varphi^{(i)} = \phi^{(i)}$ *or* $\psi^{(i)}$ *and* $\varphi^{(i)} \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$, $i = 1, 2$.

(i) *Fourier transform of* $\varphi^{(i)}$ $\widehat{\varphi}^{(i)}(\omega_i)$ *are compactly supported on finite interval* $[-C, C]$ *and Lipschitz continuous on* $\mathbb{R}$.

(ii) $\widehat{\varphi}^{(i)}(-\omega_i) = \widehat{\varphi}^{(i)}(\omega_i)$ *and* $\int_{-C}^{C} \widehat{\varphi}^{(i)}(\omega_i) d\omega_i \neq 0$.

(iii) $|\varphi^{(i)}(x_i)| \leq \frac{C'}{(1+|x_i|)^u}$ *for some* $u \geq 3$.

In the followings we employ the scaling function satisfying following Assumption 2, which enables us to estimate the amplitudes.

Assumption 2. $0 \leq \widehat{\phi}^{(i)}(\omega_i) \leq \frac{1}{\sqrt{2\pi}}$, $i = 1, 2$.

3 Hidden periodic model in two dimensionally indexed random field. In this section we consider the two dimensional hidden periodic model

$$Y_{n,m} = \sum_{r=1}^{q} A_r \exp(in\lambda_r + im\mu_r) + X_{n,m}, \quad (n, m) \in \mathbb{N}^2, \tag{3}$$

where $(\lambda_r, \mu_r) \in [-\pi, \pi)$, $r = 1, \ldots, q$ are constant vectors, called hidden periodicities, and q is the number of hidden periodicities which is an unknown nonnegative integer.

In order to lead to the consistency of our estimators, we need some assumptions on the noise $\{X_{n,m}\}$. For $s = (s_1, s_2)$, $t = (t_1, t_2) \in \mathbb{N}^2$, we will assume the usual partial order, i.e., $s \leq t$ means $s_i \leq t_i$, $i = 1, 2$ and $s < t$ means $s \leq t$ but $s \neq t$. For $i = 1, 2$, $s \leq^{(i)} t$ means $s_i < t_i$, or $s_i = t_i$ and $s_{3-i} \leq t_{3-i}$. $\left\{\mathcal{F}_t : t \in \mathbb{N}^2\right\}$ is said to be an increasing array of σ-fields under $\leq$, if $s \leq t$ implies $\mathcal{F}_s \subseteq \mathcal{F}_t$. For any increasing array of σ-fields $\left\{\mathcal{F}_t : t \in \mathbb{N}^2\right\}$, define $\mathcal{F}_{t-} = \vee_{s<t}\mathcal{F}_s = \max\left\{\mathcal{F}_s : s < t\right\}$, $\mathcal{F}_i(t_1, t_2) = \vee_{s \leq^{(i)} t}\mathcal{F}_s = \max\left\{\mathcal{F}_s : s \leq^{(i)} t\right\}$, $i = 1, 2$, and $\mathcal{F}_1(t-) = \mathcal{F}_1(t_1, t_2 - 1)$, $\mathcal{F}_2(t-) = \mathcal{F}_2(t_1 - 1, t_2)$. A random field $\left\{W_t : t \in \mathbb{N}^2\right\}$ is said adapted to $\left\{\mathcal{F}_t : t \in \mathbb{N}^2\right\}$ (or simply $\left\{W_t, \mathcal{F}_t : t \in \mathbb{N}^2\right\}$ is an adapted random field) if W_t is $\mathcal{F}_t$-measurable for each $t \in \mathbb{N}^2$. An adapted random field $\left\{W_t, \mathcal{F}_t : t \in \mathbb{N}^2\right\}$ is called a 1/4 martingale difference (MD) if $E(W_t|\mathcal{F}_{t-}) = 0$, a.s., a 1/2 leftward martingale difference (LMD) if $E(W_t|\mathcal{F}_1(t-)) = 0$, a.s. and a 1/2 rightward martingale difference (RMD) if $E(W_t|\mathcal{F}_2(t-)) = 0$, a.s., for any $t \in \mathbb{N}^2$. For any integrable random variable W if it is true that $E(E(W|\mathcal{F}_s)|\mathcal{F}_t) = E(W|\mathcal{F}_{s\wedge t})$, for any $s,t \in \mathbb{N}^2$, then we say that $\{\mathcal{F}_t : t \in \mathbb{N}^2\}$ satisfies F_4-condition, where $s \wedge t = (\min(s_1,t_1), \min(s_2,t_2))$. A 1/2 LMD (or RMD) with finite identical variance will be a white noise. But F_4-condition is required for a 1/4 MD with finite identical variance to be a white noise.

Suppose that $\{W_t : t \in \mathbb{N}^2\}$ is a real white noise random field; that is, $E(W_t) = 0$, $E(W_t^2) = \sigma^2 < \infty$ and $E(W_s W_t) = 0$, for $s \neq t$, and let $\{X_{n,m}\}$ be a linear random field given by $X_{n,m} = \sum_{t_1=0}^{\infty}\sum_{t_2=0}^{\infty} d(t_1,t_2) W_{n-t_1,m-t_2}$, where $d(t_1,t_2)$ are real constants with $\sum_{t_1=0}^{\infty}\sum_{t_2=0}^{\infty}(t_1+t_2)|d(t_1,t_2)| < \infty$. For a white noise random field $\mathbf{W} = \left\{W_t : t \in \mathbb{N}^2\right\}$, $\mathcal{F}_t = \sigma\{W_s : s \leq t\}$ denotes the information obtained by observing $\left\{W_t : t \in \mathbb{N}^2\right\}$ up to time t and we write $\mathcal{F}_{(-\infty,0)} = \cap_{t_1<0}\mathcal{F}(t_1,0)$ and $\mathcal{F}_{(0,-\infty)} = \cap_{t_2<0}\mathcal{F}_{(0,t_2)}$. We introduce the following five assumptions which are due to He (1995).

Assumption 3. *Let* $\mathbf{W}$ *be white noise random field and satisfy one of the following conditions;*

(i) $\mathbf{W}$ *is independent and there exists a nonnegative random variable* Y *such that* $E\left(Y^2\log Y\right) < \infty$*, and for all* $t \in \mathbb{N}_+^2$, $x \geq 0$, $P(|W_t| \geq x) \leq CP(Y \geq x)$.

(ii) $\mathbf{W}$ *is a strictly stationary* 1/4 *MD white noise with*

$$E\left(W_0 \log|W_0|\right)^2 < \infty, \quad E\left(W_0^2|\mathcal{F}_{(-\infty,0)}\right) = E\left(W_0^2|\mathcal{F}_{(0,-\infty)}\right) = \sigma^2.$$

(iii) $\mathbf{W}$ *is a strictly stationary ergodic* 1/4 *MD white noise with*

$$E\left(W_0^2\log|W_0|\right) < \infty. \tag{4}$$

(iv) $\mathbf{W}$ *is a strictly stationary ergodic* 1/2 *LMD white noise satisfying (4).*

(v) $\mathbf{W}$ *is a strictly stationary ergodic* 1/2 *RMD white noise satisfying (4).*

Now, we have the following lemma for a linear random field $\{X_{n,m}\}$.

Lemma 1. *Write* $S_X(\lambda,\mu,N,M)=\sum_{n=1}^{N}\sum_{m=1}^{M}X_{n,m}\exp(-in\lambda)\exp(-im\mu)$ *and suppose* $\{W_{t_1,t_2}\}$ *satisfies one of the conditions of Assumption 3, then*

$$\limsup_{N,M\to\infty}(NM\log(NM))^{-1/2}\sup_{\lambda,\mu}|S_X(\lambda,\mu,N,M)|\le 4\pi(\sup_{\lambda,\mu}f_X(\lambda,\mu))^{1/2}\quad a.s.,$$

where $f_X(\lambda,\mu)$ *is the spectral density function of* $\{X_{n,m}\}$.

The proof of Lemma 1 may be found in He (1995). Furthermore, We define the periodogram of $\{X_{n,m}\}$ as $J_X(\lambda,\mu,N,M)=\frac{1}{4NM\pi^2}|S_X(\lambda,\mu,N,M)|^2$.

Corollary 1. *With the same condition as that of Lemma 1, we have*

$$\limsup_{N,M\to\infty}\frac{1}{\log(NM)}\sup_{\lambda,\mu}(J_X(\lambda,\mu,N,M))\le 4(\sup_{\lambda,\mu}f_X(\lambda,\mu))\quad a.s.$$

For convenience, we give an alphabetical order for the periodicities; that is, for $r_1<r_2$, we suppose either $\lambda_{r_1}<\lambda_{r_2}$, or $\lambda_{r_1}=\lambda_{r_2}$ and $\mu_{r_1}<\mu_{r_2}$, and for $q=0$ we define $\sum_{r=1}^{0}=0$. In the followings, we assume that all the amplitudes A_r, $r=1,\ldots,q$ are positive constants, which is required for simplifying the problem of estimation for the amplitudes, otherwise $\sigma_r=E|A_r|^2$ should be estimated. With regard to the estimation for the number and the locations of hidden periodicities, we can show the same results even in the case that A_r, $r=1,\ldots,q$ are zero mean random variables uncorrelated with each other and $\{X_{n,m}\}$, and $|A_r|^2>A$, $r=1,\ldots q$ a.s. with a known constant A.

4 Main theoretical results.

Assume the observation $\{Y_{n,m}:1\le n\le N,1\le m\le M\}$ is obtained from the model (3) and write $G(\lambda,\mu,N,M)=\sum_{n=1}^{N}\sum_{m=1}^{M}\sum_{r=1}^{q}A_r\exp(in(\lambda_r-\lambda)+im(\mu_r-\mu))$, then the periodogram of $\{Y_{n,m}\}$ is given by

$$\begin{aligned}(5)\qquad J_Y(\lambda,\mu,N,M)&=\frac{1}{4NM\pi^2}\Big|\sum_{n=1}^{N}\sum_{m=1}^{M}Y_{n,m}\exp(-in\lambda)\exp(-im\mu)\Big|^2\\&=\frac{1}{4NM\pi^2}\Big|G(\lambda,\mu,N,M)\Big|^2+\frac{1}{2NM\pi^2}\Re\Big\{\Big(G(\lambda,\mu,N,M)\Big)S_X(\lambda,\mu,N,M)\Big\}\\&\quad+J_X(\lambda,\mu,N,M)\equiv G_1+G_2+J_X\quad(say),\end{aligned}$$

where $\Re\{\cdot\}$ denotes the real part of $\{\cdot\}$. The empirical maximal overlap wavelet coefficients of the periodogram $J_Y(\lambda,\mu,N,M)$ using scaling function are given by equation (2) with

$\gamma_g = \beta_{J_Y}$, namely

$$\begin{aligned}\beta_{J_Y}^{(\eta_j)}(j,k,l) &\equiv \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\phi_{(j,k,l)}^{(\eta_j)per}(\lambda,\mu)J_Y(\lambda,\mu,N,M)d\lambda d\mu \\ &= \frac{2\pi}{2^j}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(x_1,x_2)J_Y\left(\frac{2\pi}{2^j}x_1+\widetilde{\lambda}_{\tau_1},\frac{2\pi}{2^j}x_2+\widetilde{\mu}_{\tau_2},N,M\right)dx_1dx_2 \\ &\equiv \beta_{J_Y}(j,\widetilde{\lambda}_{\tau_1},\widetilde{\mu}_{\tau_2}),\end{aligned}$$

where $\tau_1 = 2^{J-j}k+\eta_j^{(1)}$ and $\tau_2 = 2^{J-j}l+\eta_j^{(2)}$, which may be considered as a tool for detection of the hidden periodicities. At first glance, the condition of orthogonality on wavelet function seems to be restrictive. However, once one would like to separately detect jumps and cusps in frequency domain, this condition will be required.

4.1 The magnitudes of empirical overlap wavelet coefficients. To lead to consistency of our estimators, we first investigate the magnitudes of empirical overlap wavelet coefficients taken on several sets of lattices. Let $K>0$ and define the sets of lattices as

$$\begin{aligned}I_{r,\lambda}\left(\frac{2\pi K}{2^j}\right) &\equiv \left\{\tau_1 : |\widetilde{\lambda}_{\tau_1}-\lambda_r| \le \frac{2\pi K}{2^j} \text{ or } |\widetilde{\lambda}_{\tau_1}-\lambda_r| \ge 2\pi - \frac{2\pi K}{2^j}, \tau_1 \in I_J\right\}, \\ I_{r,\mu}\left(\frac{2\pi K}{2^j}\right) &\equiv \left\{\tau_2 : |\widetilde{\mu}_{\tau_2}-\mu_r| \le \frac{2\pi K}{2^j} \text{ or } |\widetilde{\mu}_{\tau_2}-\mu_r| \ge 2\pi - \frac{2\pi K}{2^j}, \tau_2 \in I_J\right\}, \\ I_r\left(\frac{2\pi K}{2^j}\right) &= I_r\left(\lambda_r,\mu_r,\frac{2\pi K}{2^j}\right) \\ &\equiv \left\{(\tau_1,\tau_2) : \tau_1 \in I_{r,\lambda}\left(\frac{2\pi K}{2^j}\right) \text{ and } \tau_2 \in I_{r,\mu}\left(\frac{2\pi K}{2^j}\right), \tau_1,\tau_2 \in I_J\right\}, \\ E\left(\frac{2\pi K}{2^j}\right) &\equiv \bigcap_{r=1}^{q}\left\{(\tau_1,\tau_2) : \begin{array}{c}\frac{2\pi K}{2^j} \le |\widetilde{\lambda}_{\tau_1}-\lambda_r| \le 2\pi - \frac{2\pi K}{2^j} \\ \text{or} \\ \frac{2\pi K}{2^j} \le |\widetilde{\mu}_{\tau_2}-\mu_r| \le 2\pi - \frac{2\pi K}{2^j}\end{array}, \tau_1,\tau_2 \in I_J\right\}.\end{aligned}$$

We assume the sample size satisfies the following order.

Assumption 4.

$$\lim_{N,N\to\infty} 2^j N^{-1} = 0, \lim_{N,M\to\infty} 2^j M^{-1} = 0, \lim_{N,M\to\infty} 2^{-j}\left(NM\log NM\right)^{\frac{1}{4}} = 0.$$

Then, we have the following results for the magnitudes of empirical wavelet coefficients.

Theorem 1. *Suppose that Assumptions 1-4 hold, then as $N,M\to\infty$, we have the followings.*

1 *For all* $(\tau_1,\tau_2)\in E\left(\frac{2\pi K}{2^j}\right)$*, where* $K\to\infty$*, as* $N,M\to\infty$*, we have*

$$\begin{aligned}\beta_{J_Y}(j,\widetilde{\lambda}_{\tau_1},\widetilde{\mu}_{\tau_2}) &= O\left(2^jK^{-1}+2^{2j}N^{-1}+2^{2j}M^{-1}+2^{-j}\sqrt{NM\log NM}\right)\\ &= o(2^j)\quad a.s.\end{aligned}$$

2 (a) *If* $(|\widetilde{\lambda}_{\tau_1}-\lambda_r| = \frac{2\pi}{2^j}K_1$ *or* $|\widetilde{\lambda}_{\tau_1}-\lambda_r| = 2\pi-\frac{2\pi}{2^j}K_1)$*, and* $(|\widetilde{\mu}_{\tau_2}-\mu_r| = \frac{2\pi}{2^j}K_2$ *or* $|\widetilde{\mu}_{\tau_2}-\mu_r| = 2\pi-\frac{2\pi}{2^j}K_2)$*, for some constants* $K_1,K_2>0$*, then*

$$\begin{aligned}\beta_{J_Y}(j,\widetilde{\lambda}_{\tau_1},\widetilde{\mu}_{\tau_2}) =&\frac{2^j|A_r|^2}{(2\pi)^2}\int_{-C}^{C}\int_{-C}^{C}\cos K_1\omega_1\cos K_2\omega_2\widehat{\phi}(\omega_1,\omega_2)d\omega_1d\omega_2\\ &+o(2^j)\quad a.s.\end{aligned}$$

(b) *If* $(|\widetilde{\lambda}_{\tau_1}-\lambda_r| = \frac{2\pi}{2^j}K_1$ *or* $|\widetilde{\lambda}_{\tau_1}-\lambda_r| = 2\pi-\frac{2\pi}{2^j}K_1)$*, and* $\tau_2\in I_{r,\mu}\left(\frac{2\pi K}{2^j}\right)$*,* $\Big($*or if* $(|\widetilde{\mu}_{\tau_2}-\mu_r| = \frac{2\pi}{2^j}K_1$ *or* $|\widetilde{\mu}_{\tau_2}-\mu_r| = 2\pi-\frac{2\pi}{2^j}K_1)$*, and* $\tau_1\in I_{r,\lambda}\left(\frac{2\pi K}{2^j}\right)$*,*$\Big)$ *where* K_1 *is some constant and* $K\to 0$*, as* $N,M\to\infty$*, then*

$$\begin{aligned}\beta_{J_Y}(j,\widetilde{\lambda}_{\tau_1},\widetilde{\mu}_{\tau_2}) =&\frac{2^j|A_r|^2}{(2\pi)^2}\int_{-C}^{C}\int_{-C}^{C}\cos K_1\omega_a\widehat{\phi}^{(a)}(\omega_a)\widehat{\phi}^{(b)}(\omega_b)d\omega_ad\omega_b\\ &+o(2^j)\quad a.s.,\end{aligned}$$

where $(a,b)=(1,2)$ *or* $(2,1)$*.*

3 *For all* $(\tau_1,\tau_2)\in I_r\left(\frac{2\pi K}{2^j}\right)$ *where* $K\to 0$*, as* $N,M\to\infty$*, we have*

(a)

$$\beta_{J_Y}(j,\widetilde{\lambda}_{\tau_1},\widetilde{\mu}_{\tau_2}) = \frac{2^j|A_r|^2}{(2\pi)^2}\int_{-C}^{C}\int_{-C}^{C}\widehat{\phi}(\omega_1,\omega_2)d\omega_1d\omega_2+o(2^j)\quad a.s.$$

(b)

$$\begin{aligned}\beta_{J_Y}(j,\widetilde{\lambda}_{\tau_1},\widetilde{\mu}_{\tau_2}) \le&\frac{2^j|A_r|^2}{(2\pi)^2}\int_{-C}^{C}\int_{-C}^{C}\widehat{\phi}(\omega_1,\omega_2)d\omega_1d\omega_2\\ &+O\left(1+2^{2j}N^{-1}+2^{2j}M^{-1}+2^{-j}\sqrt{NM\log NM}\right)\quad a.s.\end{aligned}$$

(c)

$$\begin{aligned}\beta_{J_Y}(j,\widetilde{\lambda}_{\tau_1},\widetilde{\mu}_{\tau_2}) \ge&\frac{2^j|A_r|^2}{(2\pi)^2}\Bigg\{\int_{-C}^{C}\int_{-C}^{C}\widehat{\phi}(\omega_1,\omega_2)d\omega_1d\omega_2\\ &-K^2\int_{-C}^{C}\int_{-C}^{C}\frac{\omega_1^2+\omega_2^2}{2}\widehat{\phi}(\omega_1,\omega_2)d\omega_1d\omega_2\Bigg\}\\ &+O\left(1+2^{2j}N^{-1}+2^{2j}M^{-1}+2^{-j}\sqrt{NM\log NM}+2^jK^4\right)\quad a.s.\end{aligned}$$

4.2 Estimation for the number and locations of hidden periodicities. Now, we construct the estimators for the number and the locations of hidden periodicities. For fixed j, we define the set $\Sigma(j)$ as

$$\Sigma(j) \equiv \{(\tau_1, \tau_2) : 2^{-j}|\beta_{J_Y}(j, \widetilde{\lambda}_{\tau_1}, \widetilde{\mu}_{\tau_2})| \geq T_0,\ \tau_1, \tau_2 \in I_J\},$$

where $T_0 = A(2\pi)^{-2}\int_{-C}^{C}\int_{-C}^{C}\widehat{\phi}(\omega_1, \omega_2)d\omega_1 d\omega_2$ with $0 < A < |A_r|^2$, $r = 1, \ldots, q$. The threshold level T_0 depends on the parameter A which is usually unknown. However, we can change scale parameter j across several levels. Then, the magnitude of $2^{-j}|\beta_{J_Y}(j, \widetilde{\lambda}_{\tau_1}, \widetilde{\mu}_{\tau_2})|$ is kept constant if $(\widetilde{\lambda}_{\tau_1}, \widetilde{\mu}_{\tau_2})$ is in the neighborhood of hidden periodicities, on the other hand decreases otherwise. So, we do not have preliminarily to know the parameter A.

If $\Sigma(j)$ is empty, put $\widehat{q} = 0$. If $\Sigma(j)$ is not empty, take an arbitrary small constant $0 < \theta_1 < 1/2$ and decompose the set $\Sigma(j)$ into subsets Σ_d, in which two lattices (τ_1, τ_2) and (τ_1', τ_2') are said to be in the same subset if and only if

$$|\widetilde{\lambda}_{\tau_1} - \widetilde{\lambda}_{\tau_1'}| \leq \frac{1}{2^{j(1-\theta_1)}} \text{ or } |\widetilde{\lambda}_{\tau_1} - \widetilde{\lambda}_{\tau_1'}| \geq 2\pi - \frac{1}{2^{j(1-\theta_1)}}$$

and

$$|\widetilde{\mu}_{\tau_2} - \widetilde{\mu}_{\tau_2'}| \leq \frac{1}{2^{j(1-\theta_1)}} \text{ or } |\widetilde{\mu}_{\tau_2} - \widetilde{\mu}_{\tau_2'}| \geq 2\pi - \frac{1}{2^{j(1-\theta_1)}}.$$

According to the following proof of this paper, we can see that for sufficiently large j, $\Sigma(j)$ is uniquely denoted by disjoint union of $\widetilde{q}(N, M)$ subsets $\Sigma(j) = \Sigma_1 \oplus \cdots \oplus \Sigma_{\widetilde{q}(N,M)}$. Put $\widehat{q} = \widetilde{q}(N, M)$, and let $(\overline{\tau}_{1,d}, \overline{\tau}_{2,d})$ be one of the maximum points of $|\beta_{J_Y}(j, \widetilde{\lambda}_{\tau_1}, \widetilde{\mu}_{\tau_2})|$ within the subset Σ_d, $d = 1, \ldots, \widehat{q}$. Denote $(\widetilde{\lambda}_{\overline{\tau}_{1,d}}, \widetilde{\mu}_{\overline{\tau}_{2,d}}) = (\overline{\lambda}_d, \overline{\mu}_d)$, then rearranging the $(\overline{\lambda}_d, \overline{\mu}_d)$ according to the following order leads to $(\widehat{\lambda}_r, \widehat{\mu}_r)$, $r = 1, \ldots, \widehat{q}$.

1 If $\frac{1}{2^{j(1-\theta_1)}} < |\overline{\lambda}_{r_1} - \overline{\lambda}_{r_2}| < 2\pi - \frac{1}{2^{j(1-\theta_1)}}$ and (a) $\overline{\lambda}_{r_1} \geq \pi - \frac{1}{22^{j(1-\theta_1)}}$, then $r_1 < r_2$, (b) $\overline{\lambda}_{r_1}, \overline{\lambda}_{r_2} < \pi - \frac{1}{22^{j(1-\theta_1)}}$ and $\overline{\lambda}_{r_1} + \frac{1}{2^{j(1-\theta_1)}} < \overline{\lambda}_{r_2}$, then $r_1 < r_2$.

2 If $|\overline{\lambda}_{r_1} - \overline{\lambda}_{r_2}| \leq \frac{1}{2^{j(1-\theta_1)}}$ or $|\overline{\lambda}_{r_1} - \overline{\lambda}_{r_2}| \geq 2\pi - \frac{1}{2^{j(1-\theta_1)}}$ and (a) $\overline{\mu}_{r_1} \geq \pi - \frac{1}{22^{j(1-\theta_1)}}$, then $r_1 < r_2$, (b) $\overline{\mu}_{r_1}, \overline{\mu}_{r_2} < \pi - \frac{1}{22^{j(1-\theta_1)}}$ and $\overline{\mu}_{r_1} + \frac{1}{2^{j(1-\theta_1)}} < \overline{\mu}_{r_2}$, then $r_1 < r_2$.

Then, we have the following results of consistency.

Theorem 2. *If the same Assumptions as that of Theorem 1 hold, then we have*

1 $\lim_{N,M\to\infty} \widehat{q} = q$ *a.s.*

2 $|\widehat{\lambda}_r - \lambda_r| \le \frac{2\pi\overline{K}}{2^j}$ *and* $|\widehat{\mu}_r - \mu_r| \le \frac{2\pi\overline{K}}{2^j}$ *a.s.,*

where in addition to Assumption 4, $\overline{K} > 0$ *satisfies* $\overline{K} \to 0$*, as* $N, M \to \infty$ *and*

$$\lim_{N,M\to\infty} 2^j\overline{K}^2 = \infty, \lim_{N,M\to\infty} 2^{-j}N\overline{K}^2 = \infty, \lim_{N,M\to\infty} 2^{-j}M\overline{K}^2 = \infty,$$
$$\lim_{N,M\to\infty} \left(2^{2j}\overline{K}^2\right)^{-1}\sqrt{NM\log NM} = 0. \tag{6}$$

4.3 Estimation for the amplitudes. Next, we construct the estimators for the amplitudes. Let $(\widehat{\tau}_{1,r}, \widehat{\tau}_{2,r})$ be the lattice $\{(\tau_1, \tau_2) : \tau_1, \tau_2 \in I_J\}$ which gives the frequency $(\widehat{\lambda}_r, \widehat{\mu}_r)$. That is,

$$\widehat{\lambda}_r \equiv \widetilde{\lambda}_{\widehat{\tau}_{1,r}} = \frac{2\pi}{2^J}\widehat{\tau}_{1,r} - \pi, \quad \widehat{\mu}_r \equiv \widetilde{\mu}_{\widehat{\tau}_{2,r}} = \frac{2\pi}{2^J}\widehat{\tau}_{2,r} - \pi.$$

We propose rescaled $\beta_{J_Y}(j', \widehat{\lambda}_r, \widehat{\mu}_r)$ for the estimators $|\widehat{A}_r|^2$, $r = 1, \ldots, \widehat{q}$ where $j' = j(1-\theta')$ satisfies

$$\lim_{N,M\to\infty} 2^{3j\theta'}\left(2^j\overline{K}^2\right)^{-1} = 0, \lim_{N,M\to\infty} 2^{j(1+\theta)'}\left(N\overline{K}^2\right)^{-1} = 0,$$
$$\lim_{N,M\to\infty} 2^{j(1+\theta)'}\left(N\overline{K}^2\right)^{-1} = 0, \lim_{N,M\to\infty} 2^{4j\theta'}\sqrt{NM\log NM}\left(2^{2j}\overline{K}^2\right)^{-1}. \tag{7}$$

Define the estimators of the amplitudes as

$$\widehat{A}_r = \left(\frac{(2\pi)^2}{2^{j'}}\left(\int_{-C}^{C}\int_{-C}^{C}\widehat{\phi}(\omega_1,\omega_2)d\omega_1 d\omega_2\right)^{-1}\beta_{J_Y}(j', \widehat{\lambda}_r, \widehat{\mu}_r)\right)^{\frac{1}{2}}, \ r = 1, \ldots, \widehat{q}.$$

Note that for each $(\widetilde{\lambda}_{\tau_1}, \widetilde{\mu}_{\tau_2})$ in the neighborhood of (λ_r, μ_r), the magnitude of the difference between $2^{-j'}\beta_{J_Y}(j', \widetilde{\lambda}_{\tau_1}, \widetilde{\mu}_{\tau_2})$ and $2^{-j'}\beta_{J_Y}(j', \lambda_r, \mu_r)$ is smaller than that of the difference between $2^{-j}\beta_{J_Y}(j, \widetilde{\lambda}_{\tau_1}, \widetilde{\mu}_{\tau_2})$ and $2^{-j}\beta_{J_Y}(j, \lambda_r, \mu_r)$. Now, we have the following results.

Theorem 3. *If the same Assumptions as that of Theorem 1 hold, then we have*

$$\lim_{N,M\to\infty} \widehat{A}_r = A_r + O\left(\left(\overline{K}2^{-j\theta'}\right)^2\right) \quad a.s.$$

5 Numerical simulations. To test the performances of proposed methods, we carry out the simulation for the following examples. The selected model is $Y_{n,m} = \sum_{r=1}^{3} A_r \exp\left(i(\lambda_r n + \mu_r m)\right)$ $+X_{n,m}$, with $X_{n,m} = 0.4X_{n-1,m} + 0.45X_{n,m-1} + W_{n,m}$ and

1. $(A_1, \lambda_1, \mu_1) = (1.5, -1, 5, -0.5)$, $(A_2, \lambda_2, \mu_2) = (1, 1, 2)$, $(A_3, \lambda_3, \mu_3) = (0, 0, 0)$, $W_{n,m} = W_{1,n,m}$,

2. $(A_1, \lambda_1, \mu_1) = (1.5, -1.5, -0.5)$, $(A_2, \lambda_2, \mu_2) = (1, 1, 2)$, $(A_3, \lambda_3, \mu_3) = (0, 0, 0)$, $W_{n,m} = W_{2,n,m}$,

3. $(A_1, \lambda_1, \mu_1) = (1.5, -1.5, -0.5)$, $(A_2, \lambda_2, \mu_2) = (1.2, -1.5, 2)$, $(A_3, \lambda_3, \mu_3) = (1, 1, 2)$, $W_{n,m} = W_{1,n,m}$,

4. $(A_1, \lambda_1, \mu_1) = (1.5, -1.5, -0.5)$, $(A_2, \lambda_2, \mu_2) = (1.2, -1.5, 2)$, $(A_3, \lambda_3, \mu_3) = (1, 1, 2)$, $W_{n,m} = W_{2,n,m}$,

where $\{W_{1,n,m}\}$ are i.i.d. $\mathcal{N}(0, 2)$ noise and $W_{2,n,m} = W_{1,n,m} W_{1,n-1,m-1}$.

We employ the scaling function of the Meyer wavelet whose Fourier transform $\widehat{\phi}^{(i)}(\omega_i)$ is given by $\widehat{\phi}^{(i)}(\omega_i) = (2\pi)^{-1/2} \cos[(\pi/2)\, \nu\, \{(3/2\pi)\, |\omega_i| - 1\}]$, where $\nu(x) = \left(\int_{-\infty}^{\infty} f_1(t)dt\right)^{-1} \int_{-\infty}^{x} f_1(t)dt$, $f_1(x) = f(x)f(1-x)$, $f(x) = e^{-1/x^2}$ if $x \geq 0$ and $f(x) = 0$ if $x \leq 0$. The graph of $\widehat{\phi}(\omega_1, \omega_2) = \widehat{\phi}^{(1)}(\omega_1)\widehat{\phi}^{(2)}(\omega_2)$ is plotted in Figure 1. Figure 2 is a observation of example 4 (signal+noise) with N=100 and M=98.

Figures 1 and 2 are about here.

It is easy to seen that

$$\beta_{J_Y}(j, \widetilde{\lambda}_{\tau_1}, \widetilde{\mu}_{\tau_2}) = \left(NM2^j\right)^{-1} \sum_{n=1-L}^{L-1} \sum_{m=1-L}^{L-1} \sum_{n'=\max\{1,1-n\}}^{\min\{N,N-n\}} \sum_{m'=\max\{1,1-m\}}^{\min\{M,M-m\}} Y_{n+n',m+m'} \overline{Y}_{n',m'} \widehat{\phi}\left(\frac{n2\pi}{2^j}, \frac{m2\pi}{2^j}\right) e^{-in\widetilde{\lambda}_{\tau_1}} e^{-im\widetilde{\mu}_{\tau_2}},$$

where $supp\{\widehat{\phi}^{(i)}(\omega)\} \subset [-C, C]$ and $L = \left[2^j C/(2\pi)\right] + 1$.

Figures 3-6 are $2^{-j}|\beta_{J_Y}(j, \widetilde{\lambda}_{\tau_1}, \widetilde{\mu}_{\tau_2})|$ of example 4 with $(j, J) = (3, 7)$, $(4, 7)$, $(5, 7)$, and $(6, 7)$, respectively. It is seen that as j is increasing, the noise components will be smaller. Tables 1 and 2 show the pair $(\overline{\tau}_{1,d}, \overline{\tau}_{2,d})$, $r = 1, 2, 3$ for each (j, J) in examples 2 and 4, which gives the maximum (second and third) peak value M1 (M2 and M3) of $2^{-j}|\beta_{J_Y}(j, \widetilde{\lambda}_{\tau_1}, \widetilde{\mu}_{\tau_2})|$, respectively. It is also seen that for large j M3 of example 2 is small enough, on the other hand M1 and M2 of example 2 and M1, M2 and M3 of example 4 are almost constant. Therefore, we employ $(\widehat{\lambda}_r, \widehat{\mu}_r)$, $r = 1, \ldots, q$ with $(j, J) = (6, 8)$ for the detection of hidden periodicities. On the other hand we construct the estimators for the amplitudes $\widehat{A}_r$, $r = 1, \ldots, q$, using this $(\widehat{\lambda}_r, \widehat{\mu}_r)$ and $j' = 4$. The performances of our estimators $(\widehat{\lambda}_r, \widehat{\mu}_r, \widehat{A}_r)$, $r = 1 \ldots, q$ for 30 times experiments in examples 1-4 are listed in Table 3.

Figures 3-6 are about here.

Tables 1-3 are about here.

6 Proofs of Theorems.

Here, we give the proofs of results in Section 4.

6.1 Proof of Theorem 1.

First, we observe that $\beta_{J_Y}(j, \widetilde{\lambda}_{\tau_1}, \widetilde{\mu}_{\tau_2}) = \beta_{G_1}(j, \widetilde{\lambda}_{\tau_1}, \widetilde{\mu}_{\tau_2}) + \beta_{G_2}(j, \widetilde{\lambda}_{\tau_1}, \widetilde{\mu}_{\tau_2}) + \beta_{J_X}(j, \widetilde{\lambda}_{\tau_1}, \widetilde{\mu}_{\tau_2})$, where β_{G_1}, β_{G_2} and β_{J_X} are overlap wavelet coefficients of G_1, G_2 and J_X in (5), respectively. For the signal component, we have

$$\begin{aligned}
&\beta_{G_1}(j, \widetilde{\lambda}_{\tau_1}, \widetilde{\mu}_{\tau_2}) = \left(2\pi NM2^j\right)^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(x_1, x_2) \\
&\qquad \left| G\left(\frac{2\pi}{2^j} x_1 + \widetilde{\lambda}_{\tau_1}, \frac{2\pi}{2^j} x_2 + \widetilde{\mu}_{\tau_2}, N, M\right)\right|^2 dx_1 dx_2 \\
&= \left(2\pi NM2^j\right)^{-1} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(x_1, x_2) \\
&\qquad \left| \sum_{n=1}^{N} \sum_{m=1}^{M} \sum_{r=1}^{q} A_r e^{in(\lambda_r - \widetilde{\lambda}_{\tau_1} - \frac{2\pi}{2^j} x_1)} e^{im(\mu_r - \widetilde{\mu}_{\tau_2} - \frac{2\pi}{2^j} x_2)} \right|^2 dx_1 dx_2 \\
&= \left(NM2^j\right)^{-1} \sum_{r_1, r_2 = 1}^{q} A_{r_1} \overline{A}_{r_2} \sum_{n=1-L}^{L-1} \sum_{m=1-L}^{L-1} e^{in(\lambda_{r_1} - \widetilde{\lambda}_{\tau_1})} e^{im(\mu_{r_1} - \widetilde{\mu}_{\tau_2})} \\
&\qquad \widehat{\phi}\left(\frac{n2\pi}{2^j}, \frac{m2\pi}{2^j}\right) \sum_{n'=\max\{1,1-n\}}^{\min\{N,N-n\}} e^{in'(\lambda_{r_1} - \lambda_{r_2})} \sum_{m'=\max\{1,1-m\}}^{\min\{M,M-m\}} e^{im'(\mu_{r_1} - \mu_{r_2})}.
\end{aligned}$$

Put $\delta_0 = \min\{|\lambda_{s_1} - \lambda_{s_2}|, |\mu_{t_1} - \mu_{t_2}| : \lambda_{s_1} \neq \lambda_{s_2}, \mu_{t_1} \neq \mu_{t_2},\ s_1, s_2, t_1, t_2 = 0, 1, \ldots, q\}$, where $\lambda_0 = \lambda_q - 2\pi$, $\mu_0 = \max_{1 \le r \le q}\{\mu_r\} - 2\pi$, then we have

$$\begin{aligned}
&\left| \sum_{n'=\max\{1,1-n\}}^{\min\{N,N-n\}} e^{in'(\lambda_{r_1} - \lambda_{r_2})} \right| \cdot \left| \sum_{m'=\max\{1,1-m\}}^{\min\{M,M-m\}} e^{im'(\mu_{r_1} - \mu_{r_2})} \right| \\
&= \left| B_{N-|n|}(\lambda_{r_1} - \lambda_{r_2}) \right| \cdot \left| B_{M-|m|}(\mu_{r_1} - \mu_{r_2}) \right| \\
&= \begin{cases}
(N-|n|)(M-|m|) & \text{for } r_1 = r_2 \\
\le \{\sin(\delta_0/2)\}^{-1} (N-|n|) = O(N) & \text{for } \lambda_{r_1} = \lambda_{r_2} \text{ and } \mu_{r_1} \neq \mu_{r_2} \\
\le \{\sin(\delta_0/2)\}^{-1} (M-|m|) = O(M) & \text{for } \lambda_{r_1} \neq \lambda_{r_2} \text{ and } \mu_{r_1} = \mu_{r_2} \\
\le |\sin(\delta_0/2)|^{-2} & \text{otherwise,}
\end{cases}
\end{aligned}$$

where $B_T(\theta) = \sin\{(T\theta)/2\}/\sin(\theta/2)$. Since

$$\left(NM2^j\right)^{-1} \sum_{r_1 \neq r_2} A_{r_1} \overline{A}_{r_2} \sum_{n=1-L}^{L-1} \sum_{m=1-L}^{L-1} e^{in(\lambda_{r_1} - \widetilde{\lambda}_{\tau_1})} e^{im(\mu_{r_1} - \widetilde{\mu}_{\tau_2})} \widehat{\phi}\left(\frac{n2\pi}{2^j}, \frac{m2\pi}{2^j}\right)$$
$$\sum_{n'=\max\{1,1-n\}}^{\min\{N,N-n\}} e^{in'(\lambda_{r_1} - \lambda_{r_2})} \sum_{m'=\max\{1,1-m\}}^{\min\{M,M-m\}} e^{im'(\mu_{r_1} - \mu_{r_2})} = O\left(2^j N^{-1} + 2^j M^{-1}\right),$$

we can see that

$$\beta_{G_1}(j, \widetilde{\lambda}_{\tau_1}, \widetilde{\mu}_{\tau_2}) = 2^{-j} \sum_{r=1}^{q} |A_r|^2 \sum_{n=1-L}^{L-1} \sum_{m=1-L}^{L-1} e^{in(\lambda_r - \widetilde{\lambda}_{\tau_1})} e^{im(\mu_r - \widetilde{\mu}_{\tau_2})} \widehat{\phi}\left(\frac{n2\pi}{2^j}, \frac{m2\pi}{2^j}\right)$$
$$\left(1 - \frac{|n|}{N}\right)\left(1 - \frac{|m|}{M}\right) + O\left(2^j N^{-1} + 2^j M^{-1}\right)$$
$$= 2^{-j} \sum_{r=1}^{q} |A_r|^2 \sum_{n=1-L}^{L-1} \sum_{m=1-L}^{L-1} e^{in(\lambda_r - \widetilde{\lambda}_{\tau_1})} e^{im(\mu_r - \widetilde{\mu}_{\tau_2})} \widehat{\phi}\left(\frac{n2\pi}{2^j}, \frac{m2\pi}{2^j}\right)$$
$$+ O\left(2^{2j} N^{-1} + 2^{2j} M^{-1}\right)$$
$$= 2^{-j} \sum_{r=1}^{q} |A_r|^2 \sum_{n=1-L}^{L-1} \sum_{m=1-L}^{L-1} \cos(n(\lambda_r - \widetilde{\lambda}_{\tau_1})) \cos(m(\mu_r - \widetilde{\mu}_{\tau_2})) \widehat{\phi}\left(\frac{n2\pi}{2^j}, \frac{m2\pi}{2^j}\right)$$
$$+ O\left(2^{2j} N^{-1} + 2^{2j} M^{-1}\right).$$

Now, we have the following Lemmas 2-4 for overlap wavelet coefficients of signal components.

Lemma 2. *For all* $(\tau_1, \tau_2) \in E(\frac{2\pi K}{2^l})$*, where* $K \to \infty$*, as* $N, M \to \infty$*, we have*

$$\beta_{G_1}(j, \widetilde{\lambda}_{\tau_1}, \widetilde{\mu}_{\tau_2}) = O\left(2^j K^{-1} + 2^{2j} N^{-1} + 2^{2j} M^{-1}\right) = o(2^j).$$

Proof. In this case, for any $|x_1| < \sqrt{K}/2$, $|x_2| < \sqrt{K}/2$ and $r = 1, \ldots, q$, we have

$$\frac{\pi K}{2^j} < \left|\lambda_r - \widetilde{\lambda}_{\tau_1} - \frac{2\pi}{2^j} x_1\right| < 2\pi - \frac{\pi K}{2^j} \text{ or } \frac{\pi K}{2^j} < \left|\mu_r - \widetilde{\mu}_{\tau_2} - \frac{2\pi}{2^j} x_2\right| < 2\pi - \frac{\pi K}{2^j},$$

therefore,

$$\left|B_{2L-1}(\lambda_r - \widetilde{\lambda}_{\tau_1} - \frac{2\pi}{2^j} x_1) B_{2L-1}(\mu_r - \widetilde{\mu}_{\tau_2} - \frac{2\pi}{2^j} x_2)\right|$$
$$\leq \frac{(2L-1)}{\sin \frac{1}{2} \frac{\pi K}{2^j}} < \frac{(2L-1)}{\frac{2}{\pi} \frac{1}{2} \frac{\pi K}{2^j}} = (2L-1) 2^j K^{-1}.$$

Thanks to (iii) of Assumption 1 and Assumption 3, we have for $K' \to \infty$,

$$\int_{|x_1| < K'} \int_{|x_2| \geq K'} \phi(x_1, x_2) dx_1 dx_2 \leq \int_{-\infty}^{\infty} \int_{|x_2| \geq 2^{l'}} \phi(x_1, x_2) dx_1 dx_2$$
$$= \left(\int_{-\infty}^{\infty} \phi^{(1)}(x_1) dx_1\right) \left(\int_{|x_2| \geq K'} \phi^{(2)}(x_2) dx_2\right) = O(K'^{-(u-1)}).$$

Hence, from

$$2^{-j}\sum_{r=1}^{q}|A_r|^2\sum_{n=1-L}^{L-1}\sum_{m=1-L}^{L-1}e^{in(\lambda_r-\widetilde{\lambda}_{\tau_1})}e^{im(\mu_r-\widetilde{\mu}_{\tau_2})}\widehat{\phi}\left(\frac{n2\pi}{2^j},\frac{m2\pi}{2^j}\right)$$
$$\leq \left\{2\pi 2^j\right)^{-1}\sum_{r=1}^{q}|A_r|^2\int_{|x_1|<\sqrt{K}/2}\int_{|x_2|<\sqrt{K}/2}$$
$$\left|B_{2L-1}(\lambda_r-\widetilde{\lambda}_{\tau_1}-\frac{2\pi}{2^j}x_1)B_{2L-1}(\mu_r-\widetilde{\mu}_{\tau_2}-\frac{2\pi}{2^j}x_2)\right|\phi(x_1,x_2)dx_1dx_2$$
$$+\frac{(2L-1)^2}{2\pi 2^j}\sum_{r=1}^{q}|A_r|^2\left\{\int_{|x_1|\geq\sqrt{K}/2}\int_{-\infty}^{\infty}\phi(x_1,x_2)dx_1dx_2\right.$$
$$\left.+\int_{|x_1|<\sqrt{K}/2}\int_{|x_2|\geq\sqrt{K}/2}\phi(x_1,x_2)dx_1dx_2\right\}$$
$$=O(2^{-j})O\left((2L-1)2^jK^{-1}\right)+O\left((2L-1)^2 2^{-j}\right)O\left(K^{-\frac{u-1}{2}}\right)$$
$$=O\left(2^jK^{-1}\right)+O\left(2^jK^{-1}K^{-\frac{u-3}{2}}\right)=O\left(2^jK^{-1}\right),$$

we have $\beta_{G_1}(j,\widetilde{\lambda}_{\tau_1},\widetilde{\mu}_{\tau_2})=O(2^jK^{-1}+2^{2j}N^{-1}+2^{2j}M^{-1})=o(2^j)$. □

Lemma 3. (a) *If* $(|\widetilde{\lambda}_{\tau_1}-\lambda_r|=\frac{2\pi}{2^j}K_1$ *or* $|\widetilde{\lambda}_{\tau_1}-\lambda_r|=2\pi-\frac{2\pi}{2^j}K_1)$, *and* $(|\widetilde{\mu}_{\tau_2}-\mu_r|=\frac{2\pi}{2^j}K_2$ *or* $|\widetilde{\mu}_{\tau_2}-\mu_r|=2\pi-\frac{2\pi}{2^j}K_2)$, *for some constants* $K_1,K_2>0$, *then*

$$\beta_{G_1}(j,\widetilde{\lambda}_{\tau_1},\widetilde{\mu}_{\tau_2})=\frac{2^j|A_r|^2}{(2\pi)^2}\int_{-C}^{C}\int_{-C}^{C}\cos K_1\omega_1\cos K_2\omega_2\widehat{\phi}(\omega_1,\omega_2)d\omega_1d\omega_2$$
$$+O\left(1+2^{2j}N^{-1}+2^{2j}M^{-1}\right).$$

(b) *If* $(|\widetilde{\lambda}_{\tau_1}-\lambda_r|=\frac{2\pi}{2^j}K_1$ *or* $|\widetilde{\lambda}_{\tau_1}-\lambda_r|=2\pi-\frac{2\pi}{2^j}K_1)$, *and* $\tau_2\in I_{r,\mu}(\frac{2\pi K}{2^j})$, $\Big($*or if* $(|\widetilde{\mu}_{\tau_2}-\mu_r|=\frac{2\pi}{2^j}K_1$ *or* $|\widetilde{\mu}_{\tau_2}-\mu_r|=2\pi-\frac{2\pi}{2^j}K_1)$, *and* $\tau_1\in I_{r,\lambda}(\frac{2\pi K}{2^j})$,$\Big)$ *where* K_1 *is some constant and* $K\to 0$, *as* $N,M\to\infty$, *then*

$$\beta_{G_1}(j,\widetilde{\lambda}_{\tau_1},\widetilde{\mu}_{\tau_2})=\frac{2^j|A_r|^2}{(2\pi)^2}\int_{-C}^{C}\int_{-C}^{C}\cos K_1\omega_1\widehat{\phi}(\omega_1,\omega_2)d\omega_1d\omega_2$$
$$+O\left(1+2^jK^2+2^{2j}N^{-1}+2^{2j}M^{-1}\right).$$

Proof. For all $(\tau_1,\tau_2)\in I_r(\frac{2\pi K'}{2^j})$ with some constant K', $|x_1|<2^{\frac{j}{2}}$, $|x_2|<2^{\frac{j}{2}}$ and $p\neq r$, we have for sufficiently large j

$$\frac{\delta_0}{2}<|\lambda_p-\widetilde{\lambda}_{\tau_1}-\frac{2\pi}{2^j}x_1|<2\pi-\frac{\delta_0}{2}\text{ or }\frac{\delta_0}{2}<|\mu_p-\widetilde{\mu}_{\tau_2}-\frac{2\pi}{2^j}x_2|<2\pi-\frac{\delta_0}{2},$$

so, $\left|B_{2L-1}(\lambda_p-\widetilde{\lambda}_{\tau_1}-\frac{2\pi}{2^j}x_1)B_{2L-1}(\mu_p-\widetilde{\mu}_{\tau_2}-\frac{2\pi}{2^j}x_2)\right|\leq(2L-1)\left(\sin(\delta_0/4)\right)^{-1}$. Therefore, it is seen that

$$
\begin{aligned}
&2^{-j}\sum_{p\neq r}|A_p|^2\sum_{n=1-L}^{L-1}\sum_{m=1-L}^{L-1}e^{in(\lambda_p-\widetilde{\lambda}_{\tau_1})}e^{im(\mu_p-\widetilde{\mu}_{\tau_2})}\widehat{\phi}\left(\frac{n2\pi}{2^j},\frac{m2\pi}{2^j}\right)\\
&\leq\left(2\pi 2^j\right)^{-1}\sum_{p\neq r}|A_p|^2\int_{|x_1|<2^{\frac{j}{2}}}\int_{|x_2|<2^{\frac{j}{2}}}\Big|B_{2L-1}(\lambda_p-\widetilde{\lambda}_{\tau_1}-\frac{2\pi}{2^j}x_1)\\
&\qquad\qquad B_{2L-1}(\mu_p-\widetilde{\mu}_{\tau_2}-\frac{2\pi}{2^j}x_2)\Big|\phi(x_1,x_2)dx_1dx_2\\
&+\frac{(2L-1)^2}{2\pi 2^j}\sum_{p\neq r}|A_p|^2\Bigg\{\int_{|x_1|\geq 2^{\frac{j}{2}}}\int_{-\infty}^{\infty}\phi(x_1,x_2)dx_1dx_2\\
&\qquad\qquad+\int_{|x_1|<2^{\frac{j}{2}}}\int_{|x_2|\geq 2^{\frac{j}{2}}}\phi(x_1,x_2)dx_1dx_2\Bigg\}\\
&\leq O(2^{-j})O(2L-1)+O\left(\frac{(2L-1)^2}{2^j}\right)O\left(2^{-\frac{j}{2}(u-1)}\right)\\
&=O\left(1+2^{-\frac{j}{2}(u-3)}\right)=O(1)
\end{aligned}
$$

and

$$
\begin{aligned}
\beta_{G_1}(j,\widetilde{\lambda}_{\tau_1},\widetilde{\mu}_{\tau_2})=\frac{|A_r|^2}{2^j}&\sum_{n=1-L}^{L-1}\sum_{m=1-L}^{L-1}\cos(n(\lambda_r-\widetilde{\lambda}_{\tau_1}))\cos(m(\mu_r-\widetilde{\mu}_{\tau_2}))\\
&\widehat{\phi}\left(\frac{n2\pi}{2^j},\frac{m2\pi}{2^j}\right)+O(2^{2j}N^{-1}+2^{2j}M^{-1}+1).
\end{aligned}
$$

Since

$$
\begin{aligned}
&\sum_{n=1-L}^{L-1}\sum_{m=1-L}^{L-1}\cos\frac{n2\pi}{2^j}K_1\cos\frac{m2\pi}{2^j}K_2\widehat{\phi}\left(\frac{n2\pi}{2^j},\frac{m2\pi}{2^j}\right)\\
&=\frac{2^{2j}}{(2\pi)^2}\int_{-C}^{C}\int_{-C}^{C}\cos K_1\omega_1\cos K_2\omega_2\widehat{\phi}(\omega_1,\omega_2)d\omega_1d\omega_2+O(2^j),
\end{aligned}
$$

(**a**) is obvious. Next, with regard to (**b**), since

$$
\begin{aligned}
&\sum_{n=1-L}^{L-1}\sum_{m=1-L}^{L-1}\cos\frac{n2\pi}{2^j}K_1\widehat{\phi}\left(\frac{n2\pi}{2^j},\frac{m2\pi}{2^j}\right)\\
&=\frac{2^{2j}}{(2\pi)^2}\int_{-C}^{C}\int_{-C}^{C}\cos K_1\omega_1\widehat{\phi}(\omega_1,\omega_2)d\omega_1d\omega_2+O(2^j),
\end{aligned}
$$

we can see that

$$\beta_{G_1}(j,\widetilde{\lambda}_{\tau_1},\widetilde{\mu}_{\tau_2}) = \frac{|A_r|^2}{2^j}\sum_{n=1-L}^{L-1}\sum_{m=1-L}^{L-1}\cos(n(\lambda_r-\widetilde{\lambda}_{\tau_1}))\cos(m(\mu_r-\widetilde{\mu}_{\tau_2}))$$
$$\widehat{\phi}\left(\frac{n2\pi}{2^j},\frac{m2\pi}{2^j}\right)+O\left(2^{2j}N^{-1}+2^{2j}M^{-1}+1\right)$$
$$=\frac{|A_r|^2}{2^j}\sum_{n=1-L}^{L-1}\sum_{m=1-L}^{L-1}\cos\frac{n2\pi}{2^j}K_1\left(1+O(m^2(\mu_r-\widetilde{\mu}_{\tau_2})^2)\right)\widehat{\phi}\left(\frac{n2\pi}{2^j},\frac{m2\pi}{2^j}\right)$$
$$+O\left(2^{2j}N^{-1}+2^{2j}M^{-1}+1\right)$$
$$=\frac{|A_r|^2}{2^j}\sum_{n=1-L}^{L-1}\sum_{m=1-L}^{L-1}\cos\frac{n2\pi}{2^j}K_1\widehat{\phi}\left(\frac{n2\pi}{2^j},\frac{m2\pi}{2^j}\right)$$
$$+O\left(2^jK^2+2^{2j}N^{-1}+2^{2j}M^{-1}+1\right)$$
$$=\frac{2^j|A_r|^2}{(2\pi)^2}\int_{-C}^{C}\int_{-C}^{C}\cos K_1\omega_1\widehat{\phi}(\omega_1,\omega_2)d\omega_1d\omega_2$$
$$+O\left(2^jK^2+2^{2j}N^{-1}+2^{2j}M^{-1}+1\right).$$

□

Lemma 4. *For all* $(\tau_1,\tau_2)\in I_r(\frac{2\pi K}{2^j})$ *where* $K\to 0$*, as* $N,M\to\infty$*, we have*

(a) $\beta_{G_1}(j,\widetilde{\lambda}_{\tau_1},\widetilde{\mu}_{\tau_2}) = \frac{2^j|A_r|^2}{(2\pi)^2}\int_{-C}^{C}\int_{-C}^{C}\widehat{\phi}(\omega_1,\omega_2)d\omega_1d\omega_2+o(2^j)$

(b) $\beta_{G_1}(j,\widetilde{\lambda}_{\tau_1},\widetilde{\mu}_{\tau_2}) \le \frac{2^j|A_r|^2}{(2\pi)^2}\int_{-C}^{C}\int_{-C}^{C}\widehat{\phi}(\omega_1,\omega_2)d\omega_1d\omega_2+O(1+2^{2j}N^{-1}+2^{2j}M^{-1})$

(c) $\beta_{G_1}(j,\widetilde{\lambda}_{\tau_1},\widetilde{\mu}_{\tau_2}) \ge \frac{2^j|A_r|^2}{(2\pi)^2}\Big\{\int_{-C}^{C}\int_{-C}^{C}\widehat{\phi}(\omega_1,\omega_2)d\omega_1d\omega_2$
$-K^2\int_{-C}^{C}\int_{-C}^{C}\frac{\omega_1^2+\omega_2^2}{2}\widehat{\phi}(\omega_1,\omega_2)d\omega_1d\omega_2\Big\}+O(1+2^{2j}N^{-1}+2^{2j}M^{-1}+2^jK^4).$

Proof. According to proof of Lemma 3, we have

$$\beta_{G_1}(j,\widetilde{\lambda}_{\tau_1},\widetilde{\mu}_{\tau_2}) = \frac{|A_r|^2}{2^j}\sum_{n=1-L}^{L-1}\sum_{m=1-L}^{L-1}\cos(n(\lambda_r-\widetilde{\lambda}_{\tau_1}))\cos(m(\mu_r-\widetilde{\mu}_{\tau_2}))$$
$$\widehat{\phi}\left(\frac{n2\pi}{2^j},\frac{m2\pi}{2^j}\right)+O\left(2^{2j}N^{-1}+2^{2j}M^{-1}+1\right).$$

Since

$$\sum_{n=1-L}^{L-1}\sum_{m=1-L}^{L-1}\widehat{\phi}\left(\frac{n2\pi}{2^j},\frac{m2\pi}{2^j}\right)=\frac{2^{2j}}{(2\pi)^2}\int_{-C}^{C}\int_{-C}^{C}\widehat{\phi}(\omega_1,\omega_2)d\omega_1d\omega_2+O(2^j), \tag{8}$$

(**a**) and (**b**) are obvious. With regard to (**c**), from equation (8) and

$$\sum_{n=1-L}^{L-1}\sum_{m=1-L}^{L-1} n^2\widehat{\phi}\left(\frac{n2\pi}{2^j},\frac{m2\pi}{2^j}\right)=\frac{2^{4j}}{(2\pi)^4}\int_{-C}^{C}\int_{-C}^{C}{\omega_1}^2\widehat{\phi}(\omega_1,\omega_2)d\omega_1 d\omega_2$$
$$+O(2^{3j}),$$

we can see that

$$\begin{aligned}
&\sum_{n=1-L}^{L-1}\sum_{m=1-L}^{L-1}\cos(n(\lambda_r-\widetilde{\lambda}_{\tau_1}))\cos(m(\mu_r-\widetilde{\mu}_{\tau_2}))\widehat{\phi}\left(\frac{n2\pi}{2^j},\frac{m2\pi}{2^j}\right)\\
&=\sum_{n=1-L}^{L-1}\sum_{m=1-L}^{L-1}\left(1-\frac{n^2}{2}|\lambda_r-\widetilde{\lambda}_{\tau_1}|^2+O(n^4|\lambda_r-\widetilde{\lambda}_{\tau_1}|^4)\right)\\
&\qquad\left(1-\frac{m^2}{2}|\mu_r-\widetilde{\mu}_{\tau_2}|^2+O(m^4|\mu_r-\widetilde{\mu}_{\tau_2}|^4)\right)\widehat{\phi}\left(\frac{n2\pi}{2^j},\frac{m2\pi}{2^j}\right)\\
&\geq\sum_{n=1-L}^{L-1}\sum_{m=1-L}^{L-1}\widehat{\phi}\left(\frac{n2\pi}{2^j},\frac{m2\pi}{2^j}\right)\\
&-\left(\frac{2\pi K}{2^j}\right)^2\sum_{n=1-L}^{L-1}\sum_{m=1-L}^{L-1}\left(\frac{n^2+m^2}{2}\right)\widehat{\phi}\left(\frac{n2\pi}{2^j},\frac{m2\pi}{2^j}\right)+O(2^{2j}K^4)\\
&=\frac{2^{2j}}{(2\pi)^2}\int_{-C}^{C}\int_{-C}^{C}\widehat{\phi}(\omega_1,\omega_2)d\omega_1 d\omega_2+O(2^j)\\
&-\frac{2^{2j}K^2}{(2\pi)^2}\int_{-C}^{C}\int_{-C}^{C}\frac{{\omega_1}^2+{\omega_2}^2}{2}\widehat{\phi}(\omega_1,\omega_2)d\omega_1 d\omega_2+O(2^jK^2)+O(2^{2j}K^4).
\end{aligned}$$

Hence,

$$\begin{aligned}
\beta_{G_1}(j,\widetilde{\lambda}_{\tau_1},\widetilde{\mu}_{\tau_2})\geq\frac{2^j|A_r|^2}{(2\pi)^2}\Big\{&\int_{-C}^{C}\int_{-C}^{C}\widehat{\phi}(\omega_1,\omega_2)d\omega_1 d\omega_2\\
-K^2\int_{-C}^{C}\int_{-C}^{C}\frac{\omega_1^2+\omega_2^2}{2}\widehat{\phi}(\omega_1,\omega_2)d\omega_1 d\omega_2\Big\}&+O\left(1+2^{2j}N^{-1}+2^{2j}M^{-1}+2^jK^4\right).
\end{aligned}$$

□

Next, we have the following result for overlap wavelet coefficients of noise components.

Lemma 5. *Assume that Assumptions 1-4 hold, then*

$$\beta_{J_X}(j,\widetilde{\lambda}_{\tau_1},\widetilde{\mu}_{\tau_2})=O\left(2^{-j}\log NM\right)\qquad a.s.$$

Proof. From Corollary 1, we can see that $\sup_{\lambda,\mu}\{J_X(\lambda,\mu,N,M)\} \le O(\log NM)$ a.s., hence,

$$\begin{aligned}
&|\beta_{J_X}(j,\widetilde{\lambda}_{\tau_1},\widetilde{\mu}_{\tau_2})| \\
&\le \frac{2\pi}{2^j}\int_{\mathbb{R}^2}\phi(x_1,x_2)\left|J_X\left(\frac{2\pi}{2^j}x_1+\widetilde{\lambda}_{\tau_1},\frac{2\pi}{2^j}x_2+\widetilde{\mu}_{\tau_2},N,M\right)\right|dx_1dx_2 \\
&\le \frac{2\pi}{2^j}\sup_{\lambda,\mu}\{J_X(\lambda,\mu,N,M)\}\int_{\mathbb{R}^2}\phi(x_1,x_2)dx_1dx_2 \\
&= \frac{2\pi}{2^j}O(\log NM)\int_{\mathbb{R}^2}\phi(x_1,x_2)dx_1dx_2 = O\left(2^{-j}\log NM\right) \qquad a.s.
\end{aligned}$$

□

Furthermore, we have the following result for overlap wavelet coefficients of cross components.

Lemma 6. *Assume that Assumptions 1-4 hold, then*

$$\beta_{G_2}(j,\widetilde{\lambda}_{\tau_1},\widetilde{\mu}_{\tau_2}) = O\left(2^{-j}\sqrt{NM\log NM}\right) \qquad a.s.$$

Proof. It can be seen that

$$\begin{aligned}
&\beta_{G_2}(j,\widetilde{\lambda}_{\tau_1},\widetilde{\mu}_{\tau_2}) = \frac{1}{NM2^j\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(x_1,x_2)\Re\Big\{ \\
&G\left(\frac{2\pi}{2^j}x_1+\widetilde{\lambda}_{\tau_1},\frac{2\pi}{2^j}x_2+\widetilde{\mu}_{\tau_2},N,M\right)S_X\left(\frac{2\pi}{2^j}x_1+\widetilde{\lambda}_{\tau_1},\frac{2\pi}{2^j}x_2+\widetilde{\mu}_{\tau_2},N,M\right)\Big\}dx_1dx_2 \\
&\le \frac{\sup_{\lambda,\mu}|S_X(\lambda,\mu,N,M)|}{NM2^j}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(x_1,x_2) \\
&\qquad \left|G\left(\frac{2\pi}{2^j}x_1+\widetilde{\lambda}_{\tau_1},\frac{2\pi}{2^j}x_2+\widetilde{\mu}_{\tau_2},N,M\right)\right|dx_1dx_2 \\
&\le \frac{\sup_{\lambda,\mu}|S_X(\lambda,\mu,N,M)|}{2^j}\sum_{r=1}^{q}|A_r|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\phi(x_1,x_2)dx_1dx_2 \\
&= O\left(2^{-j}\sqrt{NM\log NM}\right) \qquad a.s.
\end{aligned}$$

□

Combining the results of Lemmas 2-6, the proof of Theorem 1 is completed.

6.2 Proof of Theorem 2. According to 1 of Theorem 1, for all $(\tau_1,\tau_2)\in E(\frac{2\pi K}{2^j})$ with any $K\to\infty$, as $N,M\to\infty$, it follows that $\beta_{J_Y}(j,\widetilde{\lambda}_{\tau_1},\widetilde{\mu}_{\tau_2}) = o(2^j)$ a.s. Therefore, for any $(\tau_1,\tau_2)\in\Sigma(j)$, there exist r $(1\le r\le q)$ and some constant $0<\theta<\theta_1$ which satisfy

$(\tau_1,\tau_2) \in I_r(\frac{2\pi}{2^{j(1-\theta)}}) \cap \Sigma(j)$ if j is sufficiently large. If the lattices $(\tau_1,\tau_2), (\tau_1',\tau_2')$ are in the same set $I_r(\frac{2\pi}{2^{j(1-\theta)}})$, then

$$|\widetilde{\lambda}_{\tau_1} - \widetilde{\lambda}_{\tau_2'}| \le |\widetilde{\lambda}_{\tau_1} - \lambda_r| + |\widetilde{\lambda}_{\tau_1'} - \lambda_r| \le \frac{4\pi}{2^{j(1-\theta)}} = \frac{1}{2^{j(1-\theta_1)}}\frac{4\pi}{2^{j(\theta_1-\theta)}} < \frac{1}{2^{j(1-\theta_1)}},$$

$$|\widetilde{\lambda}_{\tau_1} - \widetilde{\lambda}_{\tau_2'}| \ge |\widetilde{\lambda}_{\tau_1} - \lambda_r| - |\widetilde{\lambda}_{\tau_1'} - \lambda_r| \ge 2\pi - \frac{4\pi}{2^{j(1-\theta)}} > 2\pi - \frac{1}{2^{j(1-\theta_1)}}$$

or

$$\widetilde{\lambda}_{\tau_1}, \widetilde{\lambda}_{\tau_1'} \in \left[0, \frac{2\pi}{2^{j(1-\theta)}}\right), \text{ or } \widetilde{\lambda}_{\tau_1}, \widetilde{\lambda}_{\tau_1'} \in \left(2\pi - \frac{2\pi}{2^{j(1-\theta)}}, 2\pi\right],$$

that is,

$$|\widetilde{\lambda}_{\tau_1} - \widetilde{\lambda}_{\tau_1'}| < \frac{1}{2^{j(1-\theta_1)}} \text{ or } |\widetilde{\lambda}_{\tau_1} - \widetilde{\lambda}_{\tau_1'}| > 2\pi - \frac{1}{2^{j(1-\theta_1)}}$$

and similarly

$$|\widetilde{\mu}_{\tau_2} - \widetilde{\mu}_{\tau_2'}| < \frac{1}{2^{j(1-\theta_1)}} \text{ or } |\widetilde{\mu}_{\tau_2} - \widetilde{\mu}_{\tau_2'}| > 2\pi - \frac{1}{2^{j(1-\theta_1)}}.$$

So, we can see that all the lattices $(\tau_1,\tau_2) \in I_r(\frac{2\pi}{2^j(1-\theta)}) \cap \Sigma(j)$ are in the same Σ_d, hence $\widehat{q} = \widetilde{q}(N,M) \le q$ a.s.

On the other hand for large j, $I_r(\frac{2\pi}{2^{j(1-\theta)}})$, $r = 1,\ldots,q$ are disjoint, so $I_r(\frac{2\pi}{2^{j(1-\theta)}}) \cap \Sigma(j)$, $r = 1,\ldots,q$ are disjoint, too. Since if $r_1 \ne r_2$, then $\delta_0 \le |\lambda_{r_1} - \lambda_{r_2}| \le 2\pi - \delta_0$ or $\delta_0 \le |\mu_{r_1} - \mu_{r_2}| \le 2\pi - \delta_0$, we have for any $(\tau_1,\tau_2) \in I_{r_1}(\frac{2\pi}{2^{j(1-\theta)}})$, $(\tau_1',\tau_2') \in I_{r_2}(\frac{2\pi}{2^{j(1-\theta)}})$, $r_1 \ne r_2$ and large j,

$$\frac{1}{2^{j(1-\theta_1)}} < |\widetilde{\lambda}_{\tau_1} - \widetilde{\lambda}_{\tau_1'}| < 2\pi - \frac{1}{2^{j(1-\theta_1)}} \text{ or } \frac{1}{2^{\mathrm{j}(1-\theta_1)}} < |\widetilde{\mu}_{\tau_2} - \widetilde{\mu}_{\tau_2'}| < 2\pi - \frac{1}{2^{\mathrm{j}(1-\theta_1)}}.$$

Hence, if $r_1 \ne r_2$, any lattices (τ_1,τ_2) in the set $I_{r_1}(\frac{2\pi}{2^{j(1-\theta)}})$ and (τ_1',τ_2') in the set $I_{r_2}(\frac{2\pi}{2^{j(1-\theta)}})$ will not belong to same Σ_d. According to 3(a) of Theorem 1, for any lattices $(\tau_1,\tau_2) \in I_r(\frac{2\pi 2^{-j\theta}}{2^j})$ where $\theta > 0$ is an arbitrary constant, we have $2^{-j}|\beta_{J_Y}(j,\widetilde{\lambda}_{\tau_1},\widetilde{\mu}_{\tau_2})| > T_0$. That is $I_r(\frac{2\pi}{2^{j(1+\theta)}}) \subset \Sigma(j) \cap I_r(\frac{2\pi}{2^{j(1-\theta)}})$. Since inside each $I_r(\frac{2\pi}{2^{j(1+\theta)}})$ with $j(1+\theta) \ll J$, there is at least one points coming from the set $\{(\tau_1,\tau_2) : \tau_1,\tau_2 \in I_J\}$, it follows that $\Sigma(j) \cap I_r(\frac{2\pi}{2^{j(1+\theta)}}) = I_r(\frac{2\pi}{2^{j(1+\theta)}}) \ne \{\emptyset\}$. Therefore, for large N,M, so for large j, we have $\widehat{q} = \widetilde{q}(N,M) \ge q$ a.s., which complete the proof of 1 of Theorem 2.

With regarding to 2 of Theorem 2, according to the proof of 1 of Theorem 2, for any $1 \le d \le \widetilde{q}(N,M)$, there exists $1 \le r \le q$ such that $\Sigma_d = I_r(\frac{2\pi}{2^{j(1-\theta)}}) \cap \Sigma(j)$ with some constant $\theta > 0$. Hence, the maximum points of $|\beta_{J_Y}(j, \widetilde{\lambda}_{\tau_1}, \widetilde{\mu}_{\tau_2})|$ within the subset Σ_d belong to $I_r(\frac{2\pi}{2^{j(1-\theta)}})$. According to 1, 2, 3(a) of Theorem 1 and since for any $K_1, K_2 > 0$,

$$\int_{-C}^{C}\int_{-C}^{C} \cos K_1\omega_1 \cos K_2\omega_2 \widehat{\phi}(\omega_1,\omega_2)d\omega_1 d\omega_2 < \int_{-C}^{C}\int_{-C}^{C} \widehat{\phi}(\omega_1,\omega_2)d\omega_1 d\omega_2,$$
$$\int_{-C}^{C}\int_{-C}^{C} \cos K_1\omega_1 \widehat{\phi}(\omega_1,\omega_2)d\omega_1 d\omega_2 < \int_{-C}^{C}\int_{-C}^{C} \widehat{\phi}(\omega_1,\omega_2)d\omega_1 d\omega_2,$$

there exist $K > 0$ with $K \to 0$, as $N, M \to \infty$, such that for any constant $\theta > 0$

$$\max_{(\tau_1,\tau_2)\in I_r\left(\frac{2\pi}{2^{j(1-\theta)}}\right)} \{|\beta_{J_Y}(j, \widetilde{\lambda}_{\tau_1}, \widetilde{\mu}_{\tau_2})|\} = \max_{(\tau_1,\tau_2)\in I_r(\frac{2\pi K}{2^j})} \{|\beta_{J_Y}(j, \widetilde{\lambda}_{\tau_1}, \widetilde{\mu}_{\tau_2})|\}.$$

For any $K > 0$ where $K \to 0$, as $N, M \to \infty$, we take $\overline{K} > 0$ as $\lim_{N,M\to\infty} \overline{K}K^{-1} = \lim_{N,M\to\infty} K^2\overline{K}^{-1} = 0$. Then, for all $(\tau_1,\tau_2) \in \left\{I_r(\frac{2\pi K}{2^j}) - I_r(\frac{2\pi\overline{K}}{2^j})\right\}$, we have

$$\frac{2\pi\overline{K}}{2^j} < |\lambda_r - \widetilde{\lambda}_{\tau_1}| < 2\pi - \frac{2\pi\overline{K}}{2^j}$$

or

$$\frac{2\pi\overline{K}}{2^j} < |\mu_r - \widetilde{\mu}_{\tau_2}| < 2\pi - \frac{2\pi\overline{K}}{2^j}$$

hold. Therefore, according to proof of 3(c) of Theorem 1, we can see that

$$\begin{aligned}
&\max\left\{|\beta_{J_Y}(j, \widetilde{\lambda}_{\tau_1}, \widetilde{\mu}_{\tau_2})| : (\tau_1,\tau_2) \in \left\{I_r(\frac{2\pi K}{2^j}) - I_r(\frac{2\pi\overline{K}}{2^j})\right\}\right\}\\
&< \frac{2^j|A_r|^2}{(2\pi)^2}\Bigg\{\int_{-C}^{C}\int_{-C}^{C} \widehat{\phi}(\omega_1,\omega_2)d\omega_1 d\omega_2\\
&- \overline{K}^2 \min\Big\{\int_{-C}^{C}\int_{-C}^{C} \frac{\omega_1^2}{2}\widehat{\phi}(\omega_1,\omega_2)d\omega_1 d\omega_2, \int_{-C}^{C}\int_{-C}^{C} \frac{\omega_2^2}{2}\widehat{\phi}(\omega_1,\omega_2)d\omega_1 d\omega_2\Big\}\Bigg\}\\
&+ O(1 + 2^{2j}N^{-1} + 2^{2j}M^{-1} + 2^{-j}\sqrt{NM\log NM} + 2^jK^4),
\end{aligned}$$

and if $\overline{K}$ also satisfies the order of (6), then $O(1 + 2^{2j}N^{-1} + 2^{2j}M^{-1}$ $+2^{-j}\sqrt{NM\log NM} + 2^jK^4) = o(2^j\overline{K}^2)$. Furthermore, for any $(\tau_1,\tau_2) \in I_r(\frac{2\pi\widetilde{K}}{2^j})$ where

$\widetilde{K} \to 0$ and $\tilde{K}\overline{K}^{-1} \to 0$, as $N, M \to \infty$, we have

$$\beta_{J_Y}(j, \widetilde{\lambda}_{\tau_1}, \widetilde{\mu}_{\tau_2}) = \frac{2^j |A_r|^2}{(2\pi)^2} \int_{-C}^{C} \int_{-C}^{C} \widehat{\phi}(\omega_1, \omega_2) d\omega_1 d\omega_2$$
$$+ O(1 + 2^{2j} N^{-1} + 2^{2j} M^{-1} + 2^{-j} \sqrt{NM \log NM} + 2^j \widetilde{K}^2)$$
$$= \frac{2^j |A_r|^2}{(2\pi)^2} \int_{-C}^{C} \int_{-C}^{C} \widehat{\phi}(\omega_1, \omega_2) d\omega_1 d\omega_2 + o(2^j \overline{K}^2)$$
$$> \max \left\{ |\beta_{J_Y}(j, \widetilde{\lambda}_{\tau_1}, \widetilde{\mu}_{\tau_2})| : (\tau_1, \tau_2) \in \left\{ I_r(\frac{2\pi K}{2^j}) - I_r(\frac{2\pi \overline{K}}{2^j}) \right\} \right\}.$$

Hence, we have $(\overline{\lambda}_d, \overline{\mu}_d) \in I_r(\frac{2\pi \overline{K}}{2^j})$, and so, for large j $(\overline{\lambda}_d, \overline{\mu}_d) = (\widehat{\lambda}_r, \widehat{\mu}_r)$.

6.3 Proof of Theorem 3. Since $(\widehat{\lambda}_r, \widehat{\mu}_r) \in I_r(\frac{2\pi \overline{K}}{2^j}) = I_r(\frac{2\pi \overline{K} 2^{-j\theta'}}{2^{j'}})$ and $\theta' > 0$ satisfies (7), it follows that

$$\beta_{J_Y}(j', \widetilde{\lambda}_{\widehat{\tau}_{1,r}}, \widetilde{\mu}_{\widehat{\tau}_{2,r}}) = \frac{2^{j'} |A_r|^2}{(2\pi)^2} \int_{-C}^{C} \int_{-C}^{C} \widehat{\phi}(\omega_1, \omega_2) d\omega_1 d\omega_2$$
$$+ O\left(2^{j'} \left(\overline{K} 2^{-j\theta'} \right)^2 \right) \quad a.s.$$

Therefore,

$$\widehat{A}_r = \left[\frac{(2\pi)^2}{2^{j'}} \left(\int_{-C}^{C} \int_{-C}^{C} \widehat{\phi}(\omega_1, \omega_2) d\omega_1 d\omega_2 \right)^{-1} \right.$$
$$\left. \left\{ \frac{2^{j'} |A_r|^2}{(2\pi)^2} \int_{-C}^{C} \int_{-C}^{C} \widehat{\phi}(\omega_1, \omega_2) d\omega_1 d\omega_2 + O\left(2^{j'} \left(\overline{K} 2^{-j\theta'} \right)^2 \right) \right\} \right]^{\frac{1}{2}}$$
$$= \left(|A_r|^2 + O\left(\left(\overline{K} 2^{-j\theta'} \right)^2 \right) \right)^{\frac{1}{2}} = A_r + O\left(\left(\overline{K} 2^{-j\theta'} \right)^2 \right) \quad a.s.$$

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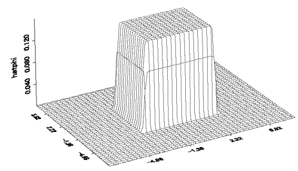

Figure 1: The graph of $\widehat{\phi}(\omega_1, \omega_2)$.

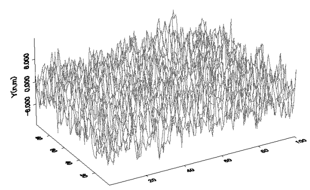

Figure 2: The observation of example 4.

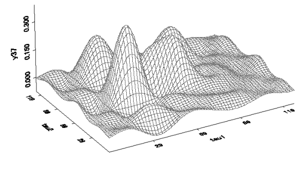

Figure 3: $2^{-j}\beta_{J_Y}(j, \widetilde{\lambda}_{\tau_1}, \widetilde{\mu}_{\tau_2})$ of example 4 with $j = 3$, $J = 7$.

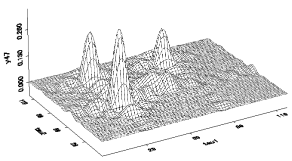

Figure 4: $2^{-j}\beta_{J_Y}(j, \widetilde{\lambda}_{\tau_1}, \widetilde{\mu}_{\tau_2})$ of example 4 with $j = 4$, $J = 7$.

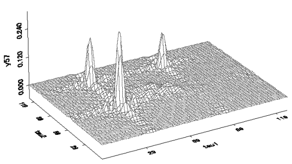

Figure 5: $2^{-j}\beta_{J_Y}(j, \widetilde{\lambda}_{\tau_1}, \widetilde{\mu}_{\tau_2})$ of example 4 with $j = 5$, $J = 7$.

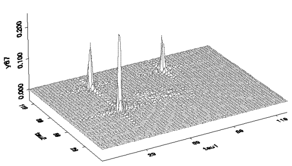

Figure 6: $2^{-j}\beta_{J_Y}(j, \widetilde{\lambda}_{\tau_1}, \widetilde{\mu}_{\tau_2})$ of example 4 with $j = 6$, $J = 7$.

(j,J)	4,6	4,7	4,8	5,6	5,7	5,8	6,6	6,7	6,8
(M1)	0.340	0.341	0.343	0.299	0.302	0.309	0.225	0.235	0.254
(M2)	0.155	0.157	0.158	0.132	0.139	0.141	0.089	0.108	0.114
(M3)	0.091	0.092	0.092	0.033	0.036	0.037	0.024	0.024	0.027
$\overline{\tau}_{1,1},\overline{\tau}_{2,1}$	17,27	33,54	67,108	17,27	33,54	67,108	17,27	33,54	67,108
	-1.473	-1.522	-1.497	-1.473	-1.522	-1.497	-1.473	-1.522	-1.497
$\overline{\lambda}_1,\overline{\mu}_1$	-0.491	-0.491	-0.491	-0.491	-0.491	-0.491	-0.491	-0.491	-0.491
$\overline{\tau}_{1,2},\overline{\tau}_{2,2}$	42,52	84,105	169,209	42,52	84,105	169,209	42,52	84,105	169,209
	0.982	0.982	1.006	0.982	0.982	1.006	0.982	0.982	1.006
$\overline{\lambda}_2,\overline{\mu}_2$	1.963	2.013	1.988	1.963	2.013	1.988	1.963	2.013	1.988

Table 1: The peak values (M1)-(M3) and $(\overline{\lambda}_d, \overline{\mu}_d)$ in example 2.

(j,J)	4,6	4,7	4,8	5,6	5,7	5,8	6,6	6,7	6,8
(M1)	0.371	0.372	0.374	0.318	0.322	0.328	0.238	0.249	0.269
(M2)	0.240	0.241	0.244	0.189	0.200	0.205	0.120	0.155	0.167
(M3)	0.143	0.156	0.159	0.127	0.134	0.136	0.085	0.104	0.110
$\overline{\tau}_{1,1},\overline{\tau}_{2,1}$	17,27	33,54	67,108	17,27	33,54	67,108	17,27	33,54	67,108
	-1.473	-1.522	-1.497	-1.473	-1.522	-1.497	-1.473	-1.522	-1.497
$\overline{\lambda}_1,\overline{\mu}_1$	-0.491	-0.491	-0.491	-0.491	-0.491	-0.491	-0.491	-0.491	-0.491
$\overline{\tau}_{1,2},\overline{\tau}_{2,2}$	17,52	34,105	67,209	17,52	33,105	67,209	17,52	33,105	67,209
	-1.473	-1.473	-1.497	-1.473	-1.522	-1.497	-1.473	-1.522	-1.497
$\overline{\lambda}_2,\overline{\mu}_2$	1.963	2.013	1.988	1.963	2.013	1.988	1.963	2.013	1.988
$\overline{\tau}_{1,3},\overline{\tau}_{2,3}$	43,52	85,105	168,209	42,52	84,105	169,209	42,52	84,105	169,209
	1.080	1.031	0.982	0.982	0.982	1.006	0.982	0.982	1.006
$\overline{\lambda}_3,\overline{\mu}_3$	1.963	2.013	1.988	1.963	2.013	1.988	1.963	2.013	1.988

Table 2: The peak values (M1)-(M3) and estimators $(\overline{\lambda}_d, \overline{\mu}_d)$ in example 4.

1	mean	sd	2	mean	sd	3	mean	sd	4	mean	sd
	-1.497	0.000		-1.497	0.000		-1.497	0.000		-1.497	0.000
$\widehat{\lambda}_1,\widehat{\mu}_1$	-0.491	0.000	$\widehat{\lambda}_1,\widehat{\mu}_1$	-0.491	0.000	$\widehat{\lambda}_1,\widehat{\mu}_1$	-0.491	0.000	$\widehat{\lambda}_1,\widehat{\mu}_1$	-0.491	0.000
	1.006	0.000		1.006	0.000		-1.497	0.000		-1.497	0.000
$\widehat{\lambda}_2,\widehat{\mu}_2$	1.991	0.008	$\widehat{\lambda}_2,\widehat{\mu}_2$	1.992	0.009	$\widehat{\lambda}_2,\widehat{\mu}_2$	1.997	0.012	$\widehat{\lambda}_2,\widehat{\mu}_2$	1.993	0.010
	-	-		-	-		1.006	0.000		1.006	0.000
$\widehat{\lambda}_3,\widehat{\mu}_3$	-	-	$\widehat{\lambda}_3,\widehat{\mu}_3$	-	-	$\widehat{\lambda}_3,\widehat{\mu}_3$	1.995	0.011	$\widehat{\lambda}_3,\widehat{\mu}_3$	1.997	0.012
$\widehat{A}_1$	1.451	0.016	$\widehat{A}_1$	1.453	0.015	$\widehat{A}_1$	1.461	0.013	$\widehat{A}_1$	1.460	0.016
$\widehat{A}_2$	0.967	0.009	$\widehat{A}_2$	0.969	0.009	$\widehat{A}_2$	1.193	0.015	$\widehat{A}_2$	1.193	0.013
$\widehat{A}_3$	-	-	$\widehat{A}_3$	-	-	$\widehat{A}_3$	0.983	0.011	$\widehat{A}_3$	0.983	0.010

Table 3: The performances of estimators in examples 1-4.

communicated by *Masanobu Taniguchi*

Department of Mathematics, Faculty of Science, Niigata University, 8050, Ikarashi

2-no-cho, Nishi-ku, Niigata City, Niigata, 950-2181, Japan

email:hirukawa@math.sc.niigata-u.ac.jp