THE SINGLE VALUED EXTENSION PROPERTY FOR HEREDITARILY NORMALOID OPERATORS

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Abstract. A bounded linear operator T on a Hilbert space (or a Banach space) is said to be normaloid if the operator norm $\|T\|$ of T equals the spectral radius $r(T) = \sup\{|z| \mid z \in \sigma(T)\}$ of T where $\sigma(T)$ denotes the spectrum of T. T is said to be hereditarily normaloid if the restriction $T|_{\mathcal{M}}$ of T to every of its invariant subspaces $\mathcal{M}$ is normaloid. Also, T is said to be totally hereditarily normaloid if T is hereditarily normaloid and for every invertible restriction $T|_{\mathcal{M}}$ of T to its invariant subspace $\mathcal{M}$ the inverse $(T|_{\mathcal{M}})^{-1}$ is also normaloid.

We shall show that every hereditarily normaloid has the single valued extension property (SVEP) and hence satisfies Browder's theorem. Also, we give an example of hereditarily normaloid which is not totally hereditarily normaloid (hence, not paranormal) and does not satisfy Weyl's theorem.

1. Introduction

Let $\mathcal{B}(\mathcal{H})$ be the set of all bounded linear operators on $\mathcal{H}$, an operator $T \in \mathcal{B}(\mathcal{H})$ is said to have the single valued extension property (SVEP) at λ if for any open neighborhood $\mathcal{U}$ of λ and analytic function $f : \mathcal{U} \to \mathcal{H}$ the zero function is only analytic solusion of the equation

$$(T - z)f(z) = 0,$$

and T is said to have the SVEP if T has the SVEP at any $\lambda \in \mathbb{C}$ (or equivalently $\lambda \in \sigma(T)$).

Let $w(T)$ be the Weyl spectrum of T, $\pi_{00}(T)$ the set of all isolated points of $\sigma(T)$ which are eigenvalues of T with finite multiplicities and $\pi_0(T)$ the set of all isolated points of $\sigma(T)$ for which the corresponding Riesz idempotent has finite-dimensional range, i.e.,

$$\begin{aligned} w(T) &= \{\lambda \in \sigma(T) \mid T - \lambda \text{ is not Fredholm with Fredholm index } 0 \,\} \\ \pi_{00}(T) &= \{\lambda \in \mathrm{iso}\sigma(T) \mid 0 < \dim\ker(T - \lambda) < \infty\} \\ \pi_0(T) &= \{\lambda \in \mathrm{iso}\sigma(T) \mid \text{The Riesz idempotent } E \text{ w. r. t. } \lambda \text{ satisfies } \dim E\mathcal{H} < \infty\}. \end{aligned}$$

An operator $T \in \mathcal{B}(\mathcal{H})$ is said to satisfy Weyl's theorem if

$$\sigma(T) \setminus w(T) = \pi_{00}(T).$$

Also, T is said to satisfy Browder's theorem if

$$\sigma(T) \setminus \pi_0(T) = w(T).$$

It is well-known that every normal, hyponormal, p-hyponormal, w-hyponormal, class A, paranormal or totally hereditarily normaloid has the SVEP and satisfies Weyl's theorem, however, normaloid operator does not necessarily have the SVEP and does not necessarily satisfy Weyl's theorem in general. For

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exsample, the adjoint U^* of the unilateral shift U is normaloid which does not have the SVEP and the operator $U \oplus U^*$ is also normaloid which does not satisfy Weyl's theorem.

$$\begin{aligned}\{\text{hyponormal},\ p\text{-hyponormal}\} &\subset \{w\text{-hyponormal}\} \subset \{\text{class } A\} \\ &\subset \{\text{paranormal}\} \\ &\subset \{\textbf{totally hereditarily normaloid}\} \\ &\subset \{\text{hereditarily normaloid}\} \\ &\subset \{\textbf{normaloid}\}\end{aligned}$$

We are very interested in the following problems;

1) Does every hereditarily normaloid have the SVEP?

2) Deos every hereditary normaloid satisfy Weyl's theorem?

In this paper, we show that every hereditarily normaloid has the SVEP in section 2. Also, we give an example of hereditarily normaloid which does not satisfy Weyl's theorem in section 3.

2. The SVEP for hereditarily normaloid.

It is known that every hereditarily normaloid operator T has the following property.

$$\ker(T-\lambda) \perp \ker(T-\mu) \quad (\lambda,\ \mu \in \sigma_p(T),\ \lambda \neq \mu).$$

In fact, the next lemma holds.

Lemma 1. If T is hereditarily normaloid and $\lambda,\ \mu \in \sigma_p(T)$ such as $\lambda \neq \mu$ and $|\mu| \leq |\lambda|$. Then

$$\|x\| \leq \|x+y\| \quad {}^\forall x \in \ker(T-\lambda),\ {}^\forall y \in \ker(T-\mu).$$

In particular, $\ker(T-\lambda) \perp \ker(T-\mu)$.

Proof) Without loss of generality, we may assume $\lambda = 1$. Let $\mathcal{M} = \ker(T-\mu) \vee \ker(T-1)$, the closed subspace generated by $\ker(T-\mu)$ and $\ker(T-1)$. Then $\mathcal{M}$ is an invariant subspace of T which satisfies $\sigma(T|_{\mathcal{M}}) = \{\mu,\ 1\}$, it follows that $\|T|_{\mathcal{M}}\| = 1$ by assumption.

(1) Suppose that $|\mu| < 1$. Then for each $x \in \ker(T-1)$ and $y \in \ker(T-\mu)$,

$$\|x+\mu^n y\| = \|(T|_{\mathcal{M}})^n(x+y)\| \leq \|x+y\| \quad ({}^\forall n \in \mathbb{N}).$$

Letting $n \to \infty$, we have the conclusion.

(2) Suppose that $|\mu| = 1$. Then for each $x \in \ker(T-1)$ and $y \in \ker(T-\mu)$,

$$\left\|x + \frac{1+\mu}{2}y\right\| \leq \frac{1}{2}(\|x+y\| + \|x+\mu y\|) = \frac{1}{2}(\|x+y\| + \|(T|_{\mathcal{M}})(x+y)\| \leq \|x+y\|.$$

Since $\alpha y \in \ker(T-\mu)$ for all $\alpha \in \mathbb{C}$, it follows that

$$\left\|x + \left(\frac{1+\mu}{2}\right)^n y\right\| \leq \|x+y\| \quad ({}^\forall n \in \mathbb{N}).$$

Letting $n \to \infty$, we have the conclusion by $\left|\dfrac{1+\mu}{2}\right| < 1$.

It is immediate that $\langle x, y\rangle = 0$ from $\|x\| \leq \|x+y\|$ for all $x \in \ker(T-1)$ and $y \in \ker(T-\mu)$, hence $\ker(T-1) \perp \ker(T-\mu)$. $\square$

Theorem 1. Every herditarily normaloid has the SVEP.

Proof) Let $\mathcal{U} \subset \mathbb{C}$ be an open set and $f : \mathcal{U} \to \mathcal{H}$ be an analytic function such that

$$(T - z)f(z) = 0 \quad ({}^{\forall} z \in \mathcal{U}).$$

Then $f(z) \perp f(w)$ $(z \neq w)$ and hence

$$\|f(z)\|^2 = \lim_{w \to z} \langle f(z), f(w) \rangle = 0.$$

Thus we have the conclusion. □

Since Banach space operators with SVEP satisfy Browder's theorem we have the following.

Corollary 1. Every herditarily normaloid satisfies Browder's theorem.

3. An example of hereditarily normaloid.

In this section, we give an elementary example of hereditarily normaloid which does not satisfy Weyl's theorem. We remark that more deeper results are obtained by B. P. Duggal, S.V. Djordjević and C.S. Kubrusly [3].

Let $V : L^2[0,1] \to L^2[0,1]$ be the Volterra integral operator defined by

$$(Vf)(x) := \int_0^x f(t)dt \quad (f \in L^2[0,1]).$$

It is known that

$$\|(V+1)^{-1}\| = 1, \ \sigma((V+1)^{-1}) = \{1\} \ \text{ and } \ \sigma_p((V+1)^{-1}) = \emptyset.$$

Let $\mathcal{H} := L^2[0,1] \oplus \mathbb{C} \oplus \mathbb{C}$ and

$$T := (V+1)^{-1} \oplus 1 \oplus 0 \quad \text{on } \mathcal{H}.$$

We shall show that T is hereditarily normaloid which does not satisfy Weyl's theorem.

Let $P : \mathcal{H} \to L^2[0,1]$ be the orthogonal projection onto $L^2[0,1]$ and $\mathcal{M}$ a T-invariant (closed) subspace of $\mathcal{H}$. If $P\mathcal{M} \neq \{0\}$ then $P\mathcal{M}$ is a $(V+1)^{-1}$-invariant subspace. Inclusion

$$\partial\sigma((V+1)^{-1}|_{P\mathcal{M}}) \subset \sigma_a((V+1)^{-1}|_{P\mathcal{M}}) \subset \sigma((V+1)^{-1}) = \{1\}$$

implies that $\partial\sigma((V+1)^{-1}|_{P\mathcal{M}}) = \{1\}$ and hence $\sigma((V+1)^{-1}) = \{1\}$. Also,

$$\{1\} \subset \sigma_a((V+1)^{-1}|_{P\mathcal{M}}) \subset \sigma_a(T|_{\mathcal{M}}) \subset \sigma(T) = \{0,1\}$$

implies that $\{1\} \subset \sigma_a((V+1)^{-1}|_{P\mathcal{M}}) = \sigma((V+1)^{-1}|_{P\mathcal{M}}) \subset \{0,1\}$. Since $\|T|_{\mathcal{M}}\| \leq 1$ and $r(T|_{\mathcal{M}}) = 1$ the restriction $T|_{\mathcal{M}}$ is normaloid.

If $P\mathcal{M} = \{0\}$ then $\mathcal{M} \subset \{0\} \oplus \mathbb{C} \oplus \mathbb{C}$ and $T|_{\mathcal{M}} = \left.\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right|_{\mathcal{M}}$ is a finite rank subnormal operator, so it is normal and hence normaloid. Therefore we obtain the following theorem.

Theorem 2. The operator $T = (V+1)^{-1} \oplus 1 \oplus 0$ on $\mathcal{H}$ is hereditarily normaloid which does not satisfy Weyl's theorem.

Proof. Since $T|_{\mathcal{M}}$ is normaloid for every T-invariant subspace $\mathcal{M}$, the operator T is hereditarily normaloid. Since $\sigma(T) = \pi_{00}(T) = \{0,1\}$ and $w(T) = \{1\}$ the operator T does not satisfy Weyl's theorem. □

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