New multiple weights and the Adams inequality on weighted Morrey spaces

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Abstract. We introduce a new multiple weights class and generalize the Adams inequality to the multilinear fractional integral operator on weighted Morrey spaces. We also investigate the boundedness of the multilinear fractional integral operator from weighted Morrey spaces to BMO or Lipschitz spaces.

1 Introduction It is well known that fractional integral operator plays important roles in harmonic analysis and other fields, such as partial differential equations and quantum mechanics. Also, weighted and Morrey norm estimates have relevance to partial differential equations and quantum mechanics. On the other hand, there are many papers with respect to multilinear operators, since A. P. Calderón proposed to the conjecture related to the bilinear Hilbert transforms(cf. [3]).

Furthermore, multilinear fractional integral operators have been studied by Grafakos [2] and Kenig and Stein [6].

In this paper, we investigate the boundedness of multilinear fractional integral operators on product of weighted Morrey spaces.

1.1 Some preliminaries and notation We will use the following notation: For $1 < p < \infty$, we define $p' := \frac{p}{p-1}$. We write a ball of radius R centered at x_0 by $B(x_0, R) := \{x; |x - x_0| < R\}$ and $aB(x_0, R) := B(x_0, aR)$, for any $a > 0$. We denote the characteristic function of E by χ_E. $|E|$ is the Lebesgue measure of E. We call a nonnegative locally integrable function w on $\mathbb{R}^n$ a weight function. We denote $w(E) = \int_E w(x)dx$. The letter C shall always denote a positive constant which is independent of essential parameters and not necessarily the same at each occurrence.

Firstly, we need some preparatory definitions and works. In 1975, Adams [1] proved the boundedness of the fractional integral operator I_α on classical Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$. The definition of ordinary fractional integral operator I_α is as follows.

Definition 1.1. For $0 < \alpha < n$, we define

$$I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} dy.$$

Moreover, the definition of classical Morrey spaces $L^{p,\lambda}(\mathbb{R}^n)$ is as follows.

Definition 1.2. For $1 < p < \infty$ and $0 \le \lambda < 1$,

$$L^{p,\lambda}(\mathbb{R}^n) := \{f \in L^1_{loc}(\mathbb{R}^n) : \|f\|_{L^{p,\lambda}(\mathbb{R}^n)} < \infty\},$$

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where

$$\|f\|_{L^{p,\lambda}(\mathbb{R}^n)} := \sup_{B\subset\mathbb{R}^n} \left(\frac{1}{|B|^\lambda} \int_B |f(x)|^p dx \right)^{\frac{1}{p}}.$$

Adams showed the celebrated theorem.

Theorem A. *Let* $0 < \alpha < n$*,* $0 < \lambda < 1 - \frac{\alpha}{n}$*,* $1 < p < \frac{n}{\alpha}(1-\lambda)$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} \cdot \frac{1}{1-\lambda}$*. Then there exists a constant* $C > 0$ *such that*

$$\|I_\alpha f\|_{L^{q,\lambda}(\mathbb{R}^n)} \le C \|f\|_{L^{p,\lambda}(\mathbb{R}^n)}.$$

Remark 1. When $\lambda = 0$, Theorem A is the Hardy, Littlewood and Sobolev theorem.

To investigate Theorem A with respect to weighted norm inequalities, we define weighted Lebesgue spaces.

Definition 1.3 (Weighted Lebesgue spaces)**.** Let $0 < p < \infty$. Suppose that w is a weight function on $\mathbb{R}^n$.

$$L^p(w) := \left\{ f : \|f\|_{L^p(w)} := \left(\int_{\mathbb{R}^n} |f(x)|^p w(x) dx \right)^{\frac{1}{p}} < \infty \right\}.$$

The definition of $A_{p,q}$ weight is as follows.

Definition 1.4. Let $1 < p, q < \infty$. One says that a weight w is in the class $A_{p,q}(\mathbb{R}^n)$ if

$$\sup_{B\subset\mathbb{R}^n} \left(\frac{1}{|B|} \int_B w(x)^q dx \right)^{\frac{1}{q}} \left(\frac{1}{|B|} \int_B w(x)^{-p'} dx \right)^{\frac{1}{p'}} < \infty.$$

In 1974, as it is well-known, Muckenhoupt and Wheeden [11] characterized the weight w, for which the fractional integral operator I_α: $L^p(w^p) \to L^q(w^q)$ for $0 < \alpha < n$, $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. That is, they showed the next theorem.

Theorem B. *Let* $0 < \alpha < n$*,* $1 < p < \frac{n}{\alpha}$ *and* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$*. Then the inequality*

$$\|I_\alpha f\|_{L^q(w^q)} \le C \|f\|_{L^p(w^p)}$$

holds if and only if w *is in the class* $A_{p,q}(\mathbb{R}^n)$*.*

To consider recent results with the boundedness of I_α, we define weighted Morrey spaces(see [7]).

Definition 1.5 (Weighted Morrey spaces)**.** Let $0 < p < \infty$, $0 \le \lambda < 1$ and u, v are weight functions on $\mathbb{R}^n$.

$$L^{p,\lambda}(u,v) := \left\{ f : \|f\|_{L^{p,\lambda}(u,v)} < \infty \right\},$$

where

$$\|f\|_{L^{p,\lambda}(u,v)} := \sup_{B\subset\mathbb{R}^n} \left(\frac{1}{v(B)^\lambda} \int_B |f(x)|^p u(x) dx \right)^{\frac{1}{p}}.$$

Recently, we have introduced $\tilde{A}_{p,\lambda}(\mathbb{R}^n)$ (see [5]).

Definition 1.6. Let $1 < p < \infty$ and $0 < \lambda < 1$. One says that a w is in the class $\tilde{A}_{p,\lambda}(\mathbb{R}^n)$ if

$$\sup_{B \subset \mathbb{R}^n} \left(\frac{1}{|B|} \int_B w(x) dx \right)^{\frac{\lambda}{p}} \left(\frac{1}{|B|} \int_B w(x)^{-\frac{\lambda p'}{p}} dx \right)^{\frac{1}{p'}} < \infty.$$

We proved the Adams inequality on weighted Morrey spaces:

Theorem C (see [5])**.** *Let $0 < \alpha < n$, $0 < \lambda < 1 - \frac{\alpha}{n}$, $1 < p < \frac{n}{\alpha}(1-\lambda)$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} \cdot \frac{1}{1-\lambda}$. If $w \in \tilde{A}_{p,\lambda}(\mathbb{R}^n)$ then there exists a constant $C > 0$ such that*

$$\|I_\alpha f\|_{L^{q,\lambda}(w^\lambda, w)} \leq C \|f\|_{L^{p,\lambda}(w^\lambda, w)}.$$

According to Kenig and Stein [6], we define the multilinear fractional integral operator $I_{\alpha,m}$ and modified multilinear fractional integral operator $\tilde{I}_{\alpha,m}$.

Definition 1.7. Let $0 < \alpha < mn$ and $\vec{f} := (f_1, \ldots, f_m)$.
(1) We define the multilinear fractional integral operator:

$$I_{\alpha,m}\left(\vec{f}\right)(x) := \int_{\mathbb{R}^{mn}} \frac{f_1(y_1) \cdots f_m(y_m)}{|(x-y_1, \ldots, x-y_m)|^{mn-\alpha}} d\vec{y},$$

where

$$|(x-y_1, \ldots, x-y_m)| := \sqrt{|x-y_1|^2 + \cdots + |x-y_m|^2},$$

and $d\vec{y} := dy_1 \cdots dy_m$.
(2) We define the modified multilinear fractional integral operator:

$$\tilde{I}_{\alpha,m}\left(\vec{f}\right)(x) := \int_{\mathbb{R}^{mn}} \left(\frac{1}{|(x-y_1, \ldots, x-y_m)|^{mn-\alpha}} - \frac{1}{|(y_1, \ldots, y_m)|^{mn-\alpha}} \chi_{\{|(y_1,\ldots,y_m)| \geq 1\}}(y_1, \ldots, y_m) \right) f_1(y_1) \cdots f_m(y_m) d\vec{y}.$$

To consider the boundedness of two multilinear operators on product of weighted Lebesgue spaces, we define two multiple weights classes:

Definition 1.8. Let $1 < p_1, \ldots, p_m < \infty$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ and $\vec{P} := (p_1, \ldots, p_m)$.
(1) (see [8]). One says that a vector of weights $\vec{w} := (w_1, \ldots, w_m)$ is in the class $A_{\vec{P}}(\mathbb{R}^n)$ if

$$\sup_{B \subset \mathbb{R}^n} \left(\frac{1}{|B|} \int_B v_{\vec{w}}(x) dx \right)^{\frac{1}{p}} \prod_{j=1}^m \left(\frac{1}{|B|} \int_B w_j(x)^{-\frac{p_j'}{p_j}} dx \right)^{\frac{1}{p_j'}} < \infty,$$

where $v_{\vec{w}}(x) := w_1(x)^{\frac{p}{p_1}} \cdots w_m(x)^{\frac{p}{p_m}}$.
(2) (see [10]). Let $1/m < p \leq q < \infty$. One says that a vector of weights $\vec{w}$ is in the class $A_{\vec{P},q}(\mathbb{R}^n)$ if

$$\sup_{B \subset \mathbb{R}^n} \left(\frac{1}{|B|} \int_B \prod_{j=1}^m w_j(x)^q dx \right)^{\frac{1}{q}} \prod_{j=1}^m \left(\frac{1}{|B|} \int_B w_j(x)^{-p_j'} dx \right)^{\frac{1}{p_j'}} < \infty.$$

Moen [10] extended Theorem B to multilinear fractional integral operator.

Theorem D (see [10])**.** *Suppose that* $0 < \alpha < mn$, $1 < p_1, \ldots, p_m < \infty$ *and* $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ *satisfies* $1/m < p < n/\alpha$. *Moreover* q *is defined by* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. *Then the inequality*

$$\left\| I_{\alpha,m}\left(\vec{f}\right)\right\|_{L^q((w_1\cdots w_m)^q)} \leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(w_j^{p_j})}$$

holds if and only if $\vec{w}$ *satisfies the* $A_{\vec{P},q}(\mathbb{R}^n)$ *condition.*

To consider the case $p \geq \frac{n}{\alpha}(1-\lambda)$, we define BMO and Lipschitz spaces.

Definition 1.9 (BMO and Lipschitz spaces)**.** We define BMO and Lipschitz spaces.

$$BMO(\mathbb{R}^n) := \left\{ f \in L^1_{loc}(\mathbb{R}^n) : \|f\|_{BMO(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|f\|_{BMO(\mathbb{R}^n)} := \sup_{B\subset\mathbb{R}^n} \inf_{c\in\mathbb{C}} \frac{1}{|B|}\int_B |f(x)-c|\,dx.$$

For $0 < \varepsilon < 1$,

$$Lip_\varepsilon(\mathbb{R}^n) := \left\{ f : \|f\|_{Lip_\varepsilon(\mathbb{R}^n)} < \infty \right\},$$

where

$$\|f\|_{Lip_\varepsilon(\mathbb{R}^n)} := \sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\varepsilon}.$$

Tang [14] proved the following theorems.

Theorem E. *Let* $0 < \alpha < mn$, $1 < p_1, \ldots, p_m < \infty$, $0 \leq \lambda < 1$ *and* $\varepsilon := \alpha + n\left(\frac{\lambda-1}{p_1} + \cdots + \frac{\lambda-1}{p_m}\right)$.
(1) If $\varepsilon = 0$ *and* $\frac{\lambda}{p_1} + \cdots + \frac{\lambda}{p_m} < 1$, *then we have*

$$\left\| \tilde{I}_{\alpha,m}\left(\vec{f}\right)\right\|_{BMO(\mathbb{R}^n)} \leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j,\lambda}(\mathbb{R}^n)}.$$

(2) If $0 < \varepsilon < 1$, *then we have*

$$\left\| \tilde{I}_{\alpha,m}\left(\vec{f}\right)\right\|_{Lip_\varepsilon(\mathbb{R}^n)} \leq C \prod_{j=1}^{m} \|f_j\|_{L^{p_j,\lambda}(\mathbb{R}^n)}.$$

In this paper, we introduce another multiple weights class which is a generalization of $w \in \tilde{A}_{p,\lambda}(\mathbb{R}^n)$ and extend the Adams inequality to $I_{\alpha,m}$ on product of weighted Morrey spaces. We also extend Tang's results to weighted Morrey spaces. These results show that our multiple weights class adapt to the multilinear fractional integral operator $I_{\alpha,m}$.

In Section 2, we define another multiple weights class and give main results. In Section 3, we prove main results.

2 Main results We define the multiple weights class(cf. [5]).

Definition 2.1. Let $1 < p_1, \ldots, p_m < \infty$, $0 < \lambda < 1$, $\vec{P} = (p_1, \ldots, p_m)$ and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. One says that a vector of weights $\vec{w}$ is in the class $\tilde{A}_{\vec{P},\lambda}(\mathbb{R}^n)$ if

$$\sup_{B \subset \mathbb{R}^n} \left(\frac{1}{|B|} \int_B v_{\vec{w}}(x) dx \right)^{\frac{\lambda}{p}} \prod_{j=1}^{m} \left(\frac{1}{|B|} \int_B w_j(x)^{-\frac{p_j'}{p_j}\lambda} dx \right)^{\frac{1}{p_j'}} < \infty,$$

where $v_{\vec{w}}(x) = w_1(x)^{\frac{p}{p_1}} \cdots w_m(x)^{\frac{p}{p_m}}$.

We have some remarks with respect to three multiple weights classes.

Remark 2.
(1) Let $1 < p_1, \ldots, p_m < \infty$ and $0 < \lambda < 1$.

$$A_{\vec{P}}(\mathbb{R}^n) \subsetneqq \tilde{A}_{\vec{P},\lambda}(\mathbb{R}^n).$$

(2) If $\vec{w} \in \tilde{A}_{\vec{P},\lambda}(\mathbb{R}^n)$ then we have $(w_1^{\lambda}, \ldots, w_m^{\lambda}) \in A_{\vec{P}}(\mathbb{R}^n)$.
(3) A vector of weights $\vec{w}$ is in the class $\tilde{A}_{\vec{P},\lambda}(\mathbb{R}^n)$ if and only if a vector of weights $\left(w_1^{\frac{\lambda}{p_1}}, \ldots, w_m^{\frac{\lambda}{p_m}} \right)$ is in the class $A_{\vec{P},p/\lambda}(\mathbb{R}^n)$.

In the following, we always assume that $0 < \alpha < mn$, $0 < \lambda < 1$ and $1 < p_1, \ldots, p_m < \infty$. We denote $\frac{1}{p} := \frac{1}{p_1} + \cdots + \frac{1}{p_m}$. We obtain the following results.

Theorem 1. *If* $0 < \lambda < 1 - \frac{\alpha}{mn}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}\frac{1}{1-\lambda} > 0$ *and* $\vec{w} \in \tilde{A}_{\vec{P},\lambda}(\mathbb{R}^n)$, *then we have*

$$\left\| I_{\alpha,m}\left(\vec{f}\right) \right\|_{L^{q,\lambda}\left(v_{\vec{w}}^{\lambda}, v_{\vec{w}}\right)} \le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j,\lambda}(w_j^{\lambda}, v_{\vec{w}})}.$$

When $p \ge \frac{n}{\alpha}(1-\lambda)$, we obtain the following results.

Theorem 2. *If* $0 < \lambda < 1 - \frac{\alpha}{mn}$, $\frac{1}{p} - \frac{\alpha}{n}\frac{1}{1-\lambda} = 0 < \frac{\lambda}{p} < 1$ *and* $\vec{w} \in \tilde{A}_{\vec{P},\lambda}(\mathbb{R}^n)$, *then we have*

$$\left\| \tilde{I}_{\alpha,m}\left(\vec{f}\right) \right\|_{BMO(\mathbb{R}^n)} \le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j,\lambda}(w_j^{\lambda}, v_{\vec{w}})}.$$

Remark 3. When $m = 1$, the above theorem is Theorem 2 for $\varepsilon = 0$ in Iida, Komori-Furuya and Sato [5]. When $w_1 = \cdots = w_m = 1$, Theorem 2 is corresponding to Theorem E (1). Moreover, when $m = 1$ and $w_1 = 1$, Theorem 2 is corresponding to the Peetre theorem([13]).

Theorem 3. *Suppose that* $0 < \varepsilon = \alpha + \frac{n(\lambda-1)}{p} < 1$ *and* $\vec{w} \in \tilde{A}_{\vec{P},\lambda}(\mathbb{R}^n)$, *then we have*

$$\left\| \tilde{I}_{\alpha,m}\left(\vec{f}\right) \right\|_{Lip_{\varepsilon}(\mathbb{R}^n)} \le C \prod_{j=1}^{m} \|f_j\|_{L^{p_j,\lambda}(w_j^{\lambda}, v_{\vec{w}})}.$$

Remark 4. When $w_1 = \cdots = w_m = 1$, Theorem 3 is corresponding to Theorem E (2).

When $m = 1$, we have a corollary:

Corollary 3.1. *Let $0 < \alpha < n$, $0 < \lambda < 1$ and $1 < p < \infty$. Suppose that $0 < \varepsilon = \alpha + \frac{n(\lambda-1)}{p} < 1$. If $w \in \tilde{A}_{p,\lambda}(\mathbb{R}^n)$, then we have*

$$\left\| \tilde{I}_\alpha f \right\|_{Lip_\varepsilon(\mathbb{R}^n)} \leq C \left\| f \right\|_{L^{p,\lambda}(w^\lambda, w)}.$$

Remark 5. Corollary 3.1 improves the condition of weights comparing with Theorem 3 and Theorem 2 for $\varepsilon > 0$ in Iida, Komori-Furuya and Sato [5]. Moreover, when $w = 1$, Corollary 3.1 is corresponding to the Peetre theorem([13]).

3 Proofs of main theorems

We use the following lemmas about weights.

Definition 3.1. One says that a weight w is in the class $A_\infty(\mathbb{R}^n)$ if there exist $C > 0, \varepsilon > 0$ such that for every balls B and all measurble subsets A of B,

$$\frac{w(A)}{w(B)} \leq C \left(\frac{|A|}{|B|} \right)^\varepsilon$$

Note that if $w \in A_\infty(\mathbb{R}^n)$ then w satisfies doubling condition: $w(2B) \leq Cw(B)$.

Lemma 3.1 ([10]; Theorem 3.4)**.** *If $\vec{w} \in A_{\vec{P},q}(\mathbb{R}^n)$ then $\prod_{i=1}^m {w_i}^q \in A_\infty(\mathbb{R}^n)$.*

By Remark 2 (3) and Lemma 3.1 we obtain the next lemma.

Lemma 3.2. *If $\vec{w} \in \tilde{A}_{\vec{P},\lambda}(\mathbb{R}^n)$ then $v_{\vec{w}} = \prod_{j=1}^m w_j^{\frac{p}{p_j}} \in A_\infty(\mathbb{R}^n)$.*

3.1 Proof of Theorem 1

To prove Theorem 1, we use the following. Firstly according to Lerner et al [8], we define two multi(sub)linear maximal operators.

Definition 3.2. Let $0 < \theta < m$ and $\vec{f} = (f_1, \ldots, f_m)$.

$$\mathcal{M}\left(\vec{f}\right)(x) := \sup_{B \ni x} \prod_{j=1}^m \frac{1}{|B|} \int_B |f_j(y_j)|\, dy_j,$$

$$\mathcal{M}_\theta\left(\vec{f}\right)(x) := \sup_{B \ni x} \frac{1}{|B|^\theta} \prod_{j=1}^m \int_B |f_j(y_j)|\, dy_j.$$

The following two lemmas are variants of the results by Adams [1] (see also [3], [4], [9]).

Lemma 3.3. *Under the condition of Theorem 1, we have the pointwise inequality:*

$$\left| I_{\alpha,m}\left(\vec{f}\right)(x) \right| \leq C \mathcal{M}_{m+\frac{\lambda-1}{p}}\left(\vec{f}\right)(x)^{1-\frac{p}{q}} \mathcal{M}\left(\vec{f}\right)(x)^{\frac{p}{q}}.$$

Proof. For any $x \in \mathbb{R}^n$ and $\delta > 0$, we have

$$\begin{aligned} \left| I_{\alpha,m}\left(\vec{f}\right)(x) \right| &\leq \int_{\mathbb{R}^{mn}} \frac{|f_1(y_1)| \cdots |f_m(y_m)|}{|(x-y_1, \ldots, x-y_m)|^{mn-\alpha}} d\vec{y} \\ &= \left(\int_{|(x-y_1,\ldots,x-y_m)|<\delta} + \int_{|(x-y_1,\ldots,x-y_m)|\geq\delta} \right) \\ &\quad \frac{|f_1(y_1)| \cdots |f_m(y_m)|}{|(x-y_1, \ldots, x-y_m)|^{mn-\alpha}} d\vec{y} \\ &= I + II. \end{aligned}$$

Firstly we estimate I:

$$\begin{aligned}
I &= \sum_{j=0}^{\infty}\int_{2^{-j-1}\delta<|(x-y_1,\ldots,x-y_m)|<2^{-j}\delta}\frac{|f_1(y_1)|\cdots|f_m(y_m)|}{|(x-y_1,\ldots,x-y_m)|^{mn-\alpha}}d\vec{y}\\
&\le C\sum_{j=0}^{\infty}\frac{1}{(2^{-j}\delta)^{mn-\alpha}}\int_{|(x-y_1,\ldots,x-y_m)|<2^{-j}\delta}|f_1(y_1)|\cdots|f_m(y_m)|\,d\vec{y}\\
&\le C\delta^{\alpha}\mathcal{M}\left(\vec{f}\right)(x).
\end{aligned}$$

Secondly we estimate II:

$$\begin{aligned}
II &= \int_{|(x-y_1,\ldots,x-y_m)|\ge\delta}\frac{|f_1(y_1)|\cdots|f_m(y_m)|}{|(x-y_1,\ldots,x-y_m)|^{mn-\alpha}}d\vec{y}\\
&= \sum_{k=0}^{\infty}\int_{2^{k}\delta\le|(x-y_1,\ldots,x-y_m)|<2^{k+1}\delta}\frac{|f_1(y_1)|\cdots|f_m(y_m)|}{|(x-y_1,\ldots,x-y_m)|^{mn-\alpha}}d\vec{y}\\
&\le C\sum_{k=0}^{\infty}\frac{1}{(2^{k}\delta)^{mn-\alpha}}(2^{k+1}\delta)^{n\left(1+\frac{\lambda-1}{p_1}\right)}\cdots\left(2^{k+1}\delta\right)^{n\left(1+\frac{\lambda-1}{p_m}\right)}\\
&\quad\times\frac{1}{|2^{k+1}B(x,\delta)|^{m+\frac{\lambda-1}{p}}}\prod_{j=1}^{m}\left(\int_{|x-y_j|\le 2^{k+1}\delta}|f_j(y_j)|\,dy_j\right)\\
&\le C\delta^{n\left(\frac{\lambda-1}{q}\right)}\mathcal{M}_{m+\frac{\lambda-1}{p}}\left(\vec{f}\right)(x).
\end{aligned}$$

By taking

$$\delta := \left(\frac{\mathcal{M}_{m+\frac{\lambda-1}{p}}\left(\vec{f}\right)(x)}{\mathcal{M}\left(\vec{f}\right)(x)}\right)^{\frac{1}{n}\cdot\frac{p}{1-\lambda}} > 0,$$

we obtain the desired result. □

Suppose that $x_0\in\mathbb{R}^n$, $r>0$, $B=B(x_0,r)$ and $x\in B$. Let $f_j^0(y_j)=f_j(y_j)\chi_{2B}(y_j)$, $f_j^{\infty}(y_j)=f_j(y_j)\chi_{(2B)^c}(y_j)$ and

$$V_x\left(\vec{f},\rho\right) := \int_{|(x-y_1,\ldots,x-y_m)|<\rho}|f_1(y_1)\cdots f_m(y_m)|\,d\vec{y}.$$

Lemma 3.4. *Under the condition of Theorem 1, we have*

$$\left|I_{\alpha,m}(f_1^0,\ldots,f_l^0,f_{l+1}^{\infty},\ldots,f_m^{\infty})(x)\right| \le C\int_r^{\infty}\rho^{\alpha-mn-1}V_x(\vec{f},\rho)d\rho,$$

for $l=0,\ldots,m-1$. *When* $l=0$, *we regard* $I_{\alpha,m}(f_1^0,\ldots,f_l^0,f_{l+1}^{\infty},\ldots,f_m^{\infty})(x)$ *as* $I_{\alpha,m}(f_1^{\infty},\ldots,f_m^{\infty})(x)$.

Proof. Since $\prod_{j=1}^{l}(2B)\times\prod_{j=l+1}^{m}(2B)^c \subset \{(y_1,\ldots,y_m):|(x-y_1,\ldots,x-y_m)|>r\}$, we have

$$\begin{aligned}
&\left|I_{\alpha,m}(f_1^0,\dots,f_l^0,f_{l+1}^\infty,\dots,f_m^\infty)(x)\right| \\
&\le \int_{|(x-y_1,\dots,x-y_m)|>r} \frac{|f_1(y_1)\cdots f_m(y_m)|}{|(x-y_1,\dots,x-y_m)|^{mn-\alpha}} d\vec{y} \\
&= \sum_{k=0}^{\infty} \int_{2^k r<|(x-y_1,\dots,x-y_m)|<2^{k+1}r} \frac{|f_1(y_1)\cdots f_m(y_m)|}{|(x-y_1,\dots,x-y_m)|^{mn-\alpha}} d\vec{y} \\
&\le \sum_{k=0}^{\infty} \frac{1}{(2^k r)^{mn-\alpha}} \left(V_x\left(\vec{f},2^{k+1}r\right) - V_x\left(\vec{f},2^k r\right)\right) \\
&\le C\sum_{k=0}^{\infty} \frac{1}{(2^{k+2}r)^{mn-\alpha}} \frac{1}{2^{k+2}r} \int_{2^{k+1}r}^{2^{k+2}r} V_x\left(\vec{f},\rho\right) d\rho \\
&\le C\int_r^\infty \rho^{\alpha-mn-1}\cdot V_x\left(\vec{f},\rho\right) d\rho.
\end{aligned}$$

Therefore we obtain the desired result. □

Proposition 1 (see [8]). *Let $1<p_1,\dots,p_m<\infty$ and $\frac{1}{p}=\frac{1}{p_1}+\cdots+\frac{1}{p_m}$. Then the inequality*

$$\left\|\mathcal{M}\left(\vec{f}\right)\right\|_{L^p(v_{\vec{w}})} \le C\prod_{j=1}^m \|f_j\|_{L^{p_j}(w_j)}$$

holds if and only if $\vec{w}$ is in the class $A_{\vec{P}}(\mathbb{R}^n)$.

Finally, we will prove Theorem 1.

Proof of Theorem 1. For any $x_0\in\mathbb{R}^n$ and $r>0$, let $B=B(x_0,r)\subset\mathbb{R}^n$, $f_j^0(x)=f_j(x)\chi_{2B}(x)$ and $f_j^\infty(x)=f_j(x)\chi_{(2B)^c}(x)$. For any $x\in B$, we obtain

$$\begin{aligned}
I_{\alpha,m}\left(\vec{f}\right)(x) &= I_{\alpha,m}(f_1^0,\dots,f_m^0)(x) + \sum_{(l_1,\dots,l_m)\ne(0,\dots,0)} I_{\alpha,m}(f_1^{l_1},\dots,f_m^{l_m})(x) \\
&= I + \sum_{(l_1,\dots,l_m)\ne(0,\dots,0)} II_{l_1,\dots,l_m}.
\end{aligned}$$

Firstly we estimate I. By Lemma 3.3 and the definition of $\tilde{A}_{\vec{P},\lambda}(\mathbb{R}^n)$ we obtain a pointwise inequality:

$$\left|I_{\alpha,m}(f_1^0,\dots,f_m^0)(x)\right| \le C\prod_{j=1}^m \|f_j\|_{L^{p_j,\lambda}(w_j^\lambda,v_{\vec{w}})}^{1-\frac{p}{q}} \mathcal{M}(f_1^0,\dots,f_m^0)(x)^{\frac{p}{q}}.$$

By Remark 2 (2), we have $(w_1^\lambda,\dots,w_m^\lambda)\in A_{\vec{P}}(\mathbb{R}^n)$. Hence, we can use Proposition 1 for the weights $(w_1^\lambda, \dots, w_m^\lambda)$. By Lemma 3.2, we obtain

$$\begin{aligned}
&\left(\int_B \left|I_{\alpha,m}(f_1^0,\dots,f_m^0)(x)\right|^q v_{\vec{w}}(x)^\lambda dx\right)^{\frac{1}{q}}\\
&\le C\prod_{j=1}^m \|f_j\|_{L^{p_j,\lambda}(w_j^\lambda,v_{\vec{w}})}^{1-\frac{p}{q}} \left(\int_B \mathcal{M}(f_1^0,\dots,f_m^0)(x)^p v_{\vec{w}}(x)^\lambda dx\right)^{\frac{1}{p}\cdot\frac{p}{q}}\\
&\le C\prod_{j=1}^m \|f_j\|_{L^{p_j,\lambda}(w_j^\lambda,v_{\vec{w}})}^{1-\frac{p}{q}} \prod_{j=1}^m \left(\int_{2B} |f_j(x)|^{p_j} w_j(x)^\lambda dx\right)^{\frac{1}{p_j}\cdot\frac{p}{q}}\\
&\le C v_{\vec{w}}(B)^{\frac{\lambda}{q}} \prod_{j=1}^m \|f_j\|_{L^{p_j,\lambda}(w_j^\lambda,v_{\vec{w}})}.
\end{aligned}$$

Secondly we estimate $II_{l_1,\dots,l_m}$. Firstly, we estimate $V_x\left(\vec{f},\rho\right)$.

$$\begin{aligned}
V_x\left(\vec{f},\rho\right) &\le \prod_{j=1}^m \left(\int_{|x-y_j|<\rho} |f_j(y_j)|\, dy_j\right)\\
&\le \prod_{j=1}^m \left(\int_{B(x,\rho)} |f_j(y_j)|^{p_j} w_j(y_j)^\lambda dy_j\right)^{\frac{1}{p_j}} w_j^{-\frac{p_j'}{p_j}\lambda}(B(x,\rho))^{\frac{1}{p_j'}}\\
&\le \prod_{j=1}^\infty \|f_j\|_{L^{p_j,\lambda}(w_j^\lambda,v_{\vec{w}})} \cdot v_{\vec{w}}(B(x,\rho))^{\frac{\lambda}{p}} \cdot \prod_{j=1}^m w_j^{-\frac{p_j'}{p_j}\lambda}(B(x,\rho))^{\frac{1}{p_j'}}\\
&\le C\rho^{n\left(m+\frac{\lambda-1}{p}\right)} \prod_{j=1}^m \|f_j\|_{L^{p_j,\lambda}(w_j^\lambda,v_{\vec{w}})}.
\end{aligned}$$

Therefore by Lemma 3.4, we obtain

$$|II_{l_1,\dots,l_m}| \le C\,|B(x_0,r)|^{\frac{\lambda-1}{q}} \prod_{j=1}^m \|f_j\|_{L^{p_j,\lambda}(w_j^\lambda,v_{\vec{w}})}.$$

Thus, by Hölder's inequality, we have the desired inequality:

$$\left(\frac{1}{v_{\vec{w}}(B)^\lambda}\int_B |II_{l_1,\dots,l_m}|^q v_{\vec{w}}(x)^\lambda dx\right)^{\frac{1}{q}} \le C\prod_{j=1}^m \|f_j\|_{L^{p_j,\lambda}(w_j^\lambda,v_{\vec{w}})}.$$

□

3.2 Proof of Theorem 2 To prove Theorem 2, we use the next proposition.

Proposition 2 (see [12]). *If $w \in A_\infty(\mathbb{R}^n)$, then we have*

$$\|f\|_{BMO(\mathbb{R}^n)} \approx \sup_{B\subset\mathbb{R}^n} \inf_{c\in\mathbb{C}} \frac{1}{w(B)}\int_B |f(x)-c|\, w(x)dx < \infty.$$

Proof of Theorem 2. For any $x_0 \in \mathbb{R}^n$, $r>0$, let $B = B(x_0,r) \subset \mathbb{R}^n$, $f_j^0(y_j) = f_j(y_j)\chi_{2B}(y_j)$ and $f_j^\infty(y_j) = f_j(y_j)\chi_{(2B)^c}(y_j)$.
Let

$$c_0 := -\int_{|(y_1,\dots,y_m)|\ge 1} \frac{f_1^0(y_1)\cdots f_m^0(y_m)}{|(y_1,\dots,y_m)|^{mn-\alpha}} d\vec{y},$$

$c_{l_1,\dots,l_m} := \tilde{I}_{\alpha,m}(f_1^{l_1},\dots,f_m^{l_m})(x_0)$ where $(l_1,\dots,l_m) \neq (0,\dots,0)$. Then we have

$$\begin{aligned}
&\left|\tilde{I}_{\alpha,m}\left(\vec{f}\right)(x) - c_0 - \sum_{(l_1,\dots,l_m)\neq(0,\dots,0)} c_{l_1,\dots,l_m}\right| \\
&\leq \left|I_{\alpha,m}(f_1^0,\dots,f_m^0)(x)\right| \\
&\quad + \sum_{(l_1,\dots,l_m)\neq(0,\dots,0)} \int_{\mathbb{R}^{mn}} \left|\frac{1}{|(x-y_1,\dots,x-y_m)|^{mn-\alpha}} - \frac{1}{|(x_0-y_1,\dots,x_0-y_m)|^{mn-\alpha}}\right| \\
&\quad \times \left|f_1^{l_1}(y_1)\right|\cdots\left|f_m^{l_m}(y_m)\right| d\vec{y} \\
&= I + \sum_{(l_1,\dots,l_m)\neq(0,\dots,0)} II_{l_1,\dots,l_m}.
\end{aligned}$$

We estimate $II_{l_1,\dots,l_m}$. For $x \in B$, we have

$$\begin{aligned}
II_{l_1,\dots,l_m} &\leq Cr \int_{|(x_0-y_1,\dots,x_0-y_m)|>r} \frac{|f_1(y_1)|\cdots|f_m(y_m)|}{|(x_0-y_1,\dots,x_0-y_m)|^{mn-\alpha+1}} d\vec{y} \\
&\leq Cr \sum_{k=0}^{\infty} \int_{2^k r<|(x_0-y_1,\dots,x_0-y_m)|\leq 2^{k+1}r} \frac{|f_1(y_1)|\cdots|f_m(y_m)|}{|(x_0-y_1,\dots,x_0-y_m)|^{mn-\alpha+1}} d\vec{y} \\
&\leq Cr \prod_{j=1}^m \|f_j\|_{L^{p_j,\lambda}(w_j^\lambda, v_{\vec{w}})} \sum_{k=0}^{\infty} \frac{v_{\vec{w}}(2^{k+1}B)^{\frac{\lambda}{p}} \prod_{j=1}^m w_j^{-\frac{p_j'}{p_j}\lambda}(2^{k+1}B)^{\frac{1}{p_j'}}}{|2^k B|^{m-\frac{\alpha}{n}+\frac{1}{n}}} \\
&\leq Cr \prod_{j=1}^m \|f_j\|_{L^{p_j,\lambda}(w_j^\lambda, v_{\vec{w}})} \sum_{k=0}^{\infty} \frac{\left|2^{k+1}B\right|^{\frac{\lambda-1}{p}+m}}{|2^k B|^{m-\frac{\alpha}{n}+\frac{1}{n}}} \\
&\leq C \prod_{j=1}^m \|f_j\|_{L^{p_j,\lambda}(w_j^\lambda, v_{\vec{w}})}.
\end{aligned}$$

We estimate I. By Remark 2 (3), we can use Theorem D for the vector of weights $\left(w_1^{\frac{\lambda}{p_1}},\dots,w_m^{\frac{\lambda}{p_m}}\right)$. By Lemma 3.2, we have

$$\begin{aligned}
&\frac{1}{v_{\vec{w}}(B)} \int_B I \cdot v_{\vec{w}}(x) dx \\
&\leq \frac{1}{v_{\vec{w}}(B)} \left(\int_B \left|I_{\alpha,m}(f_1^0,\dots,f_m^0)(x)\right|^{\frac{p}{\lambda}} v_{\vec{w}}(x) dx\right)^{\frac{\lambda}{p}} \cdot v_{\vec{w}}(B)^{\frac{1}{\left(\frac{p}{\lambda}\right)'}} \\
&= v_{\vec{w}}(B)^{-\frac{\lambda}{p}} \left(\int_B \left|I_{\alpha,m}(f_1^0,\dots,f_m^0)(x)\right|^{\frac{p}{\lambda}} \left(w_1^{\frac{\lambda}{p_1}}\cdots w_m^{\frac{\lambda}{p_m}}\right)(x)^{\frac{p}{\lambda}} dx\right)^{\frac{\lambda}{p}} \\
&\leq C v_{\vec{w}}(B)^{-\frac{\lambda}{p}} \prod_{j=1}^m \left(\int_{2B} |f_j(y_j)|^{p_j} \cdot w_j(y_j)^{\frac{\lambda}{p_j}\cdot p_j} dy_j\right)^{\frac{1}{p_j}} \\
&\leq C \prod_{j=1}^m \|f_j\|_{L^{p_j,\lambda}(w_j^\lambda, v_{\vec{w}})}.
\end{aligned}$$

Therefore we obtain

$$\left\|\tilde{I}_{\alpha,m}\left(\vec{f}\right)\right\|_{BMO(v_{\vec{w}})} \leq C\prod_{j=1}^{m}\|f_j\|_{L^{p_j,\lambda}(w_j^{\lambda},v_{\vec{w}})},$$

where

$$\|f\|_{BMO(v_{\vec{w}})} := \sup_{B\subset\mathbb{R}^n}\inf_{c\in\mathbb{C}}\left(\frac{1}{v_{\vec{w}}(B)}\int_B |f(x)-c|\, v_{\vec{w}}(x)dx\right)$$

By Proposition 2, we obtain the desired result. □

3.3 Proof of Theorem 3

Proof of Theorem 3. For any $x \neq y \in \mathbb{R}^n$, let $r = |x-y| > 0$ and $B = B(x,r)$. We will prove a inequality:

$$\left|\tilde{I}_{\alpha,m}\left(\vec{f}\right)(x) - \tilde{I}_{\alpha,m}\left(\vec{f}\right)(y)\right| \leq C\,|x-y|^{\varepsilon}\prod_{j=1}^{m}\|f_j\|_{L^{p_j,\lambda}(w_j^{\lambda},v_{\vec{w}})}.$$

Let $f_j^0(y_j) = f_j(y_j)\chi_{2B}(y_j)$ and $f_j^{\infty}(y_j) = f_j(y_j)\chi_{(2B)^c}(y_j)$. Same as in the proof of Theorem 2 we write

$$\begin{aligned}
&\left|\tilde{I}_{\alpha,m}\left(\vec{f}\right)(x) - \tilde{I}_{\alpha,m}\left(\vec{f}\right)(y)\right| \\
&\leq \sum_{l_1,\ldots,l_m\in\{0,\infty\}}\left|I_{\alpha,m}(f_1^{l_1},\ldots,f_m^{l_m})(x) - I_{\alpha,m}(f_1^{l_1},\ldots,f_m^{l_m})(y)\right| \\
&= I + \sum_{(l_1,\ldots,l_m)\neq(0,\ldots,0)} II_{l_1,\ldots,l_m}.
\end{aligned}$$

Since $\prod_{j=1}^{m} B(x,2r) \subset \{(y_1,\ldots,y_m);\ |(x-y_1,\ldots,x-y_m)| < 2\sqrt{m}r\}$, by Hölder's inequality, we obtain

$$
\begin{aligned}
&\left|I_{\alpha,m}(f_1^0,\dots f_m^0)(x)\right| \\
&\le \int_{|(x-y_1,\dots,x-y_m)|<2\sqrt{m}r} \frac{|f_1(y_1)|\cdots|f_m(y_m)|}{|(x-y_1,\dots,x-y_m)|^{mn-\alpha}}d\vec{y} \\
&\le \sum_{k=0}^{\infty} \frac{1}{(2^{-k}\sqrt{m}r)^{mn-\alpha}} \\
&\quad\times \left[\prod_{j=1}^{m}\left(\int_{|x-y_j|<2^{-k+1}\sqrt{m}r} |f_j(y_j)|^{p_j}\, w_j(y_j)^{\lambda}dy_j\right)^{\frac{1}{p_j}} \cdot \prod_{j=1}^{m} w_j^{-\frac{p'_j}{p_j}\lambda}(2^{-k+1}\sqrt{m}B)^{\frac{1}{p'_j}}\right] \\
&\le C\left(\prod_{j=1}^{m}\|f_j\|_{L^{p_j,\lambda}(w_j^{\lambda},v_{\vec{w}})}\right) \\
&\quad\times \sum_{k=0}^{\infty}\left[\frac{v_{\vec{w}}(2^{-k+1}\sqrt{m}B)^{\frac{\lambda}{p}}}{(2^{-k}\sqrt{m}r)^{mn-\alpha}} \prod_{j=1}^{m} w_j^{-\frac{p'_j}{p_j}\lambda}(2^{-k+1}\sqrt{m}B)^{\frac{1}{p'_j}}\right] \\
&\le C\,|B|^{\frac{\varepsilon}{n}}\left(\prod_{j=1}^{m}\|f_j\|_{L^{p_j,\lambda}(w_j^{\lambda},v_{\vec{w}})}\right)\sum_{k=0}^{\infty}2^{-k\varepsilon} \\
&\le C\,|x-y|^{\varepsilon}\prod_{j=1}^{m}\|f_j\|_{L^{p_j,\lambda}(w_j^{\lambda},v_{\vec{w}})}.
\end{aligned}
$$

The similarly estimate holds for $I_{\alpha,m}(f_1^0,\dots,f_m^0)(y)$.

Lastly we estimate $II_{l_1,\dots,l_m}$. We have

$$
\begin{aligned}
&|II_{l_1,\dots,l_m}| \\
&\le C|x-y|\int_{|(x-y_1,\dots,x-y_m)|\ge 2r} \frac{|f_1(y_1)|\cdots|f_m(y_m)|}{|(x-y_1,\dots,x-y_m)|^{mn-\alpha+1}}d\vec{y} \\
&\le C|x-y|\prod_{j=1}^{m}\|f_j\|_{L^{p_j,\lambda}(w_j^{\lambda},v_{\vec{w}})}\sum_{k=1}^{\infty}\frac{v_{\vec{w}}(2^{k+1}B)^{\frac{\lambda}{p}}\prod_{j=1}^{m} w_j^{-\frac{p'_j}{p_j}\lambda}(2^{k+1}B)^{\frac{1}{p'_j}}}{|2^kB|^{m-\frac{\alpha}{n}+\frac{1}{n}}} \\
&\le C|x-y|\prod_{j=1}^{m}\|f_j\|_{L^{p_j,\lambda}(w_j^{\lambda},v_{\vec{w}})}\sum_{k=1}^{\infty}\left|2^kB\right|^{\frac{\varepsilon-1}{n}} \\
&\le C|x-y|^{\varepsilon}\prod_{j=1}^{m}\|f_j\|_{L^{p_j,\lambda}(w_j^{\lambda},v_{\vec{w}})}.
\end{aligned}
$$

Therefore we obtain the desired result. □

References

[1] D. R. Adams, *A note on Riesz potentials*, Duke Math. J. 42 (1975), 765-778.

[2] L. Grafakos, *On multilinear fractional integrals*, Studia Math. 102(1992), 49-56.

[3] L. Grafakos, *Classical and modern Fourier analysis*, Prentice Hall, Upper Saddle River, NJ., 2003.

[4] L. Hedberg, *On certain convolution inequalities*, Proc. Amer. Math. Soc. 36 (1972), 505-510.

[5] T. Iida, Y. Komori-Furuya and E. Sato, *The Adams inequality on weighted Morrey spaces*, Tokyo J. Math., to appear.

[6] C. Kenig and E. Stein, *Multilinear estimates and fractional integration*, Math. Reseach Letter **6** (1999), 1-15.

[7] Y. Komori and S. Shirai, *Weighted Morrey spaces and a singular integral operator*, Math. Nachr., **282** (2009), 219-231.

[8] A. Lerner, S. Ombrosi, C. Pèrez, R. H. Torres and R. Trujillo-González, *New maximal functions and multiple weights for the multilinear Calderón-Zygmund theory*, Adv. Math. 220, (2009), 1222-1264.

[9] S. Lu, Y. Ding and D. Yan, *Singular integrals and related topics*, World Scientific publishing, Singapore, 2007.

[10] K. Moen, *Weighted inequalities for multilinear fractional integral operators*, Collect. Math. 60 (2009), 213–238.

[11] B. Muckenhoupt and R. Wheeden, *Weighted norm inequalities for fractional integrals*, Trans. Amer. Math. Soc., 192 (1974), 261-274.

[12] B. Muckenhoupt and R. Wheeden, *Weighted bounded mean oscillation and the Hilbert transform*, Studia Math., (1976) 221-237.

[13] J. Peetre, *On the theory of $\mathcal{L}_{p,\lambda}$ spaces*, J. Funct. Anal., **4** (1969), 71-87.

[14] L. Tang, *Endpoint estimates for multilinear fractional integrals*, J. Aust. Math. Soc. 84, 2008, 419-429.

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