ON FUZZY HYPERFILTERS OF HYPERLATTICES

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ABSTRACT.

This paper focuses on a new kind of fuzzy hyperfilter of a hyperlattice called $(\in, \in \sqcup q)-$fuzzy hyperfilter. These fuzzy hyperfilters are characterized by their level hyperfilters. Also the concept and properties of a fuzzy hyperfilter with thresholds are discussed.

1 Introduction Since the inception of the notion of a fuzzy set in 1965 [25] which laid the foundations of fuzzy set theory, the literature on fuzzy set theory and its applications has been growing rapidly amounting by now to several papers (see [1-5], [9-14], [16], [19], [20], [23] and [24]). These are widely scattered over many disciplines such as artificial intelligence, computer science, control engineering, expert systems, management science, operations research, pattern recognition, robotics, and others. The studies on fuzzy lattices ([1-3], [19] and [20]) can be grouped into two classes: fuzzifying the subsethood relation and maintaining ordering relations, meet operations and join operations over nonempty sets [1,20] and fuzzifying ordering relations, meet operations and join operations over nonempty sets by means of many valued equalities [3]. Because of the fact that the former class disregards the notion of many-valued equivalence relation, it does not contribute to the required theory of fuzzy lattices. Although the latter provides a useful way for developing the required theory of fuzzy lattices, it involves only many-valued equalities (separated many-valued equivalence relations) and establishes only the fuzzy counterpart to complete lattices based on many-valued equalities. In order to research the logical system whose propositional value is given in a lattice from the semantic viewpoint, Xu [21] proposed the concept of lattice implication algebras, and discussed some of their properties. Xu and Qin [22] introduced the notion of implicative filters in a lattice implication algebra, and investigated some of their properties. In [23], they applied the concept of fuzzy sets to lattice implication algebras and proposed the notions of fuzzy filters and fuzzy implicative filters. Recently, a great deal of literature has been produced on the theory of implicative filters and fuzzy implicative filters. A new type of fuzzy algebraic structures as $(\in, \in \vee q)-$fuzzy subgroups was introduced in an earlier paper of Bhakat and Das [5] by using the combined notions of 荘 belongingness 鋳 and 荘 quasi-coincidence 鋳 of fuzzy points and fuzzy sets. In fact, $(\in, \in \vee q)-$fuzzy subgroup is an important and useful generalization of Rosenfeld's fuzzy subgroup. This concept has been studied further in [4].

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The theory of algebraic hyperstructures which is a generalization of the concept of ordinary algebraic structures was first introduced by Marty [15]. Since then many researchers have worked on algebraic hyperstructures and developed it. A short review of this theory appears in [6]. A recent book [7] contains a wealth of applications. Via this book, Corsini and Leoreanu presented some of the numerous applications of algebraic hyperstructures, especially those from the last fifteen years, to the following subjects: geometry, hypergraphs, binary relations, lattices, fuzzy sets and rough sets, automata, cryptography, codes, median algebras, relation algebras, artificial intelligence and probabilities. Also A. Rahnamai-Barghi in [17] and [18] investigated on prime ideal theorem and semiprime ideals in hyperlattices and meethyperlattices. Fuzzy sets and algebraic hyperstructures introduced by Zadeh and Marty, respectively, are now used extensively from both the theoretical point of view and their many applications. The relations between fuzzy sets and hyperstructures have been already considered by Ameri, Hedayati, Koguep, Nkuimi, Lele, Corsini, Leoreanu and others, for instance see [11], [12] and [14]. This paper focuses on a new kind of fuzzy hyperfilters of a hyperlattice. The combined notions of 荘 belongingness 鋳 and 荘 quasi-coincidence 鋳 of fuzzy points and fuzzy sets are used to introduce this kind of fuzzy hyperfilters.

2 Fuzzy hyperfilters of hyperlattices

Definition 2.1. [14] Let $(\mathcal{L}, \leq)$ be a non empty partial ordered set and $\vee : \mathcal{L} \times \mathcal{L} \longrightarrow \rho(\mathcal{L})^*$ be a hyperoperation, where $\rho(\mathcal{L})$ is a power set of $\mathcal{L}$ and $\rho(\mathcal{L})^* = \rho(\mathcal{L}) \setminus \{\varnothing\}$ and $\wedge : \mathcal{L} \times \mathcal{L} \longrightarrow \mathcal{L}$ be an operation. Then $(\mathcal{L}, \vee, \wedge)$ is a *hyperlattice* if for all $a, b, c \in \mathcal{L}$,

(i) $a \in a \vee a, a \wedge a = a$;

(ii) $a \vee b = b \vee a, a \wedge b = b \wedge a$;

(iii) $(a \vee b) \vee c = a \vee (b \vee c); (a \wedge b) \wedge c = a \wedge (b \wedge c)$;

(iv) $a \in [a \wedge (a \vee b)] \cap [a \vee (a \wedge b)]$;

(v) $a \in a \vee b \Rightarrow a \wedge b = b$;

where for all non empty subsets A and B of $\mathcal{L}$, $A \wedge B = \{a \wedge b | a \in A, b \in B\}$ and $A \vee B = \bigcup \{a \vee b | a \in A, b \in B\}$.

Definition 2.2. [14] Let $(\mathcal{L}, \vee, \wedge)$ be a hyperlattice. A nonempty subset F of $\mathcal{L}$ is called a *hyperfilter* of $\mathcal{L}$ if for all $x, y \in \mathcal{L}$,

(i) $x, y \in F$ implies $x \wedge y \in F$.

(ii) If $x \in F$ and $x \leq y$, then $y \in F$.

Definition 2.3. [14] Let μ be a fuzzy set of a hyperlattice $\mathcal{L}$. Then μ is a *fuzzy hyperfilter* of $\mathcal{L}$, if for all $x, y \in \mathcal{L}$,

(i) $\mu(x \wedge y) \geq min(\mu(x), \mu(y))$,

(ii) $x \leq y$ implies $\mu(x) \leq \mu(y)$.

Proposition 2.4. [14] *Let μ be a fuzzy subset of a hyperlattice $\mathcal{L}$. Then μ is a fuzzy hyperfilter of $\mathcal{L}$, if and only if, for any $\alpha \in [0, 1]$, such that $\mu_\alpha \neq \varnothing$, μ_α is a hyperfilter of $\mathcal{L}$.*

Definition 2.5. Let $\mathcal{L}_1$ and $\mathcal{L}_2$ be two hyperlattices. A map $f : \mathcal{L}_1 \longrightarrow \mathcal{L}_2$ is called a *homomorphism* if for all $x, y \in \mathcal{L}_1$ the following conditions hold:

(i) $f(x \wedge y) = f(x) \wedge f(y)$,

(ii) $f(x \vee y) = f(x) \vee f(y)$,
(iii) $x \leq y$ implies that $f(x) \leq f(y)$.

Example 2.6. Let $\mathcal{L} = \{0, a, b, 1\}$. Then $(\mathcal{L}, \vee, \wedge)$ is a hyperlattice, where $\vee$ and $\wedge$ are defined by the following tables:

$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\vee$	0	a	b	1
0	$\{0\}$	$\{a\}$	$\{b\}$	$\{1\}$
a	$\{a\}$	$\{0, a\}$	$\{1\}$	$\{b, 1\}$
b	$\{b\}$	$\{1\}$	$\{0, b\}$	$\{a, 1\}$
1	$\{1\}$	$\{b, 1\}$	$\{a, 1\}$	$\mathcal{L}$

Also $\{\{1\}, \{a, 1\}, \{b, 1\}, \mathcal{L}\}$ is the set of all hyperfilters of $\mathcal{L}$. Now if $\mu(a) = 1 = \mu(1)$ and $\mu(0) = 0 = \mu(b)$, then it is easy to verify that μ is a fuzzy hyperfilter of $\mathcal{L}$.

3 $(\in, \in \sqcup q)-$fuzzy hyperfilters

In what follows, let $\mathcal{L}$ denote a hyperlattice unless otherwise specified.

A fuzzy subset μ of $\mathcal{L}$ of the form

$$\mu(y) = \begin{cases} t(\neq 0), & \text{if } \ y = x \\ 0, & \text{if } \ y \neq x \end{cases}$$

is said to be a *fuzzy point* with support x and value t and is denoted by x_t. A fuzzy point x_t is said to be *belong to* (resp. be *quasi-coincident with*) a fuzzy set μ, written as $x_t \in \mu$ (resp. $x_t q \mu$) if $\mu(x) \geq t$ (resp. $\mu(x) + t > 1$). If $x_t \in \mu$ or $x_t q \mu$, then we write $x_t \in \sqcup q \mu$. The symbol $\overline{\in}$ (resp. $\overline{q}$) means $\in$ (resp. q) dose not hold. The symbol $\overline{\in \sqcup q}$ means neither $\in$ nor q hold. The symbol $\overline{\in \sqcap q}$ means either $\overline{\in}$ or $\overline{q}$ holds. Now we introduce the concept of $(\in, \in \sqcup q)-$fuzzy hyperfilters of hyperlattices.

Definition 3.1. A fuzzy subset μ of $\mathcal{L}$ is called an $(\in, \in \sqcup q)-$*fuzzy hyperfilter* of $\mathcal{L}$ if for all $t, r \in (0, 1]$ and $x, y \in \mathcal{L}$,
(i) $x_t, y_r \in \mu$ implies $(x \wedge y)_{min(t,r)} \in \sqcup q \mu$,
(ii) $x_t \in \mu$ and $x \leq y$ imply $y_t \in \sqcup q \mu$.
Clearly every fuzzy hyperfilter according to the Definition 2.3, is an $(\in, \in \sqcup q)-$fuzzy hyperfilter of $\mathcal{L}$, but the following example shows that the converse is not true in general case.

Example 3.2. Let $\mathcal{L} = \{0, a, b, 1\}$. Then $(\mathcal{L}, \vee, \wedge)$ is a hyperlattice, where $\vee$ and $\wedge$ are defined by the following tables:

$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\vee$	0	a	b	1
0	$\mathcal{L}$	$\{a, 1\}$	$\{b, 1\}$	$\{1\}$
a	$\{a, 1\}$	$\{a, 1\}$	$\{1\}$	$\{1\}$
b	$\{b, 1\}$	$\{1\}$	$\{b, 1\}$	$\{1\}$
1	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$

Now if $\mu(a) = \mu(b) = \mu(1) = 0.9$ and $\mu(0) = 0.8$, then it is easy to verify that μ is an $(\in, \in \sqcup q)-$fuzzy hyperfilter of $\mathcal{L}$. However $\mu_{0.9} = \{a, b, 1\}$ is not a hyperfilter of $\mathcal{L}$ since $a \wedge b = 0 \notin \mu_{0.9}$. Therefore μ is not a fuzzy hyperfilter of $\mathcal{L}$.

Theorem 3.3. *Conditions (i) and (ii) in Definition 3.1, are equivalent to the following conditions respectively.*
(1) $min(\mu(x), \mu(y), 0.5) \leq \mu(x \wedge y)$, *for all* $x, y \in \mathcal{L}$,
(2) $min(\mu(x), 0.5) \leq \mu(y)$ *for all* $x, y \in \mathcal{L}$ *with* $x \leq y$.

Proof. $(i) \longrightarrow (1)$: Suppose that $x, y \in \mathcal{L}$. We can consider the following cases:
(a) $min(\mu(x), \mu(y)) < 0.5$. Assume that $\mu(x \wedge y) < min(\mu(x), \mu(y), 0.5)$, which implies that $\mu(x \wedge y) < min(\mu(x), \mu(y))$. Choose t such that $\mu(x \wedge y) < t < min(\mu(x), \mu(y))$. Then $x_t, y_t \in \mu$, but $(x \wedge y)_t \overline{\in \sqcup q} \mu$, which contradicts (i).
(b) $min(\mu(x), \mu(y)) \geq 0.5$. Assume that $\mu(x \wedge y) < 0.5$. Then $x_{0.5}, y_{0.5} \in \mu$, but $(x \wedge y)_{0.5} \overline{\in \sqcup q} \mu$, which is a contradiction. Hence (1) holds.

$(ii) \longrightarrow (2)$: Suppose that $x, y \in \mathcal{L}$ and $x \leq y$. We can consider the following cases:
(a) $\mu(x) < 0.5$. Assume that $\mu(y) < min(\mu(x), 0.5)$, which implies that $\mu(y) < \mu(x)$. Choose t such that $\mu(y) < t < \mu(x)$. Then $x_t \in \mu$, but $y_t \overline{\in \sqcup q} \mu$, which is a contradiction by (ii).
(b) $\mu(x) \geq 0.5$. Suppose that $\mu(y) < 0.5$. Then $x_{0.5} \in \mu$, but $y_{0.5} \overline{\in \sqcup q} \mu$, which is a contradiction. Therefore (2) is valid.

$(1) \longrightarrow (i)$: Let $x_t, y_r \in \mu$, then $\mu(x) \geq t$ and $\mu(y) \geq r$. We have $\mu(x \wedge y) \geq min(\mu(x), \mu(y), 0.5) \geq min(t, r, 0.5)$. There are two cases:
(a) $min(t, r) > 0.5$, then $\mu(x \wedge y) \geq 0.5$, which implies that $\mu(x \wedge y) + min(t, r) > 1$ and so $(x \wedge y)_{min(t,r)} q \mu$. Therefore $(x \wedge y)_{min(t,r)} \in \sqcup q \mu$.
(b) $min(t, r) \leq 0.5$, then $\mu(x \wedge y) \geq min(t, r)$ and so $(x \wedge y)_{min(t,r)} \in \mu$. Therefore $(x \wedge y)_{min(t,r)} \in \sqcup q \mu$.

$(2) \longrightarrow (ii)$: Let $x_t \in \mu$ and $x \leq y$, then $\mu(x) \geq t$. We have $\mu(y) \geq min(\mu(x), 0.5) \geq min(t, 0.5)$. Two following cases can be considered:
(a) $t > 0.5$, then $\mu(y) \geq 0.5$, which implies that $\mu(y) + t > 1$ and so $y_t q \mu$. Therefore $y_t \in \sqcup q \mu$.
(b) $t \leq 0.5$, then $\mu(y) \geq t$ and so $y_t \in \mu$. Therefore $y_t \in \sqcup q \mu$. □

Corollary 3.4. *A fuzzy subset* μ *of* $\mathcal{L}$ *is an* $(\in, \in \sqcup q)-$*fuzzy hyperfilter of* $\mathcal{L}$ *if and only if the conditions (1) and (2) in Theorem 3.3 hold.*

Proof. It is a consequence of Definition 3.1 and Theorem 3.2. □

Theorem 3.5. *Let* μ *be a fuzzy subset of* $\mathcal{L}$. *If* μ *is an* $(\in, \in \sqcup q)-$*fuzzy hyperfilter of* $\mathcal{L}$, *then for all* $0 < t \leq 0.5$, $\mu_t = \varnothing$ *or* μ_t *is a hyperfilter of* $\mathcal{L}$. *Conversely, if* $\mu_t (\neq \varnothing)$ *is a hyperfilter of* $\mathcal{L}$ *for all* $0 < t \leq 0.5$, *then* μ *is an* $(\in, \in \sqcup q)-$*fuzzy hyperfilter of* $\mathcal{L}$.

Proof. Let μ be an $(\in, \in \sqcup q)-$fuzzy hyperfilter of $\mathcal{L}$ and $0 < t \leq 0.5$. If $x, y \in \mu_t$, then $\mu(x) \geq t$ and $\mu(y) \geq t$. Thus we have

$$\mu(x \wedge y) \geq min(\mu(x), \mu(y), 0.5) \geq min(t, 0.5) = t,$$

that is $x \wedge y \in \mu_t$. Now let $x \in \mu_t$ and $x \leq y$. Thus $\mu(x) \geq t$, and so we have

$$\mu(y) \geq min(\mu(x), 0.5) \geq min(t, 0.5) = t,$$

that is $y \in \mu_t$. Therefore μ_t is a hyperfilter of $\mathcal{L}$. Conversely, let μ be a fuzzy subset of $\mathcal{L}$ such that $\mu_t(\neq \varnothing)$ is a hyperfilter of $\mathcal{L}$ for all $0 < t \leq 0.5$. If $x, y \in \mathcal{L}$, we can say that

$$\mu(x) \geq min(\mu(x), \mu(y), 0.5) = t_0, \qquad \mu(y) \geq min(\mu(x), \mu(y), 0.5) = t_0,$$

then $x, y \in \mu_{t_0}$, and so $x \wedge y \in \mu_{t_0}$, which implies that $\mu(x \wedge y) \geq t_0 = min(\mu(x), \mu(y), 0.5)$. Thus condition (1) of the Theorem 3.3 is verified. Now if $x \in \mathcal{L}$, we can write that $\mu(x) \geq min(\mu(x), 0.5) = t'_0$, then $x \in \mu_{t'_0}$, so if $x \leq y$ then $y \in \mu_{t'_0}$. Hence we can say that $\mu(y) \geq t'_0 = min(\mu(x), 0.5)$. This shows that condition (2) of the Theorem 3.3 is hold. Therefore μ is an $(\in, \in \sqcup q)-$fuzzy hyperfilter of $\mathcal{L}$. □

Let μ be a fuzzy subset of $\mathcal{L}$ and J be the set of $t \in (0, 1]$ such that $\mu_t = \varnothing$ or μ_t is a hyperfilter of $\mathcal{L}$. If $J = (0, 1]$, then by Proposition 2.4, μ is a fuzzy hyperfilter of $\mathcal{L}$. If $J = (0, 0.5]$ then by Theorem 3.5, μ is an $(\in, \in \sqcup q)-$fuzzy hyperfilter of $\mathcal{L}$. Naturally, a corresponding result should be considered when $J = (0.5, 1]$.

Definition 3.6. A fuzzy subset μ of $\mathcal{L}$ is called $(\overline{\in}, \overline{\in \sqcap q})-$*fuzzy hyperfilter* of $\mathcal{L}$ if for all $t, r \in (0, 1]$ and $x, y \in \mathcal{L}$ the following conditions hold:

(i) $(x \wedge y)_{min(t,r)} \overline{\in} \mu$ implies $x_t \overline{\in \sqcap q} \mu$ or $y_r \overline{\in \sqcap q} \mu$,

(ii) $y_t \overline{\in} \mu$ and $x \leq y$ imply $x_t \overline{\in \sqcap q} \mu$.

Theorem 3.7. *Let μ be a fuzzy subset of $\mathcal{L}$. Then μ is an $(\overline{\in}, \overline{\in \sqcap q})-$fuzzy hyperfilter of $\mathcal{L}$ if and only if for all $x, y \in \mathcal{L}$ the following conditions hold:*

(1) $max(\mu(x \wedge y), 0.5) \geq min(\mu(x), \mu(y))$,

(2) $x \leq y$ *implies* $max(\mu(y), 0.5) \geq \mu(x)$.

Proof. Let μ be an $(\overline{\in}, \overline{\in \sqcap q})-$fuzzy hyperfilter of $\mathcal{L}$. If there exist $x, y \in \mathcal{L}$ such that $max(\mu(x \wedge y), 0.5) < min(\mu(x), \mu(y)) = t$, then $t \in (0.5, 1]$, $(x \wedge y)_t \overline{\in} \mu$ and $x_t, y_t \in \mu$. By Definition 3.6, it follows that $x_t \overline{q} \mu$ or $y_t \overline{q} \mu$. Then ($\mu(x) \geq t$ and $\mu(x) + t \leq 1$) or ($\mu(y) \geq t$ and $\mu(y) + t \leq 1$). It follows that $t \leq 0.5$, which is a contradiction. Hence (1) holds. Also if there exist $x, y \in \mathcal{L}$ and $x \leq y$ such that $max(\mu(y), 0.5) < \mu(x) = t$, then $t \in (0.5, 1]$, $y_t \overline{\in} \mu$ and $x_t \in \mu$. By Definition 3.6, it follows that $x_t \overline{q} \mu$. Then $\mu(x) \geq t$ and $\mu(x) + t \leq 1$. It concludes that $t \leq 0.5$, which is a contradiction. Hence (2) holds. Conversely, Let conditions (1) and (2) be hold. Also let $x, y \in \mathcal{L}$ such that $(x \wedge y)_{min(t,r)} \overline{\in} \mu$, then $\mu(x \wedge y) < min(t, r)$. We can consider the following cases:

(a) If $\mu(x \wedge y) \geq min(\mu(x), \mu(y))$, then $min(\mu(x), \mu(y)) < min(t, r)$ and so $\mu(x) < t$ or $\mu(y) < r$. It follows that $x_t \overline{\in} \mu$ or $y_r \overline{\in} \mu$ which implies that $x_t \overline{\in \sqcap q} \mu$ or $y_r \overline{\in \sqcap q} \mu$.

(b) If $\mu(x \wedge y) < min(\mu(x), \mu(y))$, then by (1) we have $0.5 \geq min(\mu(x), \mu(y))$. Hence $max(\mu(x \wedge y), 0.5) \geq min(\mu(x), \mu(y))$. Now if $x_t, y_r \in \mu$, then $t \leq \mu(x) \leq 0.5$ or $r \leq \mu(y) \leq 0.5$. It follows that $x_t \overline{q} \mu$ or $y_r \overline{q} \mu$, which implies that $x_t \overline{\in \sqcap q} \mu$ or $y_r \overline{\in \sqcap q} \mu$.

Now let $x, y \in \mathcal{L}$ and $x \leq y$ such that $y_t \overline{\in} \mu$, then $\mu(y) < t$. We can consider the following cases:

(a) If $\mu(y) \geq \mu(x)$, then $\mu(x) < t$. It follows that $x_t \overline{\in} \mu$, which implies that $x_t \overline{\in \sqcap q} \mu$.

(b) If $\mu(y) < \mu(x)$, then by (2) we have $0.5 \geq \mu(x)$. Hence $max(\mu(y), 0.5) \geq \mu(x)$. Now if $x_t \in \mu$, then $t \leq \mu(x) \leq 0.5$. It follows $x_t \bar{q} \mu$, which implies that $x_t \overline{\in \sqcap q} \mu$. □

Theorem 3.8. *A fuzzy subset μ of $\mathcal{L}$ is an $(\overline{\in}, \overline{\in \sqcap q})-$fuzzy hyperfilter of $\mathcal{L}$ if and only if the set $\mu_t (\neq \varnothing)$ is hyperfilter of $\mathcal{L}$ for all $t \in (0.5, 1]$.*

Proof. It is immediately followed by Theorem 3.7 and the similar proof of Theorem 3.5. □

Example 3.9. Let $\mathcal{L} = \{0, a, b, 1\}$. Then $(\mathcal{L}, \vee, \wedge)$ is a hyperlattice, where $\vee$ and $\wedge$ are defined by the following tables

$\wedge$	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

$\vee$	0	a	b	1
0	$\{0\}$	$\{a\}$	$\{b\}$	$\{1\}$
a	$\{a\}$	$\mathcal{L} \setminus \{b\}$	$\{0, 1\}$	$\mathcal{L} \setminus \{a\}$
b	$\{b\}$	$\{0, 1\}$	$\mathcal{L} \setminus \{a\}$	$\mathcal{L} \setminus \{b\}$
1	$\{1\}$	$\mathcal{L} \setminus \{a\}$	$\mathcal{L} \setminus \{b\}$	$\mathcal{L}$

Now if $\mu(0) = 0.2, \mu(a) = \mu(b) = 0.1$ and $\mu(1) = 0.3$, then it is easy to verify that μ is an $(\overline{\in}, \overline{\in \sqcap q})-$fuzzy hyperfilter of $\mathcal{L}$.

4 Basic properties

Theorem 4.1. *Let μ be a nonzero $(\in, \in \sqcup q)-$fuzzy hyperfilter of $\mathcal{L}$. Then the set $supp(\mu) = \{x \in \mathcal{L} |\ \mu(x) > 0\}$ is a hyperfilter of $\mathcal{L}$.*

Proof. Straightforward. □

Theorem 4.2. *A nonempty subset $\mathcal{F}$ of $\mathcal{L}$ is a hyperfilter of $\mathcal{L}$ if and only if $\chi_{\mathcal{F}}$ is an $(\in, \in \sqcup q)-$fuzzy hyperfilter of $\mathcal{L}$.*

Proof. Let $\mathcal{F}$ be a hyperfilter of $\mathcal{L}$ and $x, y \in \mathcal{L}$. We can consider the following cases:

(1) $x \notin \mathcal{F}$ or $y \notin \mathcal{F}$, so $\chi_{\mathcal{F}}(x) = 0$ or $\chi_{\mathcal{F}}(y) = 0$, which implies that

$$min(\chi_{\mathcal{F}}(x), \chi_{\mathcal{F}}(y), 0.5) = 0 \leq \chi_{\mathcal{F}}(x \wedge y).$$

(2) $x, y \in \mathcal{F}$, so $\chi_{\mathcal{F}}(x) = 1 = \chi_{\mathcal{F}}(y)$ and $x \wedge y \in \mathcal{F}$, which implies that

$$min(\chi_{\mathcal{F}}(x), \chi_{\mathcal{I}}(y), 0.5) = 0.5 \leq 1 = \chi_{\mathcal{F}}(x \wedge y).$$

Also if $x \in \mathcal{L}$ and $x \leq y$ we can consider the following cases:

(1) $x \notin \mathcal{F}$, so $\chi_{\mathcal{F}}(x) = 0$, which implies that $min(\chi_{\mathcal{F}}(x), 0.5) = 0 \leq \chi_{\mathcal{F}}(y)$.

(2) $x \in \mathcal{F}$, so $\chi_{\mathcal{F}}(x) = 1$ and $y \in \mathcal{F}$, which implies that $min(\chi_{\mathcal{F}}(x), 0.5) = 0.5 \leq 1 = \chi_{\mathcal{F}}(y)$.

Therefore $\chi_{\mathcal{F}}$ is an $(\in, \in \sqcup q)-$fuzzy hyperfilter of $\mathcal{L}$. Conversely, let $\chi_{\mathcal{F}}$ be an $(\in, \in \sqcup q)-$fuzzy hyperfilter of $\mathcal{L}$ and $x, y \in \mathcal{F}$. So $\chi_{\mathcal{F}}(x) = 1 = \chi_{\mathcal{I}}(y)$, which implies that

$$min(\chi_{\mathcal{F}}(x), \chi_{\mathcal{F}}(y), 0.5) = 0.5 \leq \chi_{\mathcal{F}}(x \wedge y).$$

Thus $x \wedge y \in \mathcal{F}$. Also if $x \in \mathcal{F}$ and $x \leq y$, then $\chi_{\mathcal{F}}(x) = 1$, which implies that

$$min(\chi_{\mathcal{F}}(x), 0.5) = 0.5 \leq \chi_{\mathcal{F}}(y).$$

Thus $y \in \mathcal{F}$. Therefore $\mathcal{F}$ is a hyperfilter of $\mathcal{L}$. $\square$

Theorem 4.3. *Let* $\{\mu_i\}_{i\in I}$ *be a family of* $(\in, \in \sqcup q)-$*fuzzy hyperfilters of* $\mathcal{L}$. *Then* $\bigcap_{i\in I} \mu_i$ *is an* $(\in, \in \sqcup q)-$*fuzzy hyperfilter of* $\mathcal{L}$, *where* $(\bigcap_{i\in I} \mu_i)(x) = \inf_{i\in I} \mu_i(x)$.

Proof. Straightforward. $\square$

Theorem 4.4. *Let* $\{\mu_i\}_{i\in I}$ *be a family of* $(\in, \in \sqcup q)-$*fuzzy hyperfilters of* $\mathcal{L}$ *such that* $\mu_i \subseteq \mu_j$ *or* $\mu_j \subseteq \mu_i$ *for all* $i, j \in I$. *Then* $\bigcup_{i\in I} \mu_i$ *is an* $(\in, \in \sqcup q)-$*fuzzy hyperfilter of* $\mathcal{L}$, *where* $(\bigcup_{i\in I} \mu_i)(x) = \sup_{i\in I} \mu_i(x)$.

Proof. For all $x, y \in \mathcal{L}$, we have

$$(\bigcup_{i\in I} \mu_i)(x \wedge y) = \sup_{i\in I} \mu_i(x \wedge y) \geq \sup_{i\in I} min(\mu_i(x), \mu_i(y), 0.5) =$$

$$\sup_{i\in I} min(min(\mu_i(x), \mu_i(y)), 0.5) = min((\bigcup_{i\in I} \mu_i)(x), (\bigcup_{i\in I} \mu_i)(y), 0.5).$$

It is clear that $\sup_{i\in I} min(\mu_i(x), \mu_i(y), 0.5) \leq min((\bigcup_{i\in I} \mu_i)(x), (\bigcup_{i\in I} \mu_i)(y), 0.5)$. Assume that $\sup_{i\in I} min(\mu_i(x), \mu_i(y), 0.5) \neq min((\bigcup_{i\in I} \mu_i)(x), (\bigcup_{i\in I} \mu_i)(y), 0.5)$. Then there exists r such that $\sup_{i\in I} min(\mu_i(x), \mu_i(y), 0.5) < r < min((\bigcup_{i\in I} \mu_i)(x), (\bigcup_{i\in I} \mu_i)(y), 0.5)$. Since $\mu_i \subseteq \mu_j$ or $\mu_j \subseteq \mu_i$ for all $i, j \in I$, there exists $k \in I$ such that $r < min(\mu_k(x), \mu_k(y), 0.5)$. On the other hand, $min(\mu_i(x), \mu_i(y), 0.5) < r$ for all $i \in I$, which is a contradiction. Hence

$$\sup_{i\in I} min(\mu_i(x), \mu_i(y), 0.5) = min((\bigcup_{i\in I} \mu_i)(x), (\bigcup_{i\in I} \mu_i)(y), 0.5).$$

Let $x, y \in \mathcal{L}$ and $x \leq y$, we obtain

$$(\bigcup_{i\in I} \mu_i)(y) = \sup_{i\in I} \mu_i(y) \geq \sup_{i\in I} min(\mu_i(x), 0.5) =$$

$$min(\sup_{i\in I} \mu_i(x), 0.5) = min((\bigcup_{i\in I} \mu_i)(x), 0.5).$$

It is clear that $\sup_{i\in I} min(\mu_i(x), 0.5) \leq min((\bigcup_{i\in I} \mu_i)(x), 0.5)$. Assume that

$$\sup_{i\in I} min(\mu_i(x), 0.5) \neq min((\bigcup_{i\in I} \mu_i)(x), 0.5).$$

Then there exists r such that

$$\sup_{i\in I} min(\mu_i(x), 0.5) < r < min((\bigcup_{i\in I} \mu_i)(x), 0.5).$$

Since $\mu_i \subseteq \mu_j$ or $\mu_j \subseteq \mu_i$ for all $i, j \in I$, there exists $k \in I$ such that $r < min(\mu_k(x), 0.5)$. On the other hand, $min(\mu_i(x), 0.5) < r$ for all $i \in I$, which is a contradiction. Hence $\sup_{i \in I} min(\mu_i(x), 0.5) = min((\bigcup_{i \in I} \mu_i)(x), 0.5)$. Therefore $\bigcup_{i \in I} \mu_i$ is an $(\in, \in \sqcup q)-$fuzzy hyperfilter of $\mathcal{L}$. □

Theorem 4.5. *Let $f : \mathcal{L}_1 \longrightarrow \mathcal{L}_2$ be a homomorphism of hyperlattices and μ and ν be $(\in, \in \sqcup q)-$fuzzy hyperfilters of $\mathcal{L}_1$ and $\mathcal{L}_2$ respectively.*

(i) Then $f^{-1}(\nu)$ is an $(\in, \in \sqcup q)-$fuzzy hyperfilter of $\mathcal{L}_1$.

(ii) If μ satisfies the sup *property, that is, for any subset T of $\mathcal{L}_1$ there exists $x_0 \in T$ such that $\mu(x_0) = \sup\{\mu(x) \mid x \in T\}$, then $f(\mu)$ is an $(\in, \in \sqcup q)-$fuzzy hyperfilter of $\mathcal{L}_2$, when f is onto.*

Proof. (i) Let $x, y \in \mathcal{L}_1$ and $t, r \in (0, 1]$ be such that $x_t \in f^{-1}(\nu)$ and $y_r \in f^{-1}(\nu)$. Then $(f(x))_t \in \nu$ and $(f(y))_r \in \nu$. Since ν is an $(\in, \in \sqcup q)-$fuzzy hyperfilter of $\mathcal{L}_2$, it follows that $(f(x \wedge y))_{min(t,r)} = (f(x) \wedge f(y))_{min(t,r)} \in \sqcup q\nu$, so that $(x \wedge y)_{min(t,r)} \in \sqcup q f^{-1}(\nu)$. Now let $x, y \in \mathcal{L}_1$, $t \in (0, 1]$ and $x \leq y$ be such that $x_t \in f^{-1}(\nu)$. Then $(f(x))_t \in \nu$, which implies that $(f(y))_t \in \sqcup q\nu$ since ν is an $(\in, \in \sqcup q)-$fuzzy hyperfilter of $\mathcal{L}_2$. Hence $y_t \in \sqcup q f^{-1}(\nu)$. Therefore $f^{-1}(\nu)$ is an $(\in, \in \sqcup q)-$fuzzy hyperfilter of $\mathcal{L}_1$.

(ii) Let $a, b \in \mathcal{L}_2$ and $t, r \in (0, 1]$ be such that $a_t \in f(\mu)$ and $b_r \in f(\mu)$. Then $(f(\mu))(a) \geq t$ and $(f(\mu))(b) \geq r$. Since μ has the sup property, there exists $x \in f^{-1}(a)$ and $y \in f^{-1}(b)$ such that $\mu(x) = sup\{\mu(z) \mid z \in f^{-1}(a)\}$ and $\mu(y) = sup\{\mu(w) \mid w \in f^{-1}(b)\}$. Then $x_t \in \mu$ and $y_t \in \mu$. Since μ is an $(\in, \in \sqcup q)-$fuzzy hyperfilter of $\mathcal{L}_1$, we have $(x \wedge y)_{min(t,r)} \in \sqcup q\mu$. Now $x \wedge y \in f^{-1}(a \wedge b)$ and so $(f(\mu))(a \wedge b) \geq \mu(x \wedge y)$. Thus $(f(\mu))(a \wedge b) \geq min(t, r)$ or $(f(\mu))(a \wedge b) + min(t, r) > 1$, which means that $(a \wedge b)_{min(t,r)} \in \sqcup q f(\mu)$. Now let $x, y \in \mathcal{L}_2$, $x \leq y$ and $t \in (0, 1]$ be such that $x_t \in f(\mu)$. Then $(f(\mu))(x) \geq t$. Since μ has the sup property, $\mu(z) = \{\mu(w) \mid w \in f^{-1}(x)\}$ for some $z \in f^{-1}(x)$. Then $z_t \in \mu$ and hence $u_t \in \sqcup q\mu$ for every $u \in f^{-1}(y)$. Since $u \in f^{-1}(y)$, we get $f(\mu)(y) \geq \mu(u)$. It follows that $(f(\mu))(y) \geq t$ or $(f(\mu))(y) + t > 1$ so that $y_t \in \sqcup q f(\mu)$. Therefore $f(\mu)$ is an $(\in, \in \sqcup q)-$fuzzy hyperfilter of $\mathcal{L}_2$. □

Definition 4.6. Let $\alpha, \beta \in [0, 1], \alpha < \beta$ and μ be a fuzzy subset of $\mathcal{L}$. Then μ is said to be a *fuzzy hyperfilter with thresholds* (α, β) of $\mathcal{L}$ if the following conditions hold:

(1) $min(\mu(x), \mu(y), \beta) \leq max(\mu(x \wedge y), \alpha)$ for all $x, y \in \mathcal{L}$,

(2) $min(\mu(x), \beta) \leq max(\mu(y), \alpha)$ for all $x, y \in \mathcal{L}$ with $x \leq y$.

Clearly every fuzzy hyperfilter with thresholds (α, β) of $\mathcal{L}$ is an ordinary fuzzy hyperfilter when $\alpha = 0$ and $\beta = 1$. Also it is an $(\in, \in \sqcup q)-$fuzzy (resp. $(\overline{\in}, \overline{\in \sqcap q})-$fuzzy) hyperfilter when $\alpha = 0$ and $\beta = 0.5$ (resp. $\alpha = 0.5$ and $\beta = 1$) (see Theorems 3.3 and 3.7).

Theorem 4.7. *A fuzzy subset μ of $\mathcal{L}$ is a fuzzy hyperfilter with thresholds (α, β) of $\mathcal{L}$ if and only if $\mu_t (\neq \varnothing)$ is an hyperfilter of $\mathcal{L}$ for all $t \in (\alpha, \beta]$.*

Proof. Suppose that μ is a fuzzy hyperfilter with thresholds (α, β) of $\mathcal{L}$ and $t \in (\alpha, \beta]$. If $x, y \in \mu_t$, then $\mu(x) \geq t$ and $\mu(y) \geq t$. Thus we can write that

$$max(\mu(x \wedge y), \alpha) \geq min(\mu(x), \mu(y), \beta) \geq min(t, \beta) = t > \alpha,$$

which implies that $\mu(x\wedge y) \geq t$, and then $x\wedge y \in \mu_t$. Now if $x \in \mu_t$ and $x \leq y$, then $\mu(x) \geq t$. Thus $max(\mu(y), \alpha) \geq min(\mu(x), \beta) \geq min(t, \beta) = t > \alpha$, which implies that $\mu(y) \geq t$, and then $y \in \mu_t$. Therefore μ_t is a hyperfilter of $\mathcal{L}$. Conversely, let μ be a fuzzy subset of $\mathcal{L}$ such that $\mu_t(\neq \varnothing)$ is a hyperfilter of $\mathcal{L}$ for all $t \in (\alpha, \beta]$. If there exist $x, y \in \mathcal{L}$ such that

$$max(\mu(x \wedge y), \alpha) < min(\mu(x), \mu(y), \beta) = t,$$

then we can conclude that $t \in (\alpha, \beta]$, $\mu(x \wedge y) < t$ and $x, y \in \mu_t$. Since μ_t is a hyperfilter of $\mathcal{L}$, we have $x \wedge y \in \mu_t$. Hence $\mu(x \wedge y) \geq t$, which is a contradiction. Therefore for all $x, y \in \mathcal{L}$ we have $min(\mu(x), \mu(y), \beta) \leq max(\mu(x \wedge y), \alpha)$. Also if there exist $x, y \in \mathcal{L}$ and $x \leq y$ such that $max(\mu(y), \alpha) < min(\mu(x), \beta) = t$, then it concludes that $t \in (\alpha, \beta]$, $\mu(y) < t$ and $x \in \mu_t$. Since μ_t is a hyperfilter of $\mathcal{L}$ and $x \in \mu_t$, then $y \in \mu_t$. Hence $\mu(y) \geq t$, which is a contradiction. So for all $x, y \in \mathcal{L}$ and $x \leq y$ we have $min(\mu(x), \beta) \leq max(\mu(y), \alpha)$. Therefore μ is a fuzzy hyperfilter with thresholds (α, β) of $\mathcal{L}$. □

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References

[1] **N. Ajmal, K.V. Thomas**, *Fuzzy lattices*, Inform. Sci. 79 (1994) 271-291.

[2] **M. Attallah**, *Completely fuzzy prime ideals of distributive lattices*, J. Fuzzy Mathematics vol. 8, No. 1 (2000), 153-156.

[3] **R. Belohlavek**, *Fuzzy relational systems: Foundations and principles*, Kluwer Academic/Plenum Press, New York, 2002.

[4] **S. K. Bhakat**, $(\in, \in \vee q)-$*fuzzy normal, quasinormal and maximal subgroups*, Fuzzy Sets Syst. 112 (2000) 299-312.

[5] **S. K. Bhakat, P. Das**, $(\in, \in \vee q)-$*fuzzy subgroup*, Fuzzy Sets Syst. 80 (1996) 359-368.

[6] **P. Corsini**, *Prolegomena of hypergroup theory*, second ed., Aviani Editor, 1993.

[7] **P. Corsini, V. Leoreanu**, *Applications of hyperstructures theory*, Advanced in Mathematics, Kluwer Academic Publishers, 2003.

[8] **B. A. Davey, H. A. Priestley**, *Introduction to lattices and order*, Second edition Cambridge (2002).

[9] **T. K. Dutta, B. K. Biswas**, *On completely fuzzy semiprime ideals of a semiring*, J. Fuzzy Mathematics, Vol. 8, No. 3(2000), 577-581.

[10] **Y. B. Jun, S. Z. Song**, *Generalized fuzzy interior ideals in semigroups*, Inform. Sci. 176 (2006), 3079-3093.

[11] **H. Hedayati, R. Ameri**, *On fuzzy closed, invertible and reflexive subsets of hypergroups*, Italian J. Pure Appl. Math. 22 (2007), 95-114.

[12] **H. Hedayati, R. Ameri**, *Fuzzy isomorphism and quotient of fuzzy subpolygroups*, Quasigroups and Related Systems 13 (2005), 175-184.

[13] **O. Kazanci, S. Yamak**, *Generalized fuzzy bi-ideals of semigroups*, Soft computing 12 (2008), 1119-1124.

[14] **B. B. N. Koguep, C. Nkuimi, C. Lele**, *On fuzzy ideals of hyperlattice*, International J. Algebra, Vol. 2, No. 15(2008), 739-750

[15] **F. Marty**, *Sur une generalization de la notion de group*, in: 8th Congress Math. Scandenaves, Stockholm 1934, 45-49.

[16] **V. Murali**, *Fuzzy points of equivalent fuzzy subsets*, Inform. Sci. 158 (2004) 277-288.

[17] **A. Rahnamai-Barghi**, *The prime ideal theorem and semiprime ideals in meethyperlattices*, Ital. Journal of Pure and Applied Math., vol. 5 (1999), 53-60.

[18] **A. Rahnamai-Barghi**, *The prime ideal theorem for distributive hyperlattices*, Ital. Journal of Pure and Applied Math., vol. 10 (2001), 75-78.

[19] **U. M. Swany, D. Viswanadha Raju**, *Fuzzy ideals and congruences of Lattices*, Fuzzy Sets Syst. 95 (1998) 249-253.

[20] **A. Tepavcevic, G. Trajkovski**, *L-fuzzy lattices: an introduction*, Fuzzy Sets Syst. 123 (2001) 209-216.

[21] **Y. Xu**, *Lattice implication algebras*, J. Southwest Jiaotong Univ. 1 (1993) 20-27.

[22] **Y. Xu, K. Y. Qin**, *On filters of lattice implication algebras*, J. Fuzzy Math. 2 (1993) 251-260.

[23] **Y. Xu, K. Y. Qin**, *Fuzzy lattice implication algebras*, J. Southwest Jiaotong Univ. 2 (1995) 121-127.

[24] **X. Yuan, C. Zhang, Y. Ren**, *Generalized fuzzy groups and many-valued implications*, Fuzzy Sets Syst. 138 (2003) 205-211.

[25] **L. A. Zadeh**, *Fuzzy sets*, Inform. Control, 8 (1965), 338-353.

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