# DUALITY OF CONDITIONAL ENTROPY ASSOCIATED WITH A COMMUTATIVE HYPERGROUP

Satoe Yamanaka

Received May 31, 2012

*Dedicated to Professor Satoshi Kawakami on the occasion of his 60th birthday.*

Abstract. The purpose of the present paper is to introduce two notions of conditional entropy of a finite commutative hypergroup $\mathcal{K}$, one of which is associated with the normalized Haar measure of $\mathcal{K}$ and the other is associated with the canonical state of $M^b(\mathcal{K})$ which is a $*$-algebra consisted of all measures on $\mathcal{K}$. For a subhypergroup or a generalized orbital hypergroup, the dual relations of these conditional entropy are discussed. Moreover, it is shown that the structures of hypergroups are characterized by these entropy.

**1 Introduction** The notion of a hypergroup is a generalized one of a group. Roughly speaking, the hypergroup convolution is a probabilistic extension of the group convolution. The axiom of hypergroups was set up by C. F. Dunkl[D], R. I. Jewett[J] and R. Spector[S] around 1975.

We study extension problems in the category of finite commutative hypergroups in [HJKK], [HKKK], [IK1], [IK2], [IKS], [KI], [KKY], [KM] and [KST], and in the category of locally compact hypergroups in [HK1], [HK2], [HK3], [K], [KSY] and [KY].

In the present paper, we introduce two kinds of conditional entropy in the category of finite commutative signed hypergroups related to the extension problem. One of them is associated with the normalized Haar measure of a finite commutative signed hypergroup $\mathcal{K}$. The other is associated with the canonical state of $M^b(\mathcal{K})$ which is a $*$-algebra consisted of all measures on $\mathcal{K}$.

Let $\hat{\mathcal{K}}$ denote the set of all characters of $\mathcal{K}$ with the product as functions on $\mathcal{K}$. Then $\hat{\mathcal{K}}$ becomes a commutative signed hypergroup and the duality $\hat{\hat{\mathcal{K}}} \cong \mathcal{K}$ holds.

Let $\mathcal{K} = \{c_0, c_1, \cdots, c_n\}$ be a finite commutative signed hypergroup with unit $c_0$. The notion of a signed action of $\mathcal{K}$ has been introduced in our paper [KSTY]. For an irreducible signed action $\alpha$ of $\mathcal{K}$ on a finite set $X$, there exists uniquely the invariant probability measure $\mu^\alpha$ on $X$ under the action $\alpha$. Then the entropy $H(\alpha)$ of an irreducible action $\alpha$ of $\mathcal{K}$ on $X$ is given by $H(\alpha) := H(\mu^\alpha)$ where $H(\mu^\alpha)$ is Shannon's entropy of the probability measure $\mu^\alpha$ on $X$. Applying the entropy $H(\alpha)$, we characterize two dimensional irreducible actions of a hypergroup $\mathbb{Z}_q(2)$ of order two.

Since the regular action $\rho^{\mathcal{K}}$ of $\mathcal{K}$ is irreducible and the normalized Haar measure $e_{\mathcal{K}}$ of $\mathcal{K}$ is invariant under $\rho^{\mathcal{K}}$, the entropy $H(\mathcal{K})$ of $\mathcal{K}$ is given by $H(\mathcal{K}) := H(\rho^{\mathcal{K}}) = H(e_{\mathcal{K}})$. A state $\phi$ of $*$-algebra $M^b(\mathcal{K})$ is called the canonical state if $\phi(\delta_{c_0}) = 1$ and $\phi(\delta_{c_i}) = 0$ for $c_i \neq c_0$. We denote the entropy $H_\phi(M^b(\mathcal{K}))$ associated with the canonical state $\phi$ of $M^b(K)$ by $H_\phi(\mathcal{K})$. Let $\hat{\mathcal{K}}$ be the dual signed hypergroup of $\mathcal{K}$ and $\hat{\phi}$ be the canonical state of $M^b(\hat{\mathcal{K}})$. Then we obtain the duality: $H(\hat{\mathcal{K}}) = H_\phi(\mathcal{K})$ and $H_{\hat{\phi}}(\hat{\mathcal{K}}) = H(\mathcal{K})$.

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2000 *Mathematics Subject Classification.* 43A62,20N20.
*Key words and phrases.* hypergroup; entropy; extension.

Let $\mathcal{H}$ be a signed subhypergroup of a finite commutative signed hypergroup $\mathcal{K}$ and $\mathcal{L}$ be the quotient signed hypergroup $\mathcal{K}/\mathcal{H}$ of $\mathcal{K}$ by $\mathcal{H}$. The conditional entropy $H(\mathcal{K}|\mathcal{L})$ is given associated with the normalized Haar measure $e_{\mathcal{K}}$ of $\mathcal{K}$ and the quotient mapping $\varphi : \mathcal{K} \to \mathcal{L}$. The other conditional entropy $H_{\phi}^{E}(\mathcal{K}|\mathcal{H})$ is given associated with the canonical state $\phi$ of $M^b(\mathcal{K})$ and the conditional expectation $E : M^b(\mathcal{K}) \to M^b(\mathcal{H})$ such that $\phi \circ E = \phi$. Then we obtain the duality: $H(\hat{\mathcal{K}}|\hat{\mathcal{H}}) = H_{\phi}^{E}(\mathcal{K}|\mathcal{H})$ and $H_{\hat{\phi}}^{\hat{E}}(\hat{\mathcal{K}}|\hat{\mathcal{L}}) = H(\mathcal{K}|\mathcal{L})$.

These conditional entropy plays an important role to characterize equivalence classes of extension hypergroups of a hypergroup $\mathbb{Z}_q(2)$ by a hypergroup $\mathbb{Z}_p(2)$.

Let $\mathcal{K}^E$ be a generalized orbital hypergroup of a finite commutative signed hypergroup $\mathcal{K}$. In the similar way to the above, we consider the conditional entropy $H(\mathcal{K}|\mathcal{K}^E)$ associated with the normalized Haar measure $e_{\mathcal{K}}$ of $\mathcal{K}$ and the conditional entropy $H_{\phi}^{E}(\mathcal{K}|\mathcal{K}^E)$ associated with the canonical state $\phi$ of $M^b(\mathcal{K})$. Then we obtain the duality: $H(\hat{\mathcal{K}}|\widehat{\mathcal{K}^E}) = H_{\phi}^{E}(\mathcal{K}|\mathcal{K}^E)$ and $H_{\hat{\phi}}^{\hat{E}}(\hat{\mathcal{K}}|\widehat{\mathcal{K}^E}) = H(\mathcal{K}|\mathcal{K}^E)$.

## 2 Preliminaries

We recall some notions and facts on finite signed hypergroups from Bloom-Heyer's book [BH] and Wildberger's paper [W]. For a finite set $X = \{x_1, x_2, \cdots, x_m\}$, we denote by $M^b(X)$, $M^1_{\mathbb{R}}(X)$ and $M^1(X)$ the set of all complex valued measures, real valued probability measures and non-negative probability measures on $X$ respectively, namely,

$$M^b(X) = \Big\{\mu = \sum_{j=1}^{m} a_j \delta_{x_j} : a_j \in \mathbb{C}\Big\},$$

$$M^1_{\mathbb{R}}(X) = \Big\{\sum_{j=1}^{m} a_j \delta_{x_j} : a_j \in \mathbb{R}, \sum_{j=1}^{m} a_j = 1\Big\},$$

$$M^1(X) = \Big\{\sum_{j=1}^{m} a_j \delta_{x_j} : a_j \geq 0, \sum_{j=1}^{m} a_j = 1\Big\}$$

where the symbol $\delta_x$ stands for the Dirac measure at $x \in X$. For $\mu = \sum_{j=1}^{m} a_j \delta_{x_j}$ in $M^b(X)$, the *support* of $\mu$ is given by

$$\mathrm{supp}(\mu) := \{x_j \in X : a_j \neq 0\}.$$

**Definition.** A *finite signed hypergroup* $\mathcal{K}$ consists of a finite set $\mathcal{K} = \{c_0, c_1, \cdots, c_n\}$ together with a product (called *convolution*) $\circ$ and an involution $*$ in $M^b(\mathcal{K})$ satisfying the following conditions.

1. $(M^b(\mathcal{K}), \circ, *)$ is an associative $*$-algebra over $\mathbb{C}$ with unit $\delta_{c_0}$.
2. For $c_i, c_j \in \mathcal{K}$, the convolution $\delta_{c_i} \circ \delta_{c_j}$ belongs to $M^1_{\mathbb{R}}(\mathcal{K})$.
3. There exists an involutive bijection $c_i \mapsto c_i^*$ on $\mathcal{K}$ such that $\delta_{c_i^*} = \delta_{c_i}^*$ and $c_j = c_i^*$ if and only if $c_0 \in \mathrm{supp}(\delta_{c_i} \circ \delta_{c_j})$ for all $c_i, c_j \in \mathcal{K}$. Moreover, the coefficient of the unit $\delta_{c_0}$ of $\delta_{c_i} \circ \delta_{c_i^*}$ is positive.

A finite signed hypergroup $\mathcal{K} = (\mathcal{K}, M^b(\mathcal{K}), \circ, *)$ is called a *hypergroup* if the convolution $\delta_{c_i} \circ \delta_{c_j}$ belongs to $M^1(\mathcal{K})$ for any $c_i, c_j \in \mathcal{K}$.

A finite signed hypergroup $\mathcal{K}$ is said to be *commutative* if the convolution $\circ$ in $M^b(\mathcal{K})$ is commutative.

Let $\mathcal{K} = \{c_0, c_1, \cdots, c_n\}$ be a finite commutative signed hypergroup. The *weight* of the element $c_i$ of $\mathcal{K}$ is $w(c_i) := (n_i^0)^{-1}$ where $\delta_{c_i} \circ \delta_{c_i}^* = \sum_{k=0}^n n_i^k \delta_{c_k}$. The *total weight* of $\mathcal{K}$ is $w(\mathcal{K}) := \sum_{i=0}^n w(c_i)$. We note that $w(c_i) > 0$ and $w(c_i^*) = w(c_i)$. Let $e_{\mathcal{K}}$ denote the *normalized Haar measure* of $\mathcal{K}$ which is given by

$$e_{\mathcal{K}} = \sum_{k=0}^n \frac{w(c_k)}{w(\mathcal{K})} \delta_{c_k}.$$

A complex valued function $\chi$ on $\mathcal{K}$ is called a *character* of $\mathcal{K}$ if

$$(1)\ \chi(c_0) = 1, \quad (2)\ \chi(c_i^*) = \overline{\chi(c_i)}, \quad (3)\ \chi(c_i)\chi(c_j) = \sum_{k=0}^n n_{ij}^k \chi(c_k)$$

where $\delta_{c_i} \circ \delta_{c_j} = \sum_{k=0}^n n_{ij}^k \delta_{c_k}$. We denote the trivial character by $\chi_0$. The set $\hat{\mathcal{K}}$ of all characters of $\mathcal{K}$ becomes a signed hypergroup with the product of functions on $\mathcal{K}$. In the category of finite signed hypergroups, the duality $\hat{\hat{\mathcal{K}}} \cong \mathcal{K}$ holds. Then, $\hat{\mathcal{K}}$ is called the *dual* signed hypergroup of $\mathcal{K}$. When $\mathcal{K}$ is a finite hypergroup, $\hat{\mathcal{K}}$ is not necessary a finite hypergroup but a finite signed hypergroup.

We give some facts from harmonic analysis for a finite commutative signed hypergroup $\mathcal{K}$ according to Wildberger's paper [W].

We denote the dual signed hypergroup $\hat{\mathcal{K}}$ by $\hat{\mathcal{K}} = \{\chi_0, \chi_1, \cdots, \chi_n\}$. We note that $w(\hat{\mathcal{K}}) = w(\mathcal{K})$. There exist minimal projections $e_0, e_1, \cdots, e_n$ of $M^b(\mathcal{K})$ such that $\chi_i(e_j) = \delta_{ij}$ for $\chi_i \in \hat{\mathcal{K}}$. Each projection $e_i$ is given by

$$e_i = \frac{w(\chi_i)}{w(\mathcal{K})} \sum_{j=0}^n w(c_j) \overline{\chi_i(c_j)} \delta_{c_j}.$$

Let $\phi$ be a state of $M^b(\mathcal{K})$ such that $\phi(\delta_{c_0}) = 1$ and $\phi(\delta_{c_i}) = 0$ for $i = 1, 2, \cdots, n$. Then it is clear that $\phi(e_i) = \frac{w(\chi_i)}{w(\mathcal{K})}$ by the above formula. We call the state $\phi$ the *canonical* state of $M^b(\mathcal{K})$.

For a finite set $X = \{x_1, x_2, \cdots, x_m\}$, $B(M^b(X))$ denotes the algebra of all linear transformations on $M^b(X)$.

We call a *signed action* $\alpha$ of a signed hypergroup $\mathcal{K}$ on a set $X$ in our paper [KSTY] if $\alpha$ satisfies the following.

1. $\alpha$ is a homomorphism from $M^b(\mathcal{K})$ to $B(M^b(X))$ as algebras such that $\alpha(\delta_{c_0})$ is the identity on $M^b(X)$.
2. For $c_i \in \mathcal{K}$ and $\mu \in M^1(X)$, $\alpha(\delta_{c_i})\mu \in M^1_{\mathbb{R}}(X)$.
3. For the normalized Haar measure $e_{\mathcal{K}}$ of $\mathcal{K}$ and $\mu \in M^1(X)$, $\alpha(e_{\mathcal{K}})\mu \in M^1(X)$.

If $\mathcal{K}$ is a hypergroup and $\alpha(\delta_{c_i})\mu$ belongs to $M^1(X)$ for $c_i \in \mathcal{K}$ and $\mu \in M^1(X)$, $\alpha$ is called an *action* of $\mathcal{K}$ on $X$.

The dimension of a signed action $\alpha$ of $\mathcal{K}$ on $X$ is the dimension of $M^b(X)$, namely, the cardinal number $|X|$ of $X$. A subset $S$ of $X$ is called *invariant* under the signed action $\alpha$ if $\mathrm{supp}(\alpha(e_{\mathcal{K}})\delta_s) \subset S$ for all $s \in S$.

A signed action $\alpha$ of $\mathcal{K}$ on $X$ is called *irreducible* if a non-empty subset $S$ of $X$ which is invariant under the signed action $\alpha$ must be $X$.

**3 Entropy of hypergroups** Let $X = \{x_1, x_2, \cdots, x_m\}$ be a finite set. For a probability measure $\mu = a_1\delta_{x_1} + a_2\delta_{x_2} + \cdots + a_m\delta_{x_m}$ on $X$, Shannon's entropy $H(\mu)$ of $\mu$ is

$$H(\mu) = \sum_{j=1}^{m} \eta(a_j),$$

where $\eta(x)$ is the *entropy function* i.e.

$$\eta(x) = \begin{cases} -x\log x & 0 < x \le 1, \\ 0 & x = 0. \end{cases}$$

Let $\mathcal{K}$ be a finite signed hypergroup and $X$ be a finite set. For an irreducible signed action $\alpha$ of $\mathcal{K}$ on $X$, there exists the unique invariant probability measure $\mu^\alpha$ on $X$ under $\alpha$, see our paper [KSTY]. We define the *entropy* $H(\alpha)$ *of the irreducible signed action* $\alpha$ of $\mathcal{K}$ on $X$ by

$$H(\alpha) := H(\mu^\alpha).$$

Moreover, we denote the entropy $H(\rho^{\mathcal{K}})$ by $H(\mathcal{K})$ for the regular action $\rho^{\mathcal{K}}$ of $\mathcal{K}$.

Let $M$ be a finite commutative $*$-algebra with unit 1 which is generated by minimal projections $e_0, e_1, \cdots, e_n$ such that $\sum_{i=0}^{n} e_i = 1$. For a state $\phi$ of $M$, the entropy $H_\phi(M)$ of $\phi$ is given by

$$H_\phi(M) = \sum_{i=0}^{n} \eta(\phi(e_i)).$$

Let $\mathcal{K} = (\mathcal{K}, M^b(\mathcal{K}))$ be a signed hypergroup. For the canonical state $\phi$ of $M^b(\mathcal{K})$, we denote $H_\phi(M^b(\mathcal{K}))$ by $H_\phi(\mathcal{K})$.

**Proposition 3.1.** *Let $\mathcal{K} = (\mathcal{K}, M^b(\mathcal{K}))$ be a finite commutative signed hypergroup and $\hat{\mathcal{K}}$ be the dual signed hypergroup of $\mathcal{K}$. Let $\phi$ and $\hat{\phi}$ be the canonical state of $M^b(\mathcal{K})$ and $M^b(\hat{\mathcal{K}})$ respectively.*

*Then, the following formulae hold.*

1. $H(\mathcal{K}) = \log w(\mathcal{K}) - \sum_{c\in\mathcal{K}} \frac{w(c)}{w(\mathcal{K})} \log w(c),$

2. $H_\phi(\mathcal{K}) = \log w(\hat{\mathcal{K}}) - \sum_{\chi\in\hat{\mathcal{K}}} \frac{w(\chi)}{w(\hat{\mathcal{K}})} \log w(\chi),$

3. $H(\hat{\mathcal{K}}) = H_\phi(\mathcal{K}),\ H_{\hat{\phi}}(\hat{\mathcal{K}}) = H(\mathcal{K}).$

**Proof.** (1) Since the regular action $\rho^{\mathcal{K}}$ of $\mathcal{K}$ is irreducible and the $\rho^{\mathcal{K}}$-invariant probability measure $\mu^{\rho^{\mathcal{K}}}$ on $\mathcal{K}$ is the normalized Haar measure

$$e_{\mathcal{K}} = \sum_{c\in\mathcal{K}} \frac{w(c)}{w(\mathcal{K})}\delta_c$$

of $\mathcal{K}$, we have

$$\begin{aligned} H(\mathcal{K}) &= H(\rho^{\mathcal{K}}) = H(e_{\mathcal{K}}) = \sum_{c\in\mathcal{K}} \eta\left(\frac{w(c)}{w(\mathcal{K})}\right) \\ &= \sum_{c\in\mathcal{K}} \frac{w(c)}{w(\mathcal{K})} \log w(\mathcal{K}) - \sum_{c\in\mathcal{K}} \frac{w(c)}{w(\mathcal{K})} \log w(c) \\ &= \log w(\mathcal{K}) - \sum_{c\in\mathcal{K}} \frac{w(c)}{w(\mathcal{K})} \log w(c). \end{aligned}$$

(2) Let $\hat{\mathcal{K}} = \{\chi_0, \cdots, \chi_n\}$ be the dual signed hypergroup of $\mathcal{K}$. We denote the minimal projection by $e_i$ corresponding to each $\chi_i \in \hat{\mathcal{K}}$. Since $\phi(e_i) = \frac{w(\chi_i)}{w(\mathcal{K})}$ and $w(\hat{\mathcal{K}}) = w(\mathcal{K})$, we have

$$H_\phi(\mathcal{K}) = \sum_{i=0}^{n} \eta(\phi(e_i)) = \sum_{i=0}^{n} \eta\left(\frac{w(\chi_i)}{w(\mathcal{K})}\right) = \log w(\hat{\mathcal{K}}) - \sum_{\chi\in\hat{\mathcal{K}}} \frac{w(\chi)}{w(\hat{\mathcal{K}})} \log w(\chi)$$

in a similar way to the above.

(3) Applying the formula (1) to $\hat{\mathcal{K}}$, one can obtain

$$H(\hat{\mathcal{K}}) = \log w(\hat{\mathcal{K}}) - \sum_{\chi\in\hat{\mathcal{K}}} \frac{w(\chi)}{w(\hat{\mathcal{K}})} \log w(\chi).$$

Hence it is clear that $H(\hat{\mathcal{K}}) = H_\phi(\mathcal{K})$ by the formula (2).

Moreover, we have

$$H_{\hat{\phi}}(\hat{\mathcal{K}}) = H(\hat{\hat{\mathcal{K}}}) = H(\mathcal{K})$$

by the above equality and the duality $\hat{\hat{\mathcal{K}}} \cong \mathcal{K}$. □

**Remark.** It is easy to check that

$$H(\mathcal{K}) \leq \log |\mathcal{K}|.$$

The entropy $H(\mathcal{K})$ attains the maximum value $\log |\mathcal{K}|$ if and only if $\mathcal{K}$ is a group.

**Example 1.** Let $\mathcal{K} = \{0, 1\}$ be a signed hypergroup of order two with unit 0 where the structure is characterized by a parameter $q$ $(0 < q)$ as follows.

$$\delta_1 \circ \delta_1 = q\delta_0 + (1-q)\delta_1.$$

We often denote this hypergroup $\mathcal{K}$ by $\mathbb{Z}_q(2)$. Let $\alpha$ be an $m$-dimensional irreducible signed action of $\mathbb{Z}_q(2)$ on $X = \{x_1, x_2, \cdots, x_m\}$. Then the representing matrix of the action $\alpha$ associated with the basis $\delta_{x_1}, \delta_{x_2}, \cdots, \delta_{x_m}$ in $M^b(X)$ is given by

$$T_\alpha(\delta_0) = I, \quad T_\alpha(\delta_1) = \begin{pmatrix} (1+q)t_1 - q & (1+q)t_1 & \dots & (1+q)t_1 \\ (1+q)t_2 & (1+q)t_2 - q & \dots & (1+q)t_2 \\ \vdots & \vdots & \ddots & \vdots \\ (1+q)t_m & (1+q)t_m & \dots & (1+q)t_m - q \end{pmatrix}$$

where $0 < t_i < 1$ and $\sum_{i=1}^m t_i = 1$ (see [KSTY]).

The above action $\alpha$ is determined by the parameters $t := (t_1, t_2, \cdots, t_m)$ so that we denote the action $\alpha$ by $\alpha^t$.

In the case that $\mathcal{K} = \mathbb{Z}_q(2)$ is a hypergroup, namely $0 < q \le 1$, the signed action $\alpha^t$ of $\mathcal{K}$ is an action if and only if $\dim \alpha^t = m \le \frac{1+q}{q}$ and $\frac{q}{1+q} \le t_i \le \frac{1}{1+q}$ $(m \ge 2)$.

**Proposition 3.2.** *Let $\alpha^t$ be an $m$-dimensional irreducible action of $\mathbb{Z}_q(2)$ on $X$ where a parameter $t = (t_1, t_2, \cdots, t_m)$ satisfies that $\frac{q}{1+q} \le t_i \le \frac{1}{1+q}$ for all $i$ and $\sum_{i=1}^m t_i = 1$.*

*Then the following hold.*

1. *$H(\alpha^t) = \sum_{i=1}^m \eta(t_i)$.*
2. *$H(\alpha^t)$ attains the maximum value $\log m$ if and only if $\alpha^t$ is a $*$-action.*
3. *For a two-dimensional irreducible action $\alpha^t$ of $\mathbb{Z}_q(2)$, $H(\alpha^t)$ has the minimum value if and only if $\alpha^t$ is equivalent to the regular action of $\mathbb{Z}_q(2)$.*
4. *For two-dimensional irreducible actions $\alpha^t$ and $\alpha^{t'}$ of $\mathbb{Z}_q(2)$, $\alpha^t$ is equivalent to $\alpha^{t'}$ as actions if and only if $H(\alpha^t) = H(\alpha^{t'})$.*

**Proof.** (1) Since the invariant probability measure $\mu^{\alpha^t}$ under the action $\alpha^t$ of $\mathbb{Z}_q(2)$ on $X$ is

$$\mu^{\alpha^t} = t_1\delta_{x_1} + t_2\delta_{x_2} + \cdots + t_m\delta_{x_m},$$

we see that the entropy of $\alpha^t$ is

$$H(\alpha^t) = H(\mu^{\alpha^t}) = \sum_{i=1}^m \eta(t_i).$$

(2) It is known that $H(\alpha^t) \le \log m$. Moreover $H(\alpha^t) = \sum_{i=1}^m \eta(t_i) = \log m$ if and only if $t_1 = t_2 = \cdots = t_m = \frac{1}{m}$. This condition is equivalent to $T_{\alpha^t}(\delta_1)^* = T_{\alpha^t}(\delta_1)$, namely, $\alpha^t$ is a $*$-action of $\mathbb{Z}_q(2)$ in the sense of Sunder-Wildberger [SW].

(3) The two dimensional irreducible action $\alpha^t$ is parameterized by $t = (t, 1-t)$ such that $\frac{q}{1+q} \le t \le \frac{1}{1+q}$. Under the condition that $\frac{q}{1+q} \le t \le \frac{1}{1+q}$, it is easy to see that $H(\alpha^t)$ has the minimum value if and only if $t = \frac{q}{1+q}$ or $t = \frac{1}{1+q}$. This condition implies that $\alpha^t$ is equivalent to the regular action of $\mathbb{Z}_q(2)$.

(4) It is easy to see the statement (4) by the fact that $\alpha^t \cong \alpha^{t'}$ if and only if $t = t'$ or $t = 1 - t'$, see [KMTY].

□

**Remark.** Let $\alpha^t$ $(0 < t < 1)$ be a two-dimensional irreducible signed action of $\mathbb{Z}_q(2) = \{0, 1\}$ and $\pi^t$ be the representation of $\mathbb{Z}_q(2)$ associated with the action $\alpha^t$. The representing matrix of $\pi^t(\delta_1)$ is given by

$$T_{\pi^t}(\delta_1) = \begin{pmatrix} (1+q)t - q & (1+q)\sqrt{t}\sqrt{1-t} \\ (1+q)\sqrt{t}\sqrt{1-t} & (1+q)(1-t) - q \end{pmatrix}.$$

Let $u^t$ be the unitary matrix such that

$$(u^t)^* T_{\pi^t}(\delta_1) u^t = \begin{pmatrix} 1 & 0 \\ 0 & -q \end{pmatrix}.$$

Then $u^t$ is given by

$$u^t = \begin{pmatrix} \sqrt{t} & -\sqrt{1-t} \\ \sqrt{1-t} & \sqrt{t} \end{pmatrix}.$$

The entropy $H(b^t)$ of the unistochastic matrix $b^t$ defined by $u^t$ is

$$H(b^t) = \eta(t) + \eta(1-t).$$

Let $A^t$ be the maximal abelian $*$-subalgebra of $M_2(\mathbb{C})$ which is generated by $T_{\pi^t}(\delta_0)$ and $T_{\pi^t}(\delta_1)$, and $B$ be the diagonal algebra of $M_2(\mathbb{C})$. Here we note that $B = (u^t)^* A^t u^t$. By the paper [C], M. Choda introduced the conditional entropy $h(A^t|B)$ and showed that $h(A^t|B) = H(u^t)$ under the above situation. Then we have a remarkable fact :

$$H(\alpha^t) = H(u^t) = h(A^t|B).$$

**4 Conditional entropy associated with a subhypergroup.** First, we recall the classical conditional entropy. Let $\mu$ be a probability measure of a finite set $X = \{x_0, x_1, \cdots, x_n\}$. For a mapping $\psi$ from $X$ onto $Y = \{y_0, y_1, \cdots, y_m\}$, we have a decomposition $\{B_0, B_1, \cdots, B_m\}$ of $X$ by $B_j = \psi^{-1}(y_j)$ and the conditional probability measure $\mu_j$ on $B_j$ by

$$\mu_j(x) = \frac{\mu(x)}{\mu(B_j)}$$

for $x \in B_j$. Then the conditional entropy of the decomposition of $(X, \mu)$ given by $\psi : X \longrightarrow Y$ is defined by

$$H_\mu(\psi : X|Y) = \sum_{j=0}^{m} \mu(B_j) H(\mu_j)$$

where

$$H(\mu_j) = \sum_{x \in B_j} \eta(\mu_j(x)) = \sum_{x \in B_j} \eta\left(\frac{\mu(x)}{\mu(B_j)}\right).$$

Let $M$ be a finite commutative $*$-algebra with unit 1 such that $M$ consists of linear hulls of the minimal projections $e_0, e_1, \cdots, e_n$ such that $\sum_{i=0}^{n} e_i = 1$. Let $N$ be a $*$-subalgebra of $M$ with the unit 1 of $M$. We denote the minimal projections of $N$ by $f_0, f_1 \cdots, f_m$ such that $\sum_{j=0}^{m} f_j = 1$. For each minimal projection $e_i$ of $M$, there exists the unique minimal projection $f_j$ of $N$ such that $e_i \circ f_j = e_i$. Then, we define a mapping $\sigma$ from $\{0, 1, \cdots, n\}$ onto $\{0, 1, \cdots, m\}$ by $e_i \circ f_{\sigma(i)} = e_i$. We note that $f_j = \sum_{i \in \sigma^{-1}(j)} e_i$. Let $\phi$ be a state of $M$. Then, the conditional entropy of the conditional expectation $E$ from $M$ onto $N$ such that $\phi \circ E = \phi$ is defined by

$$H_\phi^E(M|N) = \sum_{i=0}^{n} \phi(\eta(E(e_i))) = \sum_{j=0}^{m} \phi(f_j) H_\phi(\sigma^{-1}(j))$$

where

$$H_\phi(\sigma^{-1}(j)) := \sum_{i \in \sigma^{-1}(j)} \eta\left(\frac{\phi(e_i)}{\phi(f_j)}\right).$$

Let $\mathcal{H}, \mathcal{K}, \mathcal{L}$ be finite commutative hypergroups. Let $\mathcal{H}$ be a subhypergroup of $\mathcal{K}$ and $\varphi$ be a hypergroup homomorphism from $\mathcal{K}$ onto $\mathcal{L}$ such that $\mathrm{Ker}\varphi = \mathcal{H}$, namely,

$$1 \longrightarrow \mathcal{H} \longrightarrow \mathcal{K} \xrightarrow{\varphi} \mathcal{L} \longrightarrow 1$$

is exact. Then the hypergroup $\mathcal{K}$ is called an extension of $\mathcal{L}$ by $H$. Let $e_{\mathcal{K}}$ be the normalized Haar measure of $\mathcal{K}$.

Under the above situation, we define the conditional entropy $H(\mathcal{K}|\mathcal{L})$ of the decomposition of $(\mathcal{K}, e_{\mathcal{K}})$ given by $\varphi : \mathcal{K} \longrightarrow \mathcal{L}$ by

$$H(\mathcal{K}|\mathcal{L}) := H_{e_{\mathcal{K}}}(\varphi : \mathcal{K}|\mathcal{L}).$$

We denote the conditional entropy $H_\phi^E(\mathcal{K}|\mathcal{H})$ of the conditional expectation $E$ from $M^b(\mathcal{K})$ onto the $*$-subalgebra $M^b(\mathcal{H})$ such that $\phi \circ E = \phi$ for the canonical state $\phi$ of $M^b(\mathcal{K})$ by

$$H_\phi^E(\mathcal{K}|\mathcal{H}) := H_\phi^E(M^b(\mathcal{K})|M^b(\mathcal{H})).$$

**Remark.** In the case that $\mathcal{K}, \mathcal{H}, \mathcal{L}$ are finite commutative signed hypergroups, the above two definitions of conditional entropy are also well-defined.

Let $\hat{\mathcal{H}}, \hat{\mathcal{K}}, \hat{\mathcal{L}}$ be the dual signed hypergroups of $\mathcal{H}, \mathcal{K}, \mathcal{L}$ respectively. Then, we have the dual exact sequence:

$$1 \longrightarrow \hat{\mathcal{L}} \longrightarrow \hat{\mathcal{K}} \xrightarrow{\hat{\varphi}} \hat{\mathcal{H}} \longrightarrow 1.$$

Let $\hat{E}$ be the conditional expectation from $M^b(\hat{\mathcal{K}})$ onto $M^b(\hat{\mathcal{L}})$ such that $\hat{\phi} \circ \hat{E} = \hat{\phi}$ for the canonical state $\hat{\phi}$ of $M^b(\hat{\mathcal{K}})$.

**Theorem 4.1.** *Let $\mathcal{H}$ be a subhypergroup of a finite commutative hypergroup $\mathcal{K}$ and $\mathcal{L}$ be the quotient hypergroup $\mathcal{K}/\mathcal{H}$ of $\mathcal{K}$ by $\mathcal{H}$. Under the above situation, the following formulae hold.*

1. $H(\mathcal{K}|\mathcal{L}) = \displaystyle\sum_{\ell \in \mathcal{L}} \sum_{c \in \varphi^{-1}(\ell)} \frac{w(c)}{w(\mathcal{K})} \log \frac{w(\ell)w(\mathcal{H})}{w(c)} = H(\mathcal{K}) - H(\mathcal{L}).$

2. $H_\phi^E(\mathcal{K}|\mathcal{H}) = \displaystyle\sum_{\tau \in \hat{\mathcal{H}}} \sum_{\chi \in \hat{\varphi}^{-1}(\tau)} \frac{w(\chi)}{w(\hat{\mathcal{K}})} \log \frac{w(\tau)w(\hat{\mathcal{L}})}{w(\chi)} = H_\phi(\mathcal{K}) - H_\phi(\mathcal{H}).$

3. $H(\hat{\mathcal{K}}|\hat{\mathcal{H}}) = H_\phi^E(\mathcal{K}|\mathcal{H})$ *and* $H_{\hat{\phi}}^{\hat{E}}(\hat{\mathcal{K}}|\hat{\mathcal{L}}) = H(\mathcal{K}|\mathcal{L}).$

**Proof.** (1) For each $\ell \in \mathcal{L}$, the conditional probability measure $\mu_\ell$ of $e_{\mathcal{K}}$ on $\varphi^{-1}(\ell)$ is given by

$$\mu_\ell = \sum_{c \in \varphi^{-1}(\ell)} \frac{w(c)}{w(\varphi^{-1}(\ell))} \delta_c.$$

Then we have

$$\begin{aligned} H(\mathcal{K}|\mathcal{L}) &= \sum_{\ell \in \mathcal{L}} e_{\mathcal{K}}(\varphi^{-1}(\ell)) H(\mu_\ell) = \sum_{\ell \in \mathcal{L}} \sum_{c \in \varphi^{-1}(\ell)} \frac{w(\varphi^{-1}(\ell))}{w(\mathcal{K})} \eta\left(\frac{w(c)}{w(\varphi^{-1}(\ell))}\right) \\ &= \sum_{\ell \in \mathcal{L}} \sum_{c \in \varphi^{-1}(\ell)} \frac{w(c)}{w(\mathcal{K})} \log \frac{w(\varphi^{-1}(\ell))}{w(c)}. \end{aligned}$$

By the fact that $w(\varphi^{-1}(\ell)) = w(\ell)w(\mathcal{H})$ (see [IK2]), we get the desired formula.

(2) Let $\hat{\mathcal{K}} = \{\chi_0, \cdots, \chi_n\}$ and $\hat{\mathcal{H}} = \{\tau_0, \cdots, \tau_m\}$. Then we have minimal projections $\{e_i\}_{i=0}^n$ in $M^b(\mathcal{K})$ and $\{f_j\}_{j=0}^m$ in $M^b(\mathcal{H})$ which satisfy

$$\chi_p(e_i) = \delta_{pi}, \ \tau_q(f_j) = \delta_{qj}$$

for $\chi_p \in \hat{\mathcal{K}}$ and $\tau_q \in \hat{\mathcal{H}}$ respectively. We note that $\phi(e_i) = \frac{w(\chi_i)}{w(\hat{\mathcal{K}})}$ and $\phi(f_j) = \frac{w(\tau_j)}{w(\hat{\mathcal{H}})}$. Let $\sigma$ be the mapping from $\{0, 1, \cdots, n\}$ onto $\{0, 1, \cdots, m\}$ given by $e_i \circ f_{\sigma(i)} = e_i$. Hence,

$$\begin{aligned} H_\phi^E(\mathcal{K}|\mathcal{H}) &= \sum_{j=0}^m \phi(f_j) \sum_{i\in\sigma^{-1}(j)} \eta\left(\frac{\phi(e_i)}{\phi(f_j)}\right) = \sum_{j=0}^m \sum_{i\in\sigma^{-1}(j)} \phi(e_i) \log \frac{\phi(f_j)}{\phi(e_i)} \\ &= \sum_{j=0}^m \sum_{i\in\sigma^{-1}(j)} \frac{w(\chi_i)}{w(\hat{\mathcal{K}})} \log\left(\frac{w(\tau_j)}{w(\hat{\mathcal{H}})} \cdot \frac{w(\hat{\mathcal{K}})}{w(\chi_i)}\right). \end{aligned}$$

It is easy to see that $e_i \circ f_j = e_i$ if and only if $\hat{\varphi}(\chi_i) = \tau_j$. This means that $i \in \sigma^{-1}(j)$ if and only if $\chi_i \in \hat{\varphi}^{-1}(\tau_j)$. By the fact that $w(\hat{\mathcal{K}}) = w(\hat{\mathcal{H}})w(\hat{\mathcal{L}})$ (see [IK2]), we get the desired conclusion.

(3) Applying the formula (1) to the exact sequence: $1 \longrightarrow \hat{\mathcal{L}} \longrightarrow \hat{\mathcal{K}} \xrightarrow{\hat{\varphi}} \hat{\mathcal{H}} \longrightarrow 1$, one can obtain

$$H(\hat{\mathcal{K}}|\hat{\mathcal{H}}) = \sum_{\tau\in\hat{\mathcal{H}}} \sum_{\chi\in\hat{\varphi}^{-1}(\tau)} \frac{w(\chi)}{w(\hat{\mathcal{K}})} \log \frac{w(\tau)w(\hat{\mathcal{L}})}{w(\chi)}.$$

Hence it is clear that $H(\hat{\mathcal{K}}|\hat{\mathcal{H}}) = H_\phi^E(\mathcal{K}|\mathcal{H})$ by the formula (2).

Moreover, we have

$$H_{\hat{\phi}}^{\hat{E}}(\hat{\mathcal{K}}|\hat{\mathcal{L}}) = H(\hat{\hat{\mathcal{K}}}|\hat{\hat{\mathcal{L}}}) = H(\mathcal{K}|\mathcal{L})$$

by the above formula and the duality. □

**Remark.** (1) In the category of finite commutative signed hypergroups, the above statements are also valid.

(2) For the regular action $\rho^{\mathcal{K}}$ of a finite hypergroup $\mathcal{K}$, let $\rho_{\mathcal{H}}^{\mathcal{K}}$ be the action of $\mathcal{K}$ which is the restriction of $\rho^{\mathcal{K}}$ to $\mathcal{H}$. Then $\rho_{\mathcal{H}}^{\mathcal{K}}$ is decomposed as

$$(\rho_{\mathcal{H}}^{\mathcal{K}}, \mathcal{K}) = \sum_{\ell\in\mathcal{L}} \oplus(\rho_\ell, \varphi^{-1}(\ell))$$

where $\rho_\ell$ is an irreducible action of $\mathcal{H}$ on $\varphi^{-1}(\ell)$ for each $\ell \in \mathcal{L}$ and $\rho_{\ell_0} = \rho^{\mathcal{H}}$ because $\varphi^{-1}(\ell_0) = \mathcal{H}$ for the unit $\ell_0$ of $\mathcal{L}$. Then, we know that the invariant probability measure under the action $\rho_\ell$ on $\varphi^{-1}(\ell)$ is the conditional probability measure of $e_{\mathcal{K}}$ on $\varphi^{-1}(\ell)$. Therefore, the conditional entropy $H(\mathcal{K}|\mathcal{L})$ of the decomposition can be rewritten as

$$H(\mathcal{K}|\mathcal{L}) = \sum_{\ell\in\mathcal{L}} \frac{w(\ell)}{w(\mathcal{L})} H(\rho_\ell).$$

**An application and an example for the extension problem.**

We consider the exact sequence

$$1 \longrightarrow \mathcal{H} \longrightarrow \mathcal{K} \xrightarrow{\varphi} \mathcal{L} \longrightarrow 1$$

in the case of $\mathcal{H} = \mathbb{Z}_q(2)(0 < q \leq 1)$ and $\mathcal{L} = \mathbb{Z}_p(2)(0 < p \leq 1)$ where the order of an extension hypergroup $\mathcal{K}$ is four. In the paper [KMTY], an extension $\mathcal{K} = \mathcal{K}(t, r)$ is determined by two-dimensional irreducible actions $\rho^t$ and $\rho^r$ of $\mathbb{Z}_q(2)$ and $\mathbb{Z}_p(2)$ which are parameterized by $\frac{q}{1+q} \leq t \leq \frac{1}{1+q}$ and $\frac{p}{1+p} \leq r \leq \frac{1}{1+p}$ respectively. Let $\phi$ and $\phi'$ be the canonical states of $M^b(\mathcal{K})$ and $M^b(\mathcal{L})$ respectively. By the formula in Theorem 4.1, we have

$$H(\mathcal{H}) = H_\phi(\mathcal{H}) = \log(1+q) + \frac{1}{1+q}\eta(q),$$

$$H(\mathcal{L}) = H_{\phi'}(\mathcal{L}) = \log(1+p) + \frac{1}{1+p}\eta(p),$$

$$H(\mathcal{K}) = H(\mathcal{K}|\mathcal{L}) + H(\mathcal{L}) = \frac{p}{1+p}H(\mathcal{H}) + \frac{1}{1+p}(\eta(t) + \eta(1-t)) + H(\mathcal{L}),$$

$$H_\phi(\mathcal{K}) = H_\phi^E(\mathcal{K}|\mathcal{H}) + H(\mathcal{H}) = \frac{q}{1+q}H_{\phi'}(\mathcal{L}) + \frac{1}{1+q}(\eta(r) + \eta(1-r)) + H_\phi(\mathcal{H}).$$

**Proposition 4.2.** *Under the above situation, For two extensions $\mathcal{K}_1 = \mathcal{K}(t_1, r_1)$ and $\mathcal{K}_2 = \mathcal{K}(t_2, r_2)$ of $\mathbb{Z}_p(2)$ by $\mathbb{Z}_q(2)$, $\mathcal{K}_1$ is equivalent to $\mathcal{K}_2$ if and only if $H(\mathcal{K}_1) = H(\mathcal{K}_2)$ and $H_\phi(\mathcal{K}_1) = H_\phi(\mathcal{K}_2)$ hold.*

**Proof.** By the paper [IK1], it is known that $\mathcal{K}_1 = \mathcal{K}(t_1, r_1)$ is equivalent to $\mathcal{K}_2 = \mathcal{K}(t_2, r_2)$ if and only if $t_2 = t_1$ or $t_2 = 1 - t_1$, and $r_2 = r_1$ or $r_2 = 1 - r_1$. The latter condition is equivalent to $H(\mathcal{K}_1) = H(\mathcal{K}_2)$ and $H_\phi(\mathcal{K}_1) = H_\phi(\mathcal{K}_2)$. □

**Remark.** Two extensions $\mathcal{K}(t)$ and $\mathcal{K}(t')$ of $\mathbb{Z}_2$ by $\mathbb{Z}_q(2)$ are equivalent as extensions if and only if $H(\mathcal{K}(t)) = H(\mathcal{K}(t'))$ holds.

## 5 conditional entropy associated with a generalized orbital hypergroup

We modify the definition of a generalized orbital hypergroup in [FK].

**Definition.** Let $\mathcal{K} = (\mathcal{K}, M^b(\mathcal{K}))$ be a finite hypergroup and $\phi$ be the canonical state of $M^b(\mathcal{K})$. Let $N$ be a $*$-subalgebra with the unit of $M^b(\mathcal{K})$. Let $E$ be the conditional expectation from $M^b(\mathcal{K})$ onto $N$ such that $\phi \circ E = \phi$. For a finite hypergroup $\mathcal{K}_1 = (\mathcal{K}_1, M^b(\mathcal{K}_1))$, if $M^b(\mathcal{K}_1)$ is isomorphic to $N$ by a $*$-isomorphism $\Psi$ from $M^b(\mathcal{K}_1)$ onto $N$ and for $c \in \mathcal{K}$ there exists $b \in \mathcal{K}_1$ such that $E(c) = \Psi(b)$, then we say $\mathcal{K}_1$ a *generalized orbital hypergroup* of $\mathcal{K}$ by $E$ and denote $\mathcal{K}_1$ by $\mathcal{K}^E$.

We note that the above definition of a generalized orbital hypergroup is well-defined for a finite signed hypergroup.

In the present paper, we identify $N$ with $M^b(\mathcal{K}^E)$ hereafter.

**Lemma 5.1.** *Let $\psi$ be a mapping from $\mathcal{K}$ onto $\mathcal{K}^E$ which is the restriction to $\mathcal{K}$ of the conditional expectation $E$. Then we have,*

1. $w(\psi^{-1}(b)) = w(b)$ *for* $b \in \mathcal{K}^E$,
2. $w(\mathcal{K}) = w(\mathcal{K}^E)$.

**Proof.** Take the Haar measure $\mu_{\mathcal{K}} = \sum_{c\in\mathcal{K}} w(c)\delta_c$ of $\mathcal{K}$ and $\mu_{\mathcal{K}^E} = \sum_{b\in\mathcal{K}^E} w(b)\delta_b$ of $\mathcal{K}^E$ respectively. For any $\nu \in M^b(\mathcal{K}^E)$, $\nu \circ E(\mu_{\mathcal{K}}) = E(\nu \circ \mu_{\mathcal{K}}) = E(\mu_{\mathcal{K}})$ holds. Hence one can write $E(\mu_{\mathcal{K}}) = a\mu_{\mathcal{K}^E}$ for some $a \geq 0$. Since $\phi(E(\mu_{\mathcal{K}})) = \phi(\mu_{\mathcal{K}}) = 1$ and $\phi(\mu_{\mathcal{K}^E}) = 1$, we get $a = 1$, namely $E(\mu_{\mathcal{K}}) = \mu_{\mathcal{K}^E}$. We obtain

$$E(\mu_{\mathcal{K}}) = \sum_{c\in\mathcal{K}} w(c)E(\delta_c) = \sum_{b\in\mathcal{K}^E} \sum_{c\in\psi^{-1}(b)} w(c)\delta_b,$$

so that we arrive at the equation (1). Moreover, it is easy to see the equality (2) by (1). □

In a similar way to the section 4, two kinds of entropy associated with a generalized orbital hypergroup $\mathcal{K}^E$ of $\mathcal{K}$ are defined by

$$H(\mathcal{K}|\mathcal{K}^E) := H_{e_{\mathcal{K}}}(\psi : \mathcal{K}|\mathcal{K}^E) \text{ and } H^E_\phi(\mathcal{K}|\mathcal{K}^E) := H^E_\phi(M^b(\mathcal{K})|M^b(\mathcal{K}^E)).$$

Let $\hat{\mathcal{K}}$ and $\widehat{\mathcal{K}^E}$ be the dual signed hypergroups of $\mathcal{K}$ and $\mathcal{K}^E$ respectively. Then we have a conditional expectation $\hat{E}$ from $M^b(\hat{\mathcal{K}})$ onto $M^b(\widehat{\mathcal{K}^E})$ given by $\hat{E}(\chi) = \chi|_{M^b(\mathcal{K}^E)}$ for a character $\chi$ of $M^b(\mathcal{K})$ and a mapping $\hat{\psi}$ from $\hat{\mathcal{K}}$ onto $\widehat{\mathcal{K}^E}$ by the restriction of $\hat{E}$ to $\hat{\mathcal{K}}$. We note that $\hat{\phi} \circ \hat{E} = \hat{\phi}$ for the canonical state $\hat{\phi}$ of $M^b(\hat{\mathcal{K}})$.

**Theorem 5.2.** *Let $\mathcal{K}^E$ be a generalized orbital hypergroup of a finite commutative hypergroup $\mathcal{K}$ by the conditional expectation $E$ such that $\phi \circ E = \phi$ for the canonical state $\phi$ of $M^b(\mathcal{K})$. Under the above situation, the following formulae hold.*

1. $H(\mathcal{K}|\mathcal{K}^E) = \displaystyle\sum_{b\in\mathcal{K}^E} \sum_{c\in\psi^{-1}(b)} \frac{w(c)}{w(\mathcal{K})} \log \frac{w(b)}{w(c)} = H(\mathcal{K}) - H(\mathcal{K}^E).$

2. $H^E_\phi(\mathcal{K}|\mathcal{K}^E) = \displaystyle\sum_{\tau\in\widehat{\mathcal{K}^E}} \sum_{\chi\in\hat{\psi}^{-1}(\tau)} \frac{w(\chi)}{w(\hat{\mathcal{K}})} \log \frac{w(\tau)}{w(\chi)} = H_\phi(\mathcal{K}) - H_\phi(\mathcal{K}^E).$

3. $H(\hat{\mathcal{K}}|\widehat{\mathcal{K}^E}) = H^E_\phi(\mathcal{K}|\mathcal{K}^E)$ *and* $H^{\hat{E}}_{\hat{\phi}}(\hat{\mathcal{K}}|\widehat{\mathcal{K}^E}) = H(\mathcal{K}|\mathcal{K}^E).$

**Proof.** (1) For each $b \in \mathcal{K}^E$, the conditional probability measure $\mu_b$ of $e_{\mathcal{K}}$ on $\psi^{-1}(b)$ is given by

$$\mu_b = \sum_{c\in\psi^{-1}(b)} \frac{w(c)}{w(\psi^{-1}(b))}\delta_c.$$

Then we have

$$\begin{aligned} H(\mathcal{K}|\mathcal{K}^E) &= \sum_{b\in\mathcal{K}^E} e_{\mathcal{K}}(\psi^{-1}(b))H(\mu_b) = \sum_{b\in\mathcal{K}^E} \sum_{c\in\psi^{-1}(b)} \frac{w(\psi^{-1}(b))}{w(\mathcal{K})}\eta\left(\frac{w(c)}{w(\psi^{-1}(b))}\right) \\ &= \sum_{b\in\mathcal{K}^E} \sum_{c\in\psi^{-1}(b)} \frac{w(c)}{w(\mathcal{K})} \log \frac{w(\psi^{-1}(b))}{w(c)}. \end{aligned}$$

Since $w(\psi^{-1}(b)) = w(b)$ by (1) of Lemma 5.1, we get the desired formula.

(2) Let $\hat{\mathcal{K}}$ and $\widehat{\mathcal{K}^E}$ be $\hat{\mathcal{K}} = \{\chi_0, \cdots, \chi_n\}$ and $\widehat{\mathcal{K}^E} = \{\tau_0, \cdots, \tau_m\}$ respectively. Then we have minimal projections $\{e_i\}_{i=0}^n$ in $M^b(\mathcal{K})$ and $\{f_j\}_{j=0}^m$ in $M^b(\mathcal{K}^E)$ which satisfy

$$\chi_p(e_i) = \delta_{pi},\ \tau_q(f_j) = \delta_{qj}$$

for $\chi_p \in \hat{\mathcal{K}}$ and $\tau_q \in \widehat{\mathcal{K}^E}$ respectively. We note that $\phi(e_i) = \frac{w(\chi_i)}{w(\hat{\mathcal{K}})}$ and $\phi(f_j) = \frac{w(\tau_j)}{w(\widehat{\mathcal{K}^E})}$. Let $\sigma$ be the mapping from $\{0, 1, \cdots, n\}$ onto $\{0, 1, \cdots, m\}$ given by $e_i \circ f_{\sigma(i)} = e_i$. Hence,

$$\begin{aligned} H^E_\phi(\mathcal{K}|\mathcal{K}^E) &= \sum_{j=0}^m \phi(f_j) \sum_{i\in\sigma^{-1}(j)} \eta\left(\frac{\phi(e_i)}{\phi(f_j)}\right) = \sum_{j=0}^m \sum_{i\in\sigma^{-1}(j)} \phi(e_i) \log \frac{\phi(f_j)}{\phi(e_i)} \\ &= \sum_{j=0}^m \sum_{i\in\sigma^{-1}(j)} \frac{w(\chi_i)}{w(\hat{\mathcal{K}})} \log\left(\frac{w(\tau_j)}{w(\widehat{\mathcal{K}^E})} \cdot \frac{w(\hat{\mathcal{K}})}{w(\chi_i)}\right). \end{aligned}$$

It is easy to see that $e_i \circ f_j = e_i$ if and only if $\hat{\psi}(\chi_i) = \tau_j$. This means that $i \in \sigma^{-1}(j)$ if and only if $\chi_i \in \hat{\psi}^{-1}(\tau_j)$. Since $w(\mathcal{K}) = w(\mathcal{K}^E)$ by (2) of Lemma 5.1, we get the desired conclusion.

(3) We can show that $\widehat{\mathcal{K}^E} = \hat{\mathcal{K}}^{\hat{E}}$ holds. Applying the formula (1) to $\hat{\psi} : \hat{\mathcal{K}} \to \hat{\mathcal{K}}^{\hat{E}}$, one can obtain

$$H(\hat{\mathcal{K}}|\widehat{\mathcal{K}^E}) = H(\hat{\mathcal{K}}|\hat{\mathcal{K}}^{\hat{E}}) = \sum_{\tau\in\hat{\mathcal{K}}^{\hat{E}}} \sum_{\chi\in\hat{\psi}^{-1}(\tau)} \frac{w(\chi)}{w(\hat{\mathcal{K}})} \log \frac{w(\tau)}{w(\chi)}.$$

Hence it is clear that $H(\hat{\mathcal{K}}|\widehat{\mathcal{K}^E}) = H^E_\phi(\mathcal{K}|\mathcal{K}^E)$ by the formula (2).

Moreover, we have

$$H^{\hat{E}}_{\hat{\phi}}(\hat{\mathcal{K}}|\widehat{\mathcal{K}^E}) = H^{\hat{E}}_{\hat{\phi}}(\hat{\mathcal{K}}|\hat{\mathcal{K}}^{\hat{E}}) = H(\hat{\hat{\mathcal{K}}}|\widehat{\hat{\mathcal{K}}^{\hat{E}}}) = H(\hat{\hat{\mathcal{K}}}|\widehat{\widehat{\mathcal{K}^E}}) = H(\mathcal{K}|\mathcal{K}^E)$$

by the above equality and the duality $\hat{\hat{\mathcal{K}}} \cong \mathcal{K}$ and $\widehat{\widehat{\mathcal{K}^E}} \cong \mathcal{K}^E$. □

**Remark.** Let $\mathcal{K}^\alpha = \{b_0, b_1, \cdots, b_m\}$ be the orbital hypergroup by an action $\alpha$ of a finite group $G$ on a finite commutative hypergroup $\mathcal{K}$. Let $\hat{\alpha}$ be the action of $G$ on the dual signed hypergroup $\hat{\mathcal{K}}$ defined by $\hat{\alpha}_g(\chi)(c) := \chi(\alpha_{g^{-1}}(c))$ for $g \in G, \chi \in \hat{\mathcal{K}}$ and $c \in \mathcal{K}$. We denote by $O_j$ $\alpha$-orbit corresponding to $b_j \in \mathcal{K}^\alpha$. Let $\psi$ be a mapping from $\mathcal{K}$ onto $\mathcal{K}^\alpha$ such that $\psi^{-1}(b_j) = O_j$ and $E$ be the conditional expectation from $M^b(\mathcal{K})$ onto $M^b(\mathcal{K}^\alpha)$ such that $E|_\mathcal{K} = \psi$ and $\phi \circ E = \phi$ for the canonical state $\phi$ of $M^b(\mathcal{K})$. We note that $M^b(\mathcal{K}^\alpha)$ is equal to the fixed point algebra $M^b(\mathcal{K})^\alpha$ of $M^b(\mathcal{K})$ by $\alpha$. Let $O'_j$ be the $\hat{\alpha}-$orbit in $\hat{\mathcal{K}}$ corresponding to $\tau_j \in \widehat{\mathcal{K}^\alpha}$. We denote $|O_j|$ and $|O'_j|$ by $d_j$ and $d'_j$ respectively.

Then we remark the following.

1. $H(\mathcal{K}|\mathcal{K}^\alpha) = \sum_{j=0}^m \frac{w(c^{(j)})}{w(\mathcal{K})} d_j \log d_j$, where $c^{(j)} \in O_j$.

2. $H^E_\phi(\mathcal{K}|\mathcal{K}^\alpha) = \sum_{j=0}^m \frac{w(\chi^{(j)})}{w(\mathcal{K})} d'_j \log d'_j$, where $\chi^{(j)} \in O'_j$.

**Acknowledgement** The author would like to thank Professor Marie Choda for her warm encouragements and valuable suggestions and to thank Professor Satoshi Kawakami for useful discussions, information and references.

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Communicated by *Moto O'uchi*

S. Yamanaka : Visiting Researcher, Osaka Prefecture University,
1-1, Gakuen-cho, naka-ku, Sakai, Osaka 590-0075, Japan
e-mail : dt301001@edu.osakafu-u.ac.jp