UPPER ESTIMATIONS ON INTEGRAL OPERATOR MEANS

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Abstract. For an interpolational path of symmetric operator means, one of the author introduced the integral mean and showed that it is not less than the original mean, which is a generalization of the fact that the logarithmic operator mean is not less than the geometric operator mean. In this paper, we show estimations for the integral mean from the above.

1 Introduction. Recently in [5], we discussed again interpolational paths for Kubo-Ando operator means and introduced the integral means for the interpolational paths: For a Kubo-Ando symmetric operator mean m (cf. [11]) and positive operators A, B on a Hilbert space, the *path* m_t can be defined by inductive relation

$$A\,\mathrm{m}_{(2k+1)/2^{n+1}}B = (A\,\mathrm{m}_{k/2^n}B)\,\mathrm{m}\,(A\,\mathrm{m}_{(k+1)/2^n}B) = (A\,\mathrm{m}_{(k+1)/2^n}B)\,\mathrm{m}\,(A\,\mathrm{m}_{k/2^n}B) \quad (1)$$

with the initial conditions

$$A\,\mathrm{m}_0 B = A, \quad A\,\mathrm{m}_{1/2}B = A\,\mathrm{m}\,B, \quad A\,\mathrm{m}_1 B = B$$

so that the map $t \mapsto A\,\mathrm{m}_t B$ should be continuous. If m_t satisfies

$$(A\,\mathrm{m}_r B)\,\mathrm{m}_t(A\,\mathrm{m}_s B) = A\,\mathrm{m}_{(1-t)r+ts}B$$

for all weights r, s, $t \in [0,1]$, then we call it an *interporational path* and also call the original mean an *interpolational* one as in [6, 7, 9]. The interpolational paths are closely related to the geodesics of geometry or to the relative entropies, see [1, 2, 3, 4, 6, 9, 12, 13]. Then we defined the *integral mean* $\widetilde{\mathrm{m}}$ for m (or m_t) by

$$A\,\widetilde{\mathrm{m}}\,B = \int_0^1 A\,\mathrm{m}_t B\,dt.$$

For example, let $A\#_t^{(r)}B$ be the quasi-arithmetic mean $A^{1/2}f_t^{(r)}(A^{-1/2}BA^{-1/2})B$ with the representing function

$$f_t^{(r)}(x) = (1-t+tx^r)^{\frac{1}{r}}$$

for $-1 \leqq r \leqq 1$. Then, the representing function $\widetilde{f^{(r)}}$ of the integral mean $\widetilde{\#^{(r)}}$ is obtained by

$$\widetilde{f^{(r)}}(x) = \int_0^1 (1-t+tx^r)^{\frac{1}{r}}\,dt = \left[\frac{(1-t+tx^r)^{\frac{1+r}{r}}}{(x^r-1)\frac{1+r}{r}}\right]_0^1 = \frac{r}{1+r}\frac{x^{r+1}-1}{x^r-1}.$$

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Typical operator means we obtain here are:

$$(r=1) \quad \text{arithmetic mean: } \widetilde{f^{(1)}}(x) = \frac{1+x}{2},$$

$$(r=0) \quad \text{logarithmic mean: } \widetilde{f^{(0)}}(x) \equiv \lim_{\varepsilon\downarrow 0} \widetilde{f^{(\varepsilon)}}(x) = \frac{x-1}{\log x},$$

$$(r=-1/2) \quad \text{geometric mean: } \widetilde{f^{(-1/2)}}(x) = \sqrt{x},$$

$$(r=-1) \quad \text{adjoint logarithmic mean: } \widetilde{f^{(-1)}}(x) \equiv \lim_{\varepsilon\downarrow 0} \widetilde{f^{(\varepsilon-1)}}(x) = \frac{x\log x}{x-1}.$$

Then we showed in [5] that it is a symmetric operator mean and dominates the original one; $A\,\widetilde{\mathrm{m}}\,B \geq A\,\mathrm{m}\,B$. In this paper, we show its upper bound inspired by Kittaneh's method [10] for integral inequalities.

2 Estimation by operator convexity. An hermitian operator-valued function $\phi(t)$ is *operator-valued convex* on $\mathcal{I} \subset \mathbb{R}$ if

$$\phi((1-t)s+tr) \leq (1-t)\phi(s) + t\phi(r)$$

holds for all $s, r \in \mathcal{I}$ and $t \in [0,1]$ for the usual order of operators. Though Kittaneh's method in [10] is based on the existance of the minimum for a convex function, it is not suitable for operator functions. So we dare to set an arbitrary dividing point s:

Lemma 1. *If a parametrized operator $\phi(t)$ is operator-valued convex on $[0,1]$, then for each $s \in [0,1]$, $\phi(t)$ is dominated by*

$$\begin{cases} \frac{t}{s}(\phi(s)-\phi(0)) + \phi(0) & \text{if } 0 \leqq t \leqq s, \text{ and} \\ \frac{t-s}{1-s}(\phi(1)-\phi(s)) + \phi(s) & \text{if } s \leqq t \leqq 1. \end{cases}$$

Proof. Since $t = (1-p)\cdot 0 + ps$ for $p = t/s \in [0,1]$ for the former case, we have

$$\phi(t) = \phi((1-p)\cdot 0 + ps) \leq (1-p)\phi(0) + p\phi(s) = \frac{t}{s}\,(\phi(s)-\phi(0)) + \phi(0).$$

For the latter case, putting $t = (1-q)s + q\cdot 1$ for $q = (t-s)/(1-s) \in [0,1]$, we have

$$\phi(t) = \phi((1-q)s + q\cdot 1) \leq (1-q)\phi(s) + q\phi(1) = \frac{t-s}{1-s}\,(\phi(1)-\phi(s)) + \phi(s). \qquad \square$$

Since the arithmetic mean $A\nabla B = (A+B)/2$ is the maximum in symmetric operator means, we also have the weighted inequalities:

Lemma 2. $A\nabla_t B \equiv (1-t)A + tB \geq A\,\mathrm{m}_t B$ *for an interpolational path* m_t.

Proof. It holds for initial points $t = 0, 1/2, 1$. Then it also holds for binary points $k/2^n$ inductively by (1), so that it holds for all $t \in [0,1]$. $\square$

Though the following property is known, we give a proof for completeness:

Lemma 3. *Each interporational path* m_t *is operator-valued convex for* $t \in [0,1]$.

Proof. By interpolationality and Lemma 2, we have

$A\,\mathrm{m}_{(1-t)r+ts}B = (A\,\mathrm{m}_r B)\,\mathrm{m}_t(A\,\mathrm{m}_s B) \leq (A\,\mathrm{m}_r B)\nabla_t(A\,\mathrm{m}_s B) = (1-t)(A\,\mathrm{m}_r B)+t(A\,\mathrm{m}_s B)$,

which shows the operator-valued convexity. □

So we have the following upper estimation of integral operator means:

Theorem 4. *For the integral mean* $\widetilde{\mathrm{m}}$ *for an interpolational path* m_t,

$$A\,\widetilde{\mathrm{m}}B \leq \frac{sA+(1-s)B+A\,\mathrm{m}_s B}{2}$$

for all $s \in [0,1]$. *In particular,* $A\,\widetilde{\mathrm{m}}B \leq \dfrac{A\nabla B + A\,\mathrm{m}\,B}{2}$.

Proof. By the above lemma, we have

$$\begin{aligned}\int_0^s A\,\mathrm{m}_t B dt &\leq \int_0^s \left(\frac{t}{s}(A\,\mathrm{m}_s B - A)+A\right)dt = \left[\frac{t^2}{2s}(A\,\mathrm{m}_s B - A)+tA\right]_0^s \\ &= \frac{s^2}{2s}\,(A\,\mathrm{m}_s B - A) + sA = \frac{s}{2}\,(A\,\mathrm{m}_s B + A) \quad \text{and} \\ \int_s^1 A\,\mathrm{m}_t B dt &\leq \int_s^1 \left(\frac{t-s}{1-s}\,(B - A\,\mathrm{m}_s B) + A\,\mathrm{m}_s B\right)dt \\ &= \left[\frac{t^2/2-ts}{1-s}\,(B - A\,\mathrm{m}_s B) + tA\,\mathrm{m}_s B\right]_s^1 \\ &= \frac{1/2-s-s^2/2+s^2}{1-s}\,(B-A\,\mathrm{m}_s B) + (1-s)A\,\mathrm{m}_s B = \frac{1-s}{2}\,(B + A\,\mathrm{m}_s B)\,.\end{aligned}$$

Therefore, $A\widetilde{\mathrm{m}}B = \displaystyle\int_0^1 A\,\mathrm{m}_t B dt \leq \frac{sA+(1-s)B+A\,\mathrm{m}_s B}{2}$. □

3 Estimation for many dividing points.

Here we generalize Lemma 1 similarly:

Lemma 5. *If a parametrized operator* $\phi(t)$ *is operator-valued convex on* $[0,1]$, *then for each* $0 \leqq s \leqq t \leqq r \leqq 1$, *the operator* $\phi(t)$ *is dominated by*

$$\frac{t-s}{r-s}(\phi(r)-\phi(s))+\phi(s).$$

Proof. The operator-valued convexity shows

$$\phi(t) \leq \frac{t-s}{r-s}\phi(r) + \frac{r-t}{r-s}\phi(s) = \frac{t-s}{r-s}\big(\phi(r)-\phi(s)\big)+\phi(s).$$ □

Then we have a better estimation for integral means:

Theorem 6. *For the integral mean* $\widetilde{\mathrm{m}}$ *for an interpolational path* m_t *and* $0 \equiv t_0 < t_1 < \cdots < t_n < t_{n+1} \equiv 1$,

$$A\,\widetilde{\mathrm{m}}B \leq \frac{1}{2}\left(t_1 A + (1-t_n)B + \sum_{k=1}^{n}(t_{k+1}-t_{k-1})A\,\mathrm{m}_{t_k}B\right).$$

In particular, $A\,\widetilde{\mathrm{m}}B \leq \dfrac{1}{n+1}\left(A\nabla B + \displaystyle\sum_{k=1}^{n} A\,\mathrm{m}_{k/(n+1)}B\right)$.

Proof. It is shown that

$$\int_0^{t_1} A\,\mathrm{m}_t B dt \le \frac{t_1}{2}(A\,\mathrm{m}_{t_1} B + A) \quad \text{and} \quad \int_{t_n}^1 A\,\mathrm{m}_t B dt \le \frac{1-t_n}{2}(B + A\,\mathrm{m}_{t_n} B).$$

Since

$$\begin{aligned}\int_{t_k}^{t_{k+1}} A\,\mathrm{m}_t B dt &\le \int_{t_k}^{t_{k+1}} \left(\frac{t-t_k}{t_{k+1}-t_k}(A\,\mathrm{m}_{t_{k+1}} B - A\,\mathrm{m}_{t_k} B) + A\,\mathrm{m}_{t_k} B\right) dt \\ &= \left[\frac{t^2/2 - tt_k}{t_{k+1}-t_k}(A\,\mathrm{m}_{t_{k+1}} B - A\,\mathrm{m}_{t_k} B) + tA\,\mathrm{m}_{t_k} B\right]_{t_k}^{t_{k+1}} \\ &= \frac{t_{k+1}-t_k}{2}(A\,\mathrm{m}_{t_{k+1}} B - A\,\mathrm{m}_{t_k} B) + (t_{k+1}-t_k)A\,\mathrm{m}_{t_k} B \\ &= \frac{t_{k+1}-t_k}{2}(A\,\mathrm{m}_{t_{k+1}} B + A\,\mathrm{m}_{t_k} B),\end{aligned}$$

we have

$$A\,\widetilde{\mathrm{m}}\,B = \int_0^1 A\,\mathrm{m}_t B\, dt \le \frac{1}{2}\left(t_1 A + (1-t_n)B + \sum_{k=1}^n (t_{k+1}-t_{k-1})A\,\mathrm{m}_{t_k} B\right).$$

The last inequality follows from $t_k = k/(n+1)$. □

Finally we compare two estimations of Theorems 4 and 6. Put the (2-times) difference

$$\begin{aligned}D_s &= sA + (1-s)B + A\,\mathrm{m}_s B - t_1 A - (1-t_n)B - \sum_{k=1}^n (t_{k+1}-t_{k-1})A\,\mathrm{m}_{t_k} B \\ &= (s-t_1)A + (t_n - s)B + A\,\mathrm{m}_s B - \sum_{k=1}^n (t_{k+1}-t_{k-1})A\,\mathrm{m}_{t_k} B.\end{aligned}$$

Then the difference D_s is positive for $s = t_k$ for each $k = 0, \cdots, n+1$:

Theorem 7. *If $s = t_k$ for some $k = 0, \cdots, n+1$ in Theorem 6,*

$$\begin{aligned}A\,\widetilde{\mathrm{m}}\,B &\le \frac{1}{2}\left(t_1 A + (1-t_n)B + \sum_{k=1}^n (t_{k+1}-t_{k-1})A\,\mathrm{m}_{t_k} B\right) \\ &\le sA + (1-s)B + A\,\mathrm{m}_s B.\end{aligned}$$

Proof. For convenience' sake, we use $\sum_{k=a}^b \cdot = 0$ if $b < a$. Let $1 \leqq K \leqq n$. Then Lemma 1 or 5 shows

$$A\,\mathrm{m}_{t_k} B \le \frac{t_k}{t_{k+1}}(A\,\mathrm{m}_{t_{k+1}} B - A) + A = \frac{t_k}{t_{k+1}} A\,\mathrm{m}_{t_{k+1}} B + \frac{t_{k+1}-t_k}{t_{k+1}} A,$$

that is,

$$t_{k+1} A\,\mathrm{m}_{t_k} B \le t_k A\,\mathrm{m}_{t_{k+1}} B + (t_{k+1}-t_k)A.$$

Then, summing up from $k = 1$ to $k = K-1$, we have

$$\sum_{k=1}^{K-1} (t_{k+1}-t_{k-1})A\,\mathrm{m}_{t_k} B \le t_{K-1} A\,\mathrm{m}_{t_K} B + (t_K - t_1)A \qquad (a)$$

On the other hand, summing up from $k = K + 1$ to $k = n$ for other type of inequalities

$$(1 - t_{k-1})A\,\mathrm{m}_{t_k}B \le (t_k - t_{k-1})B + (1 - t_k)A\,\mathrm{m}_{t_{k-1}}B,$$

we have

$$\sum_{k=K+1}^{n}(t_{k+1} - t_{k-1})A\,\mathrm{m}_{t_k}B \le (t_n - t_K)B + (1 - t_{K+1})A\,\mathrm{m}_{t_K}B. \qquad (b)$$

It follows that

$$\begin{aligned} D_{t_K} &= (t_K - t_1)A + (t_n - t_K)B + A\,\mathrm{m}_{t_K}B - \sum_{k=1}^{n}(t_{k+1} - t_{k-1})A\,\mathrm{m}_{t_k}B \\ &= (t_K - t_1)A + (t_n - t_K)B + (1 - t_{K+1} + t_{K-1})A\,\mathrm{m}_{t_K}B - \sum_{k\ne K}(t_{k+1} - t_{k-1})A\,\mathrm{m}_{t_k}B \\ &\ge (t_K - t_1)A + (t_n - t_K)B + (1 - t_{K+1} + t_{K-1})A\,\mathrm{m}_{t_K}B \\ &\qquad -t_{K-1}A\,\mathrm{m}_{t_K}B - (t_K - t_1)A - (t_n - t_K)B - (1 - t_{K+1})A\,\mathrm{m}_{t_K}B = 0. \end{aligned}$$

Note that

$$D_{t_0} = D_0 = D_{t_{n+1}} = D_1 = t_nB + (1 - t_1)A - \sum_{k=1}^{n}(t_{k+1} - t_{k-1})A\,\mathrm{m}_{t_k}B.$$

Similarly to (a) or (b), we have

$$\sum_{k=1}^{n}(t_{k+1} - t_{k-1})A\,\mathrm{m}_{t_k}B \le t_nB + (1 - t_1)A,$$

which implies $D_{t_0} = D_{t_{n+1}} \ge 0$. □

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