NAZAROV TYPE UNCERTAINTY INEQUALITY FOR FOURIER SERIES

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Abstract. We shall obtain an analogue of Nazarov's uncertainty inequality for n-dimensional Fourier series from the one for n-dimensional Fourier transform. Some inequalities are new and better than ones deduced from a classical local uncertainty inequality.

1 Introduction The uncertainty principle generally asserts that a non-zero function and its Fourier transform cannot be too small simultaneously. Although interpretations of the smallness are very broad, various versions of the uncertainty principle exist and also, analogous versions for Fourier series exist. However, as for the Nazarov uncertainty principle, it is known only for Fourier transform. In this paper we shall obtain an analogous uncertainty inequality for n-dimensional Fourier series. Nazarov's uncertainty inequality is originally appeared in [2] and P. Jaming [1] extends it to a higher dimensional Fourier transform: There exists a constant C such that, if S, Σ are subset of $\mathbb{R}^n$ of finite measure, then for every $f \in L^2(\mathbb{R}^n)$,

$$(1) \qquad \int_{\mathbb{R}^n} |f(x)|^2 dx \leq \gamma(S,\Sigma)\Big(\int_{\mathbb{R}^n\setminus S} |f(x)|^2 dx + \int_{\mathbb{R}^n\setminus\Sigma} |\mathcal{F}f(x)|^2 dx\Big),$$

where

$$\gamma(S,\Sigma) = Ce^{C\min\{|S||\Sigma|, |S|^{\frac{1}{n}} w(\Sigma), |\Sigma|^{\frac{1}{n}} w(S)\}},$$

$\mathcal{F}f$ is the Fourier transform of f, $|\cdot|$ is the measure and $w(\cdot)$ is the mean width. We shall rewrite this inequality for a Fourier series version. For $f \in l^1(\mathbb{Z}^n)\cap l^2(\mathbb{Z}^n)$, we denote by $\check{f}$ the trigonometric series

$$\check{f}(\lambda) = \sum_{m\in\mathbb{Z}^n} f(m)e^{2\pi i\lambda m}, \quad \lambda \in \mathbb{R}^n,$$

and for any function $g \in L^p(\mathbb{R}^n)$, $1 \leq p \leq \infty$, we denote by $f \boxtimes g$ a convolution

$$f \boxtimes g(x) = \sum_{m\in\mathbb{Z}^n} f(m)g(x-m), \quad x \in \mathbb{R}^n.$$

Applying the above inequality (1) to the function $f \boxtimes g$ on $\mathbb{R}^n$, we can deduce a Nazarov type uncertainty inequality for f on $\mathbb{T}^n$. Let I be a closed measurable subset in $[-\frac{1}{2}, \frac{1}{2}]^n$, S a finite subset in $\mathbb{Z}^n$ and Σ a measurable subset in I. Then for all functions $g_I \in L^2(\mathbb{R}^n)$

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supported on I and vanishing on the boundary of I, the following inequality follows (see Theorem 3.4 and let $J = \{0\}$):

$$\sum_{m\in\mathbb{Z}^n} |f(m)|^2$$
$$\leq \gamma(S+I,\Sigma)\Big(\sum_{m\in\mathbb{Z}^n\setminus S} |f(m)|^2 + \int_{\mathbb{T}^n\setminus\Sigma} |\check{f}(x)|^2 dx + \int_\Sigma |\check{f}(x)|^2 \Big(1 - \frac{|\mathcal{F}(g_I)(x)|^2}{\|g_I\|^2_{L^2(\mathbb{R}^n)}}\Big) dx\Big).$$

Especially, letting $I = [-\frac{1}{2}, \frac{1}{2}]^n$ and taking the characteristic function χ of $[-\frac{1}{2}, \frac{1}{2}]^n$ as g_I, we see that

$$1 - \frac{|\mathcal{F}(\chi)(x)|^2}{\|\chi\|^2_{L^2(\mathbb{R}^n)}} = 1 - \prod_{i=1}^n \Big|\frac{\sin \pi x_i}{\pi x_i}\Big|^2 = O(|x|^2), \quad x = (x_1, x_2, \cdots, x_n),$$

and therefore, we can deduce that

$$\sum_{m\in\mathbb{Z}^n} |f(m)|^2 \leq \gamma(S+I,\Sigma)\Big(\sum_{m\in\mathbb{Z}^n\setminus S} |f(m)|^2 + \int_{\mathbb{T}^n\setminus\Sigma} |\check{f}(x)|^2 dx + C\int_\Sigma |\check{f}(x)|^2 |x|^2 dx\Big) \tag{2}$$

(see §4, (I)). Clearly, if we can find a function g_I satisfying

$$\frac{|\mathcal{F}(g_I)(x)|^2}{\|g_I\|^2_{L^2(\mathbb{R}^n)}} = 1 + O(x^\gamma),$$

then we can replace $|x|^2$ in the last integral of (2) by $|x|^\gamma$. However, we see that $\gamma = 2$ associated to χ is maximal when I is a subset in $[-\frac{1}{2}, \frac{1}{2}]^n$ (see Remark 4.1). Therefore, in §3 we shall consider a general subset $I \subset \mathbb{R}^n$ and obtain a modified inequality of (2). In §4, we find an example of the modified inequality where $|x|^2$ in (2) is replaced by $|x|^{10}$ (see Corollary 4.2).

On the other hand, J. F. Price and P. C. Racki [3] obtain the so-called local uncertainty inequalities for Fourier series. As a special case, for $\beta < \frac{1}{2}$, $\alpha \geq 0$ and $\alpha \geq \beta$, there exists a constant K such that for all $\check{f} \in L^1(\mathbb{T}^n)$ and all finite S of $\mathbb{Z}^n$,

$$\sum_{m\in S} |f(m)|^2 \leq K^2 |S|^{2\alpha} \int_{\mathbb{T}^n} |\check{f}(x)|^2 |x|^{2n\beta} dx.$$

Hence, it easily follows that

$$\sum_{m\in\mathbb{Z}^n} |f(m)|^2 \leq \sum_{m\in\mathbb{Z}^n\setminus S} |f(m)|^2 + K^2|S|^{2\alpha} \int_{\mathbb{T}^n\setminus\Sigma} |\check{f}(x)|^2 dx + K^2 |S|^{2\alpha} \int_\Sigma |\check{f}(x)|^2 |x|^{2n\beta} dx. \tag{3}$$

Here the weight function $|x|^{2n\beta}$, which reflects the localization of $\check{f}$ around zero, is better than $|x|^2$ in (2) when $n \geq 3$. In our modified inequalities (see Theorem 3.4 and Remark 3.5), there is some possibility of replacing $|x|^{2n\beta}$ by $|x|^\gamma$. At this stage, $\gamma = 10$ is our best result (see §4, (IV)).

2 Notations.

We identify $[-\frac{1}{2}, \frac{1}{2}]^n$ with $\mathbb{T}^n$. We denote by $L^p(\mathbb{T}^n)$ the space of all measurable functions $f(t)$ on $\mathbb{T}^n$ such that $\|f\|^p_{L^p(\mathbb{T}^n)} = \int_{\mathbb{T}^n} |f(t)|^p dt < \infty$ for $1 \leq p < \infty$ and $\|f\|_{L^\infty(\mathbb{T}^n)} = \text{ess.sup}_{t\in\mathbb{T}^n} |f(t)| < \infty$ for $p = \infty$. Similarly, we denote by $l^p(\mathbb{Z}^n)$ the space of all functions $F(m)$ on $\mathbb{Z}^n$ such that $\|F\|^p_{l^p(\mathbb{Z}^n)} = \sum_{m\in\mathbb{Z}^n} |F(m)|^p < \infty$ for $1 \le p < \infty$ and $\|F\|_{L^\infty(\mathbb{Z}^n)} = \sup_{m\in\mathbb{Z}^n} |F(m)| < \infty$ for $p = \infty$. For $f \in L^1(\mathbb{T}^n)$, the Fourier coefficients of f are defined by

$$\hat{f}(m) = \int_{\mathbb{T}^n} f(t)e^{-2\pi i t m} dt, \quad m \in \mathbb{Z}^n.$$

and $\|\hat{f}\|_{L^\infty(\mathbb{Z}^n)} \le \|f\|_{L^1(\mathbb{T}^n)}$. For $F \in l^1(\mathbb{Z}^n)$, we denote by $\check{F}$ the trigonometric series

$$\check{F}(\lambda) = \sum_{m\in\mathbb{Z}^n} F(m)e^{2\pi i\lambda m}, \quad \lambda \in \mathbb{R}^n.$$

$\check{F}$ is a 1-periodic function on $\mathbb{R}^n$ that can be regarded as a function on $\mathbb{T}^n$ and $\|\check{F}\|_{L^\infty(\mathbb{T}^n)} \le \|F\|_{L^1(\mathbb{Z}^n)}$. If f and $\hat{f}$ belong to $L^1(\mathbb{T}^n)$ and $l^1(\mathbb{Z}^n)$ respectively, then $f = \check{\hat{f}}$. Moreover, if $f \in L^2(\mathbb{T}^n)$, then $\hat{f} \in l^2(\mathbb{Z}^n)$ and $\|f\|_{L^2(\mathbb{T}^n)} = \|\hat{f}\|_{L^2(\mathbb{Z}^n)}$. We define a convolution $\otimes$ on $\mathbb{Z}^n$ as follows: For $F, G \in l^1(\mathbb{Z}^n)$,

$$F \otimes G(n) = \sum_{m\in\mathbb{Z}^n} F(m)G(n-m).$$

Then $F \otimes G \in l^1(\mathbb{Z}^n)$ and $(F \otimes G)\check{} = \check{F}\check{G}$.

We denote by $L^p(\mathbb{R}^n)$ the space of all measurable functions $f(x)$ on $\mathbb{R}^n$ such that $\|f\|^p_{L^p(\mathbb{R}^n)} = \int_{\mathbb{R}^n} |f(x)|^p dx < \infty$ for $1 \le p < \infty$ and $\|f\|_{L^\infty(\mathbb{R}^n)} = \text{ess.sup}_{x\in\mathbb{R}^n}|f(x)| < \infty$ for $p = \infty$. For $f \in L^1(\mathbb{R}^n)$, we denote the Fourier transform of f as

$$\mathcal{F}(f)(\lambda) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i\lambda x} dx, \quad \lambda \in \mathbb{R}^n.$$

Clearly $\|\mathcal{F}(f)\|_{L^\infty(\mathbb{R}^n)} \le \|f\|_{L^1(\mathbb{R}^n)}$. For $F \in L^1(\mathbb{R})$, the inverse Fourier transform $\mathcal{F}^{-1}$ is given by

$$\mathcal{F}^{-1}(F)(x) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} F(\lambda)e^{2\pi i\lambda x} d\lambda, x \in \mathbb{R}^n.$$

If f and $\mathcal{F}(f) \in L^1(\mathbb{R}^n)$, then $f = \mathcal{F}^{-1}(\mathcal{F}(f))$. Moreover, if $f \in L^2(\mathbb{R}^n)$, then $\mathcal{F}(f) \in l^2(\mathbb{R}^n)$ and $(2\pi)^{\frac{n}{2}}\|f\|_{L^2(\mathbb{R}^n)} = \|\mathcal{F}(f)\|_{L^2(\mathbb{R}^n)}$. We define a convolution $*$ on $\mathbb{R}^n$ as follows: For $f, g \in L^1(\mathbb{R}^n)$,

$$f * g(x) = \int_{\mathbb{R}^n} f(y)g(x-y)dy.$$

Then $f * g \in L^1(\mathbb{R}^1)$ and $\mathcal{F}(f * g) = \mathcal{F}(f)\mathcal{F}(g)$.

For $f \in l^1(\mathbb{Z}^n)$ and $g \in L^p(\mathbb{R}^n)$, we define a convolution $\boxtimes$ as

$$f \boxtimes g(x) = \sum_{m\in\mathbb{Z}^n} f(m)g(x-m), \quad x \in \mathbb{R}^n.$$

We can easily see that for $1 \le p \le \infty$,

$$\|f \boxtimes g\|_{L^p(\mathbb{R}^n)} \le \|f\|_{l^1(\mathbb{Z}^n)}\|g\|_{L^p(\mathbb{R}^n)}$$

and, if $p = 1$, then

$$\mathcal{F}(f \boxtimes g)(\lambda) = \check{f}(\lambda)\mathcal{F}(g)(\lambda). \tag{4}$$

3 Nazarov's inequality on $\mathbb{Z}^n$. Let I be a connected subset of $\mathbb{R}^n$ of finite measure and containing 0 and put

$$2I^\circ \cap \mathbb{Z}^n = J,$$

where $I^\circ = I - \partial I$. Let $J^\sharp$ denote the cardinal of J. For a finite subset S of $\mathbb{Z}^n$, let $\tilde{S}$ be a subset of $\mathbb{R}^n$ defined by

$$\tilde{S} = S + I.$$

Then $(S + 2I^\circ) \cap \mathbb{Z}^n = S + J$. In what follows g_I is a function in $L^2(\mathbb{R}^n)$ which is supported on I and vanishes on ∂I. We define

$$\|g_I\|_{2,j}^2 = \int_{\mathbb{R}^n} g_I(x)\overline{g_I(x-j)}dx$$

for $j \in J$.

Lemma 3.1 *For $f \in l^1(\mathbb{Z}^n) \cap l^2(\mathbb{Z}^n)$, it follows that*

$$\begin{aligned} \|f \boxtimes g_I\|_{L^2(\mathbb{R}^n)} &= \sum_{j\in J} \|g_I\|_{2,j}^2 \sum_{m\in\mathbb{Z}^n} f(m)\overline{f(m+j)} \\ &= \sum_{j\in J} \|g_I\|_{2,j}^2 \int_{\mathbb{T}^n} |\check{f}(\lambda)|^2 e^{2\pi i j\lambda} d\lambda \le J^\sharp \|g_I\|_{L^2(\mathbb{R}^n)}^2 \|f\|_{l^2(\mathbb{Z}^n)}^2. \end{aligned} \tag{5}$$

Proof. Since $g_I(x) = 0$ if $x \notin I^\circ$ and $2I^\circ \cap \mathbb{Z}^n = J$, it follows that

$$\begin{aligned} \|f \boxtimes g_I\|_{L^2(\mathbb{R}^n)}^2 &= \sum_{m\in\mathbb{Z}^n}\sum_{m'\in\mathbb{Z}^n} f(m)\overline{f(m')} \int_{\mathbb{R}^n} g_I(x-m)\overline{g_I(x-m')}dx \\ &= \sum_{m\in\mathbb{Z}^n}\sum_{m'\in\mathbb{Z}^n} f(m)\overline{f(m')} \int_{\mathbb{R}^n} g_I(x)\overline{g_I(x-(m'-m))}dx \\ &= \sum_{j\in J}\sum_{m\in\mathbb{Z}^n} f(m)\overline{f(m+j)} \int_{\mathbb{R}^n} g_I(x)\overline{g_I(x-j)}dx. \end{aligned}$$

The equality (5) follows from the Parseval equality for Fourier series. The inequality is obvious from the Schwartz inequality. □

Lemma 3.2 *Let $f \in l^1(\mathbb{Z}^n)$ and $g_I, S, \tilde{S}$ be as above. Then for all $x \in \mathbb{R}^n$,*

$$|f| \boxtimes |g_I|(x)\chi_{\tilde{S}}(x) \le |f\chi_{S+J}| \boxtimes |g_I|(x) \le |f| \boxtimes |g_I|(x)|\chi_{\tilde{S}+J}(x), \tag{6}$$

where χ_{S+J} and $\chi_{\tilde{S}}, \chi_{\tilde{S}+J}$ are the characteristic functions of the sets $S+J \subset \mathbb{Z}^n$ and $\tilde{S}, \tilde{S}+J \subset \mathbb{R}^n$ respectively. Especially, when $J = \{0\}$, it follows that

$$f \boxtimes g_I(x)\chi_{\tilde{S}}(x) = (f\chi_S) \boxtimes g_I(x). \tag{7}$$

Proof. We recall that $|f| \boxtimes |g_I|(x)$ is given by

$$\sum_{m\in\mathbb{Z}^n} |f(m)||g_I(x-m)|.$$

We suppose that $\chi_{\tilde{S}}(x) = 1$ and $g(x-m) \neq 0$. Then $m \in x - I^\circ \subset \tilde{S} + I^\circ = S + 2I^\circ$. Since $(S + 2I^\circ) \cap \mathbb{Z}^n = S + J$, it follows $m \in S + J$ and $\chi_{S+J}(m) = 1$. If $\chi_{S+J}(m) = 1$ and $g_I(x-m) \neq 0$, then $x \in m + I \subset S + J + I = \tilde{S} + J$ and $\chi_{\tilde{S}+J}(x) = 1$. Hence the desired inequalities (6) follow. Let $J = \{0\}$ and suppose that $g(x-m) \neq 0$. Then it follows that $\chi_{\tilde{S}}(x) = 1$ if and only if $\chi_{S+J}(m) = 1$. Hence we can deduce the equality (7). □

Corollary 3.3 *Let $I, J, g_I, S, \tilde{S}$ be as above and Σ a measurable subset in $\mathbb{R}^n$ such that $\Sigma \subset \mathbb{T}^n$. Then for $f \in l^1(\mathbb{Z}^n)$,*

$$(i) \quad \int_{\mathbb{R}^n \setminus (\tilde{S}+J)} |f \boxtimes g_I(x)|^2 dx \leq J^\sharp \|g_I\|^2_{L^2(\mathbb{R}^n)} \sum_{m \in \mathbb{Z}^n \setminus (S+J)} |f(m)|^2$$

and the equality holds when $J = \{0\}$.

$$(ii) \quad \int_{\mathbb{R}^n \setminus \Sigma} |\mathcal{F}(f \boxtimes g_I)(x)|^2 dx = \sum_{j \in J} \|g_I\|^2_{2,j} \int_{\mathbb{T}^n \setminus \Sigma} |\check{f}(\lambda)|^2 e^{2\pi i j \lambda} d\lambda + \int_\Sigma |\check{f}(\lambda)|^2 (\sum_{j \in J} \|g_I\|^2_{2,j} e^{2\pi i j \lambda} - |\mathcal{F}(g_I)(\lambda)|^2) dx.$$

Proof. (i) : By using Lemma 3.1 and Lemma 3.2, we see that

$$\begin{aligned}
\int_{\mathbb{R}^n \setminus (\tilde{S}+J)} |f \boxtimes g_I(x)|^2 dx &\leq \||f| \boxtimes |g_I|\|^2_{L^2(\mathbb{R}^n)} - \||f| \boxtimes |g_I \cdot \chi_{\tilde{S}+J}|\|^2_{L^2(\mathbb{R}^n)} \\
&\leq \||f| \boxtimes |g_I|\|^2_{L^2(\mathbb{R}^n)} - \||f\chi_{S+J}| \boxtimes |g_I|\|^2_{L^2(\mathbb{R}^n)} \\
&= \||f(1-\chi_{S+J})| \boxtimes |g_I|\|^2_{L^2(\mathbb{R}^n)} \\
&\leq J^\sharp \|g_I\|^2_{L^2(\mathbb{R}^n)} \sum_{m \in \mathbb{Z}^n \setminus (S+J)} |f(m)|^2.
\end{aligned}$$

When $J = \{0\}$, the equalities (5) and (7) yield the desired equality.

(ii) : It follows from (4) and Lemma 3.1 that

$$\begin{aligned}
&\int_{\mathbb{R}^n \setminus \Sigma} |\mathcal{F}(f \boxtimes g_I)(x)|^2 dx \\
&= \int_{\mathbb{R}^n} |\mathcal{F}(f \boxtimes g_I)(x)|^2 dx - \int_\Sigma |\mathcal{F}(f \boxtimes g_I)(x)|^2 dx \\
&= \int_{\mathbb{R}^n} |f \boxtimes g_I(x)|^2 dx - \sum_{j \in J} \|g\|^2_{2,j} \int_\Sigma |\check{f}(\lambda)|^2 e^{2\pi i j \lambda} d\lambda \\
&\quad + \sum_{j \in J} \|g\|^2_{2,j} \int_\Sigma |\check{f}(\lambda)|^2 e^{2\pi i j \lambda} d\lambda - \int_\Sigma |\check{f}(\lambda) \mathcal{F}(g_I)(\lambda)|^2 dx \\
&= \sum_{j \in J} \|g\|^2_{2,j} \int_{\mathbb{T}^n \setminus \Sigma} |\check{f}(\lambda)|^2 e^{2\pi i j \lambda} d\lambda + \int_\Sigma |\check{f}(x)|^2 (\sum_{j \in J} \|g\|^2_{2,j} e^{2\pi i j \lambda} - |\mathcal{F}(g_I)(\lambda)|^2) d\lambda.
\end{aligned}$$

□

Now applying Nazarov's inequality (1) to $f \boxtimes g_I$ and combining Lemma 3.1 and Corollary 3.3, we can deduce the following.

Theorem 3.4 *Let I be a connected subset of $\mathbb{R}^n$ of finite measure and containing 0 and put $2I^\circ \cap \mathbb{Z}^n = J$. Let S be a finite subset of $\mathbb{Z}^n$ and Σ a measurable subset in $\mathbb{R}^n$ such that $\Sigma \subset \mathbb{T}^n$. Then for all $f \in l^1(\mathbb{Z}^n) \cap l^2(\mathbb{Z}^n)$,*

$$\begin{aligned}&\sum_{j\in J} \|g_I\|_{2,j} \sum_{m\in\mathbb{Z}^n} f(m)\overline{f(m+j)}\\ &\leq \gamma(\tilde{S}+J,\Sigma)\Bigg(J^\sharp \|g_I\|^2_{L^2(\mathbb{R}^n)}\Big(\sum_{m\in\mathbb{Z}^n\setminus(S+J)} |f(m)|^2 + \int_{\mathbb{T}^n\setminus\Sigma} |\check{f}(\lambda)|^2 d\lambda\Big)\\ &\qquad + \int_\Sigma |\check{f}(\lambda)|^2\Big(\sum_{j\in J} \|g_I\|^2_{2,j} e^{2\pi i j\lambda} - |\mathcal{F}(g_I)(\lambda)|^2\Big)d\lambda\Bigg),\end{aligned}$$

where $\gamma(\tilde{S}+J,\Sigma)$ is the Nazarov constant.

Remark 3.5 Let $p \in \mathbb{Z}$ and $I = [-p-\frac{1}{2}, p+\frac{1}{2}]^n$. Then $J_p^n = 2I^\circ \cap \mathbb{Z}^n = \{j = (j_1, j_2, \cdots, j_n) \mid -2p \leq j_i \leq 2p, 1 \leq i \leq n\}$ is a representative set of $\mathbb{Z}^n/(2p+1)\mathbb{Z}^n$. For each $j \in J_p^n$ we put

$$f_j(m) = \begin{cases} f(m) & \text{if } m \in j + (2p+1)\mathbb{Z}^n \\ 0 & \text{otherwise} \end{cases}.$$

We note that $f = \sum_{j\in J_p^n} f_j$ and apply the above theorem to each f_j. Then summing up each inequality, we can deduce the following:

$$\begin{aligned}&\sum_{m\in\mathbb{Z}^n} |f(m)|^2\\ &\leq \gamma(\tilde{S}+J_p^n,\Sigma)\Bigg((2p+1)^n \|g_I\|^2_{L^2(\mathbb{R}^n)}\Big(\sum_{m\in\mathbb{Z}^n\setminus(S+J_p^n)} |f(m)|^2 + \int_{\mathbb{T}^n\setminus\Sigma} \sum_{j\in J_p^n} |\check{f}_j(\lambda)|^2 d\lambda\Big)\\ &\qquad + \int_\Sigma \sum_{j\in J_p^n} |\check{f}_j(\lambda)|^2\Big(\sum_{j\in J_p^n} \|g_I\|^2_{2,j} e^{2\pi i j\lambda} - |\mathcal{F}(g_I)(\lambda)|^2\Big)d\lambda\Bigg).\end{aligned}$$

4 Examples

We here give some examples of g_I and calculate explicitly the inequality in Theorem 3.4. Especially, the order of the weight function in the second integral is important for the localization of $\check{f}$.

(I) First we shall consider the case of $n = 1$. Let $I = [-\frac{1}{2}, \frac{1}{2}]$ and g_I be the characteristic function of I:

$$g_I(x) = \chi(x) = \begin{cases} 1 & |x| < \frac{1}{2} \\ 0 & |x| \geq \frac{1}{2}. \end{cases}$$

Then $J = \{0\}$ and $\|g_I\|_{2,0} = \|g_I\|^2_{L^2(\mathbb{R})} = 1$. Moreover, $\mathcal{F}(g_I)(\lambda) = \frac{\sin \pi\lambda}{\pi\lambda} = 1 - \frac{(\pi\lambda)^2}{6} + O(|\lambda|^4)$. Hence, it follows that

$$\|g_I\|^2_{2,0} - |\mathcal{F}(g_I)(\lambda)|^2 = \frac{(\pi\lambda)^2}{3} + O(|\lambda|^4).$$

When $n > 1$, let $I = [-\frac{1}{2}, \frac{1}{2}]^n$ and $g_I(x) = \chi(x_1)\chi(x_2)\cdots\chi(x_n)$. Then it is easy to see that

$$\|g_I\|^2_{2,0} - |\mathcal{F}(g_I)(\lambda)|^2 = 1 - \prod_{i=1}^n \Big|\frac{\sin \pi\lambda_i}{\pi\lambda_i}\Big|^2 = \frac{(\pi|\lambda|)^2}{3} + O(|\lambda|^4).$$

Remark 4.1 Let f be a function supported on $I_0 \subset I = [-\frac{1}{2}, \frac{1}{2}]^n$ and satisfy

$$\|f\|^2_{L^2(\mathbb{R}^n)} - |\mathcal{F}(f)(0)|^2 = 0. \tag{8}$$

Without loss of generality, we may suppose that $\|f\|_\infty \le 1$. Since

$$|\mathcal{F}(f)(0)| = \Big| \int_{I_0} f(x)dx \Big| \le \int_{I_0} |f(x)|dx \le |I_0|^{1/2} \|f\|_{L^2(\mathbb{R}^n)},$$

it follows that $I = I_0$ and $f(x) = 1$ for all $x \in I_0$. Therefore, functions f satisfying (8) are constant multiples of the characteristic function χ of I.

In the following we shall give some examples of g_I, whose support is larger than $[-\frac{1}{2}, \frac{1}{2}]^n$. We shall consider the case of $n = 1$, because, similarly as in (I), we can obtain a general case from their products.
(II) Let $n = 1$, $I = [-1, 1]$ and

$$g_I = \chi * \chi.$$

Then $J = \{-1, 0, 1\}$ and

$$\|g_I\|^2_{2,0} = \frac{2}{3}, \quad \|g_I\|^2_{2,\pm 1} = \frac{1}{6}.$$

Hence, it follows that

$$\sum_{j \in J} \|g_I\|^2_{2,j} e^{2\pi i j \lambda} = \frac{2}{3} + \frac{1}{3}\cos 2\pi\lambda = 1 - \frac{2}{3}(\pi\lambda)^2 + \frac{2}{9}(\pi\lambda)^4 + O(|\lambda|^6).$$

On the other hand,

$$\mathcal{F}(g_I)(\lambda)^2 = \Big(\frac{\sin \pi\lambda}{\pi\lambda}\Big)^4 = 1 - \frac{2}{3}(\pi\lambda)^2 + \frac{1}{5}(\pi\lambda)^4 + O(|\lambda|^6).$$

Hence, it follows that

$$\sum_{j \in J} \|g_I\|^2_{2,j} e^{2\pi i j \lambda} - |\mathcal{F}(g_I)(\lambda)|^2 = \frac{(\pi\lambda)^4}{45} + O(|\lambda|^6).$$

(III) Let $n = 1$, $I_p = [-\frac{1}{2} - p, \frac{1}{2} + p]$ and

$$g_{I_p} = \chi(x - p) + \chi(x + p) - 2\chi(x)$$

for $p = 1, 2, \cdots$. Then $J_p = \{-2p, -2p + 1, \cdots, 2p - 1, 2p\}$ and

$$\|g_{I_p}\|^2_{2,0} = 6,$$
$$\|g_{I_p}\|^2_{2,\pm j} = \begin{cases} -4 & j = p \\ 0 & j \ne p \end{cases} \qquad \text{if } j \equiv 1, 3 \pmod 4,$$
$$\|g_{I_p}\|^2_{2,\pm j} = \begin{cases} 1 & j = \frac{p}{2} \\ -4 & j = p \\ 0 & j \ne \frac{p}{2}, p \end{cases} \qquad \text{if } j \equiv 0, 2 \pmod 4.$$

Hence, it follows that

$$\sum_{j\in J_p} \|g_{I_p}\|_{2,j}^2 e^{2\pi ij\lambda} = 16(p\pi\lambda)^4 + \frac{32}{3}(p\pi\lambda)^6 + O(|\lambda|^8).$$

On the other hand,

$$\begin{aligned}\mathcal{F}(g_{I_p})(\lambda)^2 &= \Big(\frac{2e^{-i2p\pi\lambda}(-1+e^{i2p\pi\lambda})^2\sin\pi\lambda}{2\pi\lambda}\Big)^2\\ &= 16(p\pi\lambda)^4 + \frac{16}{3}(p^4+2p^6)(\pi\lambda)^6 + O(|\lambda|^8).\end{aligned}$$

Therefore, it follows that

$$\sum_{j\in J_p} \|g_{I_p}\|_{2,j}^2 e^{2\pi ij\lambda} - |\mathcal{F}(g_{I_p})(\lambda)|^2 = \frac{16p^4(\pi\lambda)^6}{3} + O(|\lambda|^8).$$

(IV) Let $n=1$, $I_p = [-p-\frac{1}{2}, p+\frac{1}{2}]$ and

$$g_{I_{p,q}} = (\chi(x-p)+\chi(x+p)-2\chi(x)) - \frac{p^2}{q^2}(\chi(x-q)+\chi(x+q)-2\chi(x)),$$

where $p>4q$, $p+q$ is even, $q\ge 2$. Then $J_p = \{j \mid -2p\le j\le 2p\}$ and

$$\begin{aligned}&\|g_{I_{p,q}}\|_{2,0}^2 = 2+\tfrac{2p^4}{q^4}+4\Big(\tfrac{p^2}{q^2}-1\Big)^2, && \|g_{I_{p,q}}\|_{2,\pm q}^2 = -\tfrac{4p^2}{q^2}\Big(\tfrac{p^2}{q^2}-1\Big),\\ &\|g_{I_{p,q}}\|_{2,\pm 2q}^2 = \tfrac{p^4}{q^4}, &&\\ &\|g_{I_{p,q}}\|_{2,\pm(p-q)}^2 = -\tfrac{2p^2}{q^2}, && \|g_{I_{p,q}}\|_{2,\pm p}^2 = 4\Big(\tfrac{p^2}{q^2}-1\Big),\\ &\|g_{I_{p,q}}\|_{2,\pm(p+q)}^2 = -\tfrac{2p^2}{q^2}, && \|g_{I_{p,q}}\|_{2,\pm 2p}^2 = 1\end{aligned}$$

and $\|g_{I_{p,q}}\|_{2,\pm j}^2 = 0$ otherwise. Hence, it follows that

$$\begin{aligned}&\sum_{j\in J} \|g_{I_{p,q}}\|_{2,j}^2 e^{2\pi ij\lambda}\\ =&\frac{16}{9}p^4(p^2-q^2)^2(\pi\lambda)^8 - \frac{64}{135}p^4(p^2-q^2)^2(p^2+q^2)(\pi\lambda)^{10} + O(|\lambda|^{12}).\end{aligned}$$

On the other hand,

$$\begin{aligned}&\mathcal{F}(g_{I_{p,q}})(\lambda)^2 = \Big(e^{-2\pi(p+q)i\lambda}\frac{\sin\pi\lambda}{2\pi q^2\lambda}\Big)^2\\ &\times(-q^2e^{2\pi qi\lambda} - q^2e^{2\pi(2p+q)i\lambda} + p^2e^{2\pi pi\lambda} + p^2e^{2\pi(2q+p)i\lambda} - 2(p^2-q^2)e^{2\pi(p+q)i\lambda})^2\\ &= \frac{16}{9}p^4(p^2-q^2)^2(\pi\lambda)^8 - \frac{16}{135}p^4(p^2-q^2)^2(5+4p^2+4q^2)(\pi\lambda)^{10} + O(|\lambda|^{12}).\end{aligned}$$

Therefore, it follows that

$$\sum_{j\in J} \|g_{I_{p,q}}\|_{2,j}^2 e^{2\pi ij\lambda} - |\mathcal{F}(g_{I_{p,q}})(\lambda)|^2 = \frac{16}{27}p^4(p^2-q^2)(\pi\lambda)^{10} + O(|\lambda|^{12}).$$

At this stage, this $|\lambda|^{10}$ is our best weight function. When $n > 1$, as said before, let $I = [-p - \frac{1}{2}, p + \frac{1}{2}]^n$ and $g_I(x) = g_{I_{p,q}}(x_1)g_{I_{p,q}}(x_2)\cdots g_{I_{p,q}}(x_n)$. Then $J = J_p^n = \{j = (j_1, j_2, \cdots, j_n) \in \mathbb{Z}^n \mid -2p \leq j_i \leq 2p, 1 \leq i \leq n\}$ and it is easy to see that $\|g_I\|_{2,j}^2 = \prod_{i=1}^n \|g_{I_{p,q}}\|_{2,j_i}$ and

$$\sum_{j\in J_p^n} \|g_I\|_{2,j}^2 e^{2\pi i j\lambda} = \prod_{i=1}^n (\sum_{j_i\in J_p} \|g_{I_{p,q}}\|_{2,j_i} e^{2\pi i j_i\lambda_i}).$$

Since $\mathcal{F}(g_I)(\lambda) = \mathcal{F}(g_{I_{p,q}})(\lambda_1)\mathcal{F}(g_{I_{p,q}})(\lambda_2)\cdots\mathcal{F}(g_{I_{p,q}})(\lambda_n)$ and $\|\mathcal{F}(g_{I_{p,q}})\|_\infty \leq \|\mathcal{F}(g_{I_{p,q}})\|_1 = \frac{4p^2}{q^2}$, it follows that

$$\begin{aligned}&\Big|\sum_{j\in J_p^n} \|g_I\|_{2,j}^2 e^{2\pi i j\lambda} - |\mathcal{F}(g_I)(\lambda)|^2\Big| \\ \leq&\Big(\frac{4p^2}{q^2}\Big)^{n-1}\frac{16}{27}p^4(p^2-q^2)(|\lambda_1|^{10}+\cdots+|\lambda_n|^{10}) + O(|\lambda|^{12}).\end{aligned}$$

Corollary 4.2 *Let S be a finite subset in $\mathbb{Z}^n$ and Σ a measurable subset in $[-\frac{1}{2}, \frac{1}{2}]^n$. For $p > 4q$, $p+q$ even and $q \geq 2$, let $J_p^n = \{j = (j_1, j_2, \cdots, j_n) \in \mathbb{Z}^n \mid -2p \leq j_i \leq 2p, 1 \leq i \leq n\}$ and $c_{p,q,i} = \|g_{I_{p,q}}\|_{2,i}^2$ listed above. Then there exists a constant $c, c_{p,q}$ such that for all $f \in l^1(\mathbb{Z}^n) \cap l^2(\mathbb{Z}^n)$,*

$$\begin{aligned}&\sum_{j\in J_p^n}\sum_{m\in\mathbb{Z}^n} c_j^{p,q} f(m)\overline{f(m+j)} \\ \leq& c\gamma(\tilde{S}+J_p^n, \Sigma)\Big(\sum_{m\in\mathbb{Z}^n\setminus(S+J_p^n)} |f(m)|^2 + \int_{\mathbb{T}^n\setminus\Sigma} |\check{f}(\lambda)|^2 d\lambda + c_{p,q}\int_\Sigma |\check{f}(\lambda)|^2|\lambda|^{10} d\lambda\Big),\end{aligned}$$

where $c_j^{p,q} = \prod_{i=1}^n c_{p,q,j_i}$ and $\gamma(\tilde{S}+J_p^n, \Sigma)$ is the Nazarov constant.

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