CONVERGENCE THEOREMS FOR THE HENSTOCK-KURZWEIL INTEGRAL TAKING VALUES IN A VECTOR LATTICE

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Abstract. In previous papers we defined a Denjoy integral and a Henstock-Kurzweil integral of mappings from a vector lattice into a complete vector lattice. In this paper we consider some convergence theorems for the Henstock-Kurzweil integral of mappings from a vector lattice with unit satisfying the principal projection property, in particular the real line, into a complete vector lattice.

1 Introduction The purpose of our researches is to consider some derivatives and some integrals of mappings in vector spaces and to study their relations, for instance, the fundamental theorem of calculus, inclusive relations between integrals and so on. To this end we consider some convergence theorems for these integrals.

In previous papers [14,15] we defined a Denjoy integral and a Henstock-Kurzweil integral of mappings from a vector lattice into a complete vector lattice. In this paper we consider some convergence theorems for the Henstock-Kurzweil integral of mappings from a vector lattice with unit satisfying the principal projection property, in particular the real line, into a complete vector lattice.

2 Preliminaries In this section we recall some notation and definitions in [14, 15] that will be used in this paper.

An element e of a vector lattice X is said to be a unit if $e \wedge x > 0$ for any $x \in X$ with $x > 0$. Let $\mathcal{K}_X$ be the class of units of X. Let $\mathcal{I}_X$ be the class of intervals of X and $\mathcal{IK}_X$ the class of intervals $[a, b]$ with $b - a \in \mathcal{K}_X$. Two elements $x_1, x_2 \in X$ are said to be orthogonal, denoted by $x_1 \perp x_2$, if $|x_1| \wedge |x_2| = 0$. Let $A^\perp$ be the class of $x_1 \in X$ satisfying $x_1 \perp x$ for any $x \in A \subset X$. Let $\mathcal{L}(X, Y)$ be the class of bounded linear mappings from X into a vector lattice Y. If Y is complete, then $\mathcal{L}(X, Y)$ is also so [1, 4, 18, 22, 23]. A subset D of a vector lattice X with unit is said to be open if for any $x \in D$ and for any $e \in \mathcal{K}_X$ there exists $\varepsilon \in \mathcal{K}_{\mathbf{R}}$ such that $[x - \varepsilon e, x + \varepsilon e] \subset D$. Let $\mathcal{O}_X$ be the class of open subsets of X. Let $\mathcal{U}_Y^s(\mathcal{K}_X, \geq)$ be the class of $\{v_e \mid e \in \mathcal{K}_X\}$ satisfying the following conditions:

(U1) $v_e \in Y$ with $v_e > 0$;

(U2)^d $v_{e_1} \geq v_{e_2}$ if $e_1 \geq e_2$;

(U3)^s For any $e \in \mathcal{K}_X$ there exists $\theta(e) \in \mathcal{K}_{\mathbf{R}}$ such that $v_{\theta(e)e} \leq \frac{1}{2} v_e$.

Let $|\mathcal{K}_X|$ be the class of x satisfying $|x| \in \mathcal{K}_X$. For any $x \in |\mathcal{K}_X|$ let $x_+^\perp = \{0 \vee x\}^\perp$, $x_-^\perp = \{0 \vee (-x)\}^\perp$,

$$Q(x) = \{x_1 \mid x_1 \in |\mathcal{K}_X|,\ (x_1)_+^\perp = x_+^\perp,\ (x_1)_-^\perp = x_-^\perp\}$$

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and

$$\overline{Q}(x)=\left(\bigcup_{x_1,x_2\in Q(x)}[0\wedge x_1,0\vee x_2]\right)\setminus\{0\}.$$

For $e\in\mathcal{K}_X,\ a,b\in D\subset X$ with $a\neq b$ let $\mathbf{CSIP}_e(a,b)$ be the class of mappings φ from $[0,1]$ into D satisfying the following conditions (P), (CP$_e$) and (SI), $\mathbf{CSDP}_e(a,b)$ the class of mappings φ form $[0,1]$ into D satisfying the following conditions (P), (CP$_e$) and (SD), and $\mathbf{CSMP}_e(a,b)=\mathbf{CSIP}_e(a,b)\cup\mathbf{CSDP}_e(a,b)$:

(P) $\varphi(0)=a$ and $\varphi(1)=b$;

(CP$_e$) for any $t\in[0,1]$ and for any $\varepsilon\in\mathcal{K}_{\mathbf{R}}$ there exists $\delta\in\mathcal{K}_{\mathbf{R}}$ such that for any $s\in[0,1]$ if $|s-t|\leq\delta$, then $|\varphi(s)-\varphi(t)|\leq\varepsilon e$;

(SI) $\varphi(t_1)<\varphi(t_2)$ if $t_1<t_2$;

(SD) $\varphi(t_1)>\varphi(t_2)$ if $t_1<t_2$.

Let $\mathbf{CSSMP}(a,b)$ be the class of mappings φ from $[0,1]$ into D satisfying the following conditions:

(CS1) there exist a natural number r_φ and $\{e^i_\varphi\mid e^i_\varphi\in\mathcal{K}_X$ for $i=1,\ldots,r_\varphi\}$ such that the following mapping

$$\begin{array}{cccc}\varphi^i: & [0,1] & \longrightarrow & D\\ & \Downarrow & & \Downarrow\\ & s & \longmapsto & \varphi(\frac{s+i-1}{r_\varphi})\end{array}$$

belongs to $\mathbf{CSMP}_{e^i_\varphi}(\varphi(\frac{i-1}{r_\varphi}),\varphi(\frac{i}{r_\varphi}))$;

(CS2) there exists $x\in|\mathcal{K}_X|$ such that $\varphi^i(1)-\varphi^i(0)\in\overline{Q}(x)$ for any $i=1,\ldots,r_\varphi$;

(CS3) $\varphi([0,1])\subset[a\wedge b,a\vee b]$.

Note that φ^i satisfies either (SI) or (SD). For convenience, φ^i is said to be **CSIP** if φ^i satiefies (SI) and φ^i is **CSDP** if φ^i satisfies (SD), respectively. A subset $D\subset X$ is said to be connected if $\mathbf{CSSMP}(a,b)\neq\emptyset$ for any $a,b\in D$ with $a\neq b$. Let $\mathcal{CO}_X$ be the class of connected open subsets of X. For $a,b\in D\in\mathcal{CO}_X$ the following subset

$$\langle a|b\rangle=\begin{cases}\bigcup_{\varphi\in\mathbf{CSSMP}(a,b)}\varphi([0,1]) & \text{if } a\neq b,\\ \{a\} & \text{if } a=b\end{cases}$$

is said to be a stepwise interval from a to b. A mapping δ form $D\times\mathcal{K}_X$ into $(0,\infty)$ is said to be a gauge. For $\xi\in D\subset X$ and δ a gauge the following subset

$$O_D(\xi,\delta)=\left(\bigcup_{e\in\mathcal{K}_X}[\xi-\delta(\xi,e)e,\xi+\delta(\xi,e)e]^e\right)\cap D$$

is said to be a δ-neighborhood of ξ in D, where

$$[a,b]^e=\{x\mid \text{ there exists }\varepsilon\in\mathcal{K}_{\mathbf{R}}\text{ such that }x-a\geq\varepsilon e\text{ and }b-x\geq\varepsilon e\}.$$

When $D = X$, it is denoted by $O(\xi, \delta)$ simply. If $a \neq b$, then the following set

$$\left\{ (\langle x_{k-1} | x_k \rangle, \xi_k) \;\middle|\; \begin{array}{l} x_k \in \langle a|b \rangle \ (k = 0, \ldots, K), x_0 = a, x_K = b, \\ \xi_k \in D \ (k = 1, \ldots, K) \end{array} \right\}$$

is said to be a δ-fine division of $\langle a|b \rangle$ if it satisfies the following (NOL) and (DF), and it is said to be a δ-fine McShane division of $\langle a|b \rangle$ if it satisfies the following (NOL) and (DFMS), respectively:

(NOL) there exists $x \in |\mathcal{K}_X|$ such that $x_k - x_{k-1} \in \overline{Q}(x)$ for any $k = 1, \ldots, K$;

(DF) $\xi_k \in \langle x_{k-1} | x_k \rangle \subset O_D(\xi_k, \delta)$ for any $k = 1, \ldots, K$;

(DFMS) $\langle x_{k-1} | x_k \rangle \subset O_D(\xi_k, \delta)$ for any $k = 1, \ldots, K$.

If $a = b$, then $\{(\langle a|a \rangle, \xi)\}$ is said to be a δ-fine division of $\langle a|b \rangle$ if it satisfies (DF), and a δ-fine McShane division of $\langle a|b \rangle$ if it satisfies (DFMS), respectively. A mapping f is said to be Henstock-Kurzweil integrable on $\langle a|b \rangle$ if there exists $I(f; a, b) \in Y$ and $\{v_e\} \in \mathcal{U}_Y^s(\mathcal{K}_X, \geq)$ such that for any $e \in \mathcal{K}_X$ there exists a gauge δ such that for any δ-fine division $\{(\langle x_{k-1} | x_k \rangle, \xi_k) \mid k = 1, \ldots, K\}$ of $\langle a|b \rangle$

$$\left| \sum_{k=1}^{K} f(\xi_k)(x_k - x_{k-1}) - I(f; a, b) \right| \leq v_e.$$

$I(f; a, b)$ is said to be a Henstock-Kurzweil integral of f on $\langle a|b \rangle$, denoted by

$$I(f; a, b) = o\text{-}(HK) \int_a^b f(x) dx.$$

If for any $a, b \in D$ a mapping f is Henstock-Kurzweil integrable on $\langle a|b \rangle$, then it is said to be Henstock-Kurzweil integrable on D. Let $\mathbf{(HK)}(\langle a|b \rangle, Y)$ and $\mathbf{(HK)}(D, Y)$ be the class of Henstock-Kurzweil integrable mappings on $\langle a|b \rangle$ and D, respectively. The following mapping

$$\begin{array}{rccccc} F_a : & D & \longrightarrow & Y & & \\ & \Psi & & \Psi & & \\ & x & \longmapsto & I(f; a, x) & = & o\text{-}(HK) \int_a^x f(x) dx \end{array}$$

is said to be a Henstock-Kurzweil primitive of f. For $\mathcal{L}(X, Y)$ we consider the following condition:

(CB) there exists $\{l_n \mid n = 1, 2, \ldots\} \subset \mathcal{L}(X, Y)$ satisfying the following conditions:

- (CB1) $l_{n_1} \leq l_{n_2}$ if $n_1 < n_2$;
- (CB2) for any $l \in \mathcal{L}(X, Y)$ there exists a natural number n such that $|l| \leq l_n$;
- (CB3) there exists $\{\varepsilon_n\} \subset \mathcal{K}_{\mathbf{R}}$ such that $\sum_{n=1}^{\infty} \varepsilon_n l_n \in \mathcal{L}(X, Y)$.

3 Convergence theorems In this section we consider some convergence theorems. Since there is some difficulty, we often consider the case of $X = \mathbf{R}$ endowed with the Lebesgue measure. See [14, Definition 3.8] for the concept of "null sets" in a vector lattice.

Lemma 3.1. *Let X be a vector lattice with unit satisfying the principal projection property, Y a complete vector lattice, $a, b \in D \in \mathcal{CO}_X$ with $a < b$ and $f \in \mathbf{(HK)}(\langle a|b\rangle, Y)$.*

If $f(x) \geq 0$ for any $x \in \langle a|b\rangle$, then

$$o\text{-}(HK)\int_a^b f(x)dx \geq 0.$$

Proof. Since $f \in \mathbf{(HK)}(\langle a|b\rangle, Y)$, there exists $\{v_e\} \in \mathcal{U}^s_{\mathcal{L}(X,Y)}(\mathcal{K}_X, \geq)$ such that for any $e \in \mathcal{K}_X$ there exists a gauge δ such that for any δ-fine division $\{(\langle x_{k-1}|x_k\rangle, \xi_k) \mid k = 1, \ldots, K\}$ of $\langle a|b\rangle$

$$\left|\sum_{k=1}^{K} f(\xi_k)(x_k - x_{k-1}) - o\text{-}(HK)\int_a^b f(x)dx\right| \leq v_e.$$

Since $x_{k-1} < x_k$ by [15, Remark 2.3] and $f(x) \geq 0$, it holds that

$$\sum_{k=1}^{K} f(\xi_k)(x_k - x_{k-1}) \geq 0.$$

Therefore

$$o\text{-}(HK)\int_a^b f(x)dx \geq \sum_{k=1}^{K} f(\xi_k)(x_k - x_{k-1}) - v_e \geq -v_e.$$

Since e is arbitrary, it holds that

$$o\text{-}(HK)\int_a^b f(x)dx \geq 0.$$

$\square$

Lemma 3.2. *Let $X = \mathbf{R}$, Y a complete vector lattice, $a, b \in D \in \mathcal{CO}_X$ with $a < b$ and $f \in \mathbf{(HK)}(\langle a|b\rangle, Y)$. Suppose that $\mathcal{L}(X, Y)$ satisfies* (CB)*.*

If $f(x) \geq 0$ for almost every $x \in \langle a|b\rangle$, then

$$o\text{-}(HK)\int_a^b f(x)dx \geq 0.$$

Proof. Let $N = \{x \mid f(x) \not\geq 0\}$, χ_N the characteristic function of N and $g(x) = \chi_N(x)|f(x)|$. Since $g(x) = 0$ for almost every $x \in D$, by [15, Lemma 4.1] it holds that $g \in \mathbf{(HK)}(\langle a|b\rangle, Y)$ and

$$o\text{-}(HK)\int_a^b g(x)dx = 0.$$

Since $f(x) + g(x) \geq 0$ for any $x \in D$, by Lemma 3.1 and [15, Theorem 2.2] it holds that

$$\begin{aligned} o\text{-}(HK)\int_a^b f(x)dx &= o\text{-}(HK)\int_a^b f(x)dx + o\text{-}(HK)\int g(x)dx \\ &= o\text{-}(HK)\int_a^b (f(x) + g(x))dx \geq 0. \end{aligned}$$

$\square$

Definition 3.1. Let X and Y be vector lattices, $D \subset X$, Λ an upward directed set, $\mathcal{U}_Y(\Lambda)$ the class of $\{v_\lambda \mid \lambda \in \Lambda\}$ satisfying the following conditions:

(U1) $v_\lambda \in Y$ with $v_\lambda > 0$;

(U2)^u $v_{\lambda_1} \geq v_{\lambda_2}$ if $\lambda_1 \leq \lambda_2$;

(U3) $\bigwedge_{\lambda\in\Lambda} v_\lambda = 0$,

and $\{f_\lambda \mid f_\lambda$ is a mapping from D into $Y, \lambda \in \Lambda\}$ a net of mappings and f a mapping from D into Y.

The net $\{f_\lambda\}$ is said to be pointwise convergent to f if for any $x \in D$ there exists $\{v_{x,\lambda}\} \in \mathcal{U}_Y(\Lambda)$ such that $|f_\lambda(x) - f(x)| \leq v_{x,\lambda}$ for any $\lambda \in \Lambda$. The net $\{f_\lambda\}$ is said to be almost pointwise convergent to f if for almost every $x \in D$ there exists $\{v_{x,\lambda}\} \in \mathcal{U}_Y(\Lambda)$ such that $|f_\lambda(x) - f(x)| \leq v_{x,\lambda}$ for any $\lambda \in \Lambda$. The net $\{f_\lambda\}$ is said to be uniformly pointwise convergent on $A \subset D$ if there exists $\{v_\lambda\} \in \mathcal{U}_Y(\Lambda)$ such that for any $x \in A$ there exists $\{v_{x,\lambda}\} \in \mathcal{U}_Y(\Lambda)$ such that for any $\lambda \in \Lambda$ there exists $\rho(x,\lambda) \in \Lambda$ such that $|f_{\rho(x,\lambda)}(x) - f(x)| \leq v_{x,\rho(x,\lambda)} \leq v_\lambda$. The net $\{f_\lambda\}$ is said to be uniformly almost pointwise convergent on $A \subset D$ if there exists $\{v_\lambda\} \in \mathcal{U}_Y(\Lambda)$ such that for almost every $x \in A$ there exists $\{v_{x,\lambda}\} \in \mathcal{U}_Y(\Lambda)$ such that for any $\lambda \in \Lambda$ there exists $\rho(x,\lambda) \in \Lambda$ such that $|f_{\rho(x,\lambda)}(x) - f(x)| \leq v_{x,\rho(x,\lambda)} \leq v_\lambda$.

For $\mathcal{U}^s_Y(\mathcal{K}_X, \geq)$ we consider the following conditions:

(U4) for any $\{v_{n,e}\} \in \mathcal{U}^s_Y(\mathcal{K}_X, \geq)$ with $n \in \mathbf{N}$ there exists $\{\eta_n\} \subset \mathcal{K}_{\mathbf{R}}$ such that $\sum_{n=1}^\infty \eta_n v_{n,e} \in \mathcal{U}^s_Y(\mathcal{K}_X, \geq)$;

(U5) for any $\{v_\lambda\} \in \mathcal{U}_Y(\Lambda)$ there exists $\{v_e\} \in \mathcal{U}^s_Y(\mathcal{K}_X, \geq)$ such that for any $e \in \mathcal{K}_X$ there exists $\lambda(e) \in \Lambda$ such that $v_{\lambda(e)} \leq v_e$.

Theorem 3.1. *Let X be a vector lattice with unit satisfying the principal projection property, Y a complete vector lattice and $a, b \in D \in \mathcal{CO}_X$ with $a < b$. Suppose that $\mathcal{U}^s_Y(\mathcal{K}_X, \geq)$ satisfies* (U4) *and* (U5) *and $\mathcal{U}^s_{\mathcal{L}(X,Y)}(\mathcal{K}_X, \geq)$ satisfies* (U5).

*If $f_n \in$ **(HK)**$(\langle a|b\rangle, Y)$, $\{f_n\}$ is monotone increasing and uniformly pointwise convergent to f on $\langle a|b\rangle$ and there exists $I \in Y$ such that*

$$o\text{-}(HK)\int_a^b f_n(x)dx \uparrow I,$$

*then $f \in$ **(HK)**$(\langle a|b\rangle, Y)$ and*

$$o\text{-}(HK)\int_a^b f(x)dx = I.$$

Proof. Let F_n be the primitive of f_n. Since $\mathcal{U}^s_Y(\mathcal{K}_X, \geq)$ satisfies (U5), there exists $\{v_e\} \in \mathcal{U}^s_Y(\mathcal{K}_X, \geq)$ such that for any $e \in \mathcal{K}_X$ there exists a natural number $n_1(e)$ such that

$$0 \leq I - (F_{n_1(e)}(b) - F_{n_1(e)}(a)) \leq v_e.$$

Since $\{f_n\}$ is uniformly convergent to f on $\langle a|b\rangle$ and $\mathcal{U}^s_{\mathcal{L}(X,Y)}(\mathcal{K}_X, \geq)$ satisfies (U5), there exists $\{w_e\} \in \mathcal{U}^s_{\mathcal{L}(X,Y)}(\mathcal{K}_X, \geq)$ such that for any $e \in \mathcal{K}_X$ there exists a natural number $n_2(e)$ such that for any $\xi \in \langle a|b\rangle$ there exists a natural number $\rho(\xi, n_2(e))$ such that

$$0 \leq f(\xi) - f_{\rho(\xi,n_2(e))}(\xi) \leq w_e.$$

Without loss of generality it may be assumed that $n_1(e) \le \rho(\xi, n_2(e))$ for any $e \in \mathcal{K}_X$ and for any $\xi \in \langle a|b\rangle$. By [15, Theorem 3.1] for any natural number n there exists $\{v_{n,e}\} \in \mathcal{U}_Y^s(\mathcal{K}_X, \ge)$ such that for any $e \in \mathcal{K}_X$ there exists a gauge δ_n such that for any δ_n-fine division $\{(\langle x_{k-1}|x_k\rangle, \xi_k) \mid k = 1, \dots, K\}$

$$\sum_{k=1}^{K} |f_n(\xi_k)(x_k - x_{k-1}) - (F_n(x_k) - F_n(x_{k-1}))| \le 4v_{n,e}.$$

Since $\mathcal{U}_Y^s(\mathcal{K}_X, \ge)$ satisfies (U4), there exists $\{\eta_n\} \subset \mathcal{K}_{\mathbf{R}}$ such that $\sum_{n=1}^{\infty} \eta_n v_{n,e} \in \mathcal{U}_Y^s(\mathcal{K}_X, \ge)$. Let $p(n)$ be a natural number with $2^{-p(n)} \le \eta_n$. Denote δ_n corresponding $\theta(e, p(n))e$ instead of e by the same symbol again, where $\theta(e, n) = \underbrace{\theta(\theta(\cdots\theta(\theta}_{n}(e)e)\cdots e)e)$. Then

$$\begin{aligned} \sum_{k=1}^{K} |f_n(\xi_k)(x_k - x_{k-1}) - (F_n(x_k) - F_n(x_{k-1}))| &\le 4v_{n,\theta(e,p(n))e} \\ &\le 4 \cdot 2^{-p(n)} v_{n,e} \le 4\eta_n v_{n,e}. \end{aligned}$$

Since $\{F_n(x_k) - F_n(x_{k-1}) \mid n = 1, 2, \dots\}$ is monotone increasing for any k, it holds that

$$\begin{aligned} 0 &\le I - \sum_{k=1}^{K} (F_{\rho(\xi_k, n_2(e))}(x_k) - F_{\rho(\xi_k, n_2(e))}(x_{k-1})) \\ &\le I - \sum_{k=1}^{K} (F_{n_1(e)}(x_k) - F_{n_1(e)}(x_{k-1})) \\ &= I - (F_{n_1(e)}(b) - F_{n_1(e)}(a)) \le v_e. \end{aligned}$$

Let $\delta(\xi, e) = \delta_{\rho(\xi, n_2(e))}(\xi, e)$. Then for any δ-fine division $\{(\langle x_{k-1}|x_k\rangle, \xi_k) \mid k = 1, \dots, K\}$ since without loss of generality it may be assumed that there exists a natural number n such that the δ-fine division is also δ_n-fine, it holds that

$$\begin{aligned} &\left| \sum_{k=1}^{K} f(\xi_k)(x_k - x_{k-1}) - I \right| \\ &\quad \le \sum_{k=1}^{K} |f(\xi_k)(x_k - x_{k-1}) - f_{\rho(\xi_k, n_2(e))}(\xi_k)(x_k - x_{k-1})| \\ &\qquad + \sum_{k=1}^{K} |f_{\rho(\xi_k, n_2(e))}(\xi_k)(x_k - x_{k-1}) \\ &\qquad\qquad - (F_{\rho(\xi_k, n_2(e))}(x_k) - F_{\rho(\xi_k, n_2(e))}(x_{k-1}))| \\ &\qquad + \left| \sum_{k=1}^{K} (F_{\rho(\xi_k, n_2(e))}(x_k) - F_{\rho(\xi_k, n_2(e))}(x_{k-1})) - I \right| \\ &\quad \le \sum_{k=1}^{K} |f(\xi_k) - f_{\rho(\xi_k, n_2(e))}(\xi_k)|(x_k - x_{k-1}) \\ &\qquad + \sum_{n=1}^{\infty} 4\eta_n v_{n,e} + \left| F_{n_1(e)}(b) - F_{n_1(e)}(a) - I \right| \end{aligned}$$

$$\leq w_e(b-a) + 4\sum_{n=1}^{\infty} \eta_n v_{n,e} + v_e.$$

By [14, Remark 2.1] $f \in \mathbf{(HK)}(\langle a|b\rangle, Y)$ and

$$o\text{-}(HK)\int_a^b f(x)dx = I.$$

$\square$

Theorem 3.2. *Let $X = \mathbf{R}$, Y a complete vector lattice and $a, b \in D \in \mathcal{CO}_X$ with $a < b$. Suppose that $\mathcal{L}(X,Y)$ satisfies* (CB) *and $\mathcal{U}^s_{\mathcal{L}(X,Y)}(\mathcal{K}_X, \geq) \cong \mathcal{U}^s_Y(\mathcal{K}_X, \geq)$ satisfies* (U4) *and* (U5).

If $f_n \in \mathbf{(HK)}(\langle a|b\rangle, Y)$, $\{f_n\}$ is monotone increasing for almost every $x \in \langle a|b\rangle$ and uniformly almost pointwise convergent to f on $\langle a|b\rangle$ and there exists $I \in Y$ such that

$$o\text{-}(HK)\int_a^b f_n(x)dx \uparrow I,$$

then $f \in \mathbf{(HK)}(\langle a|b\rangle, Y)$ and

$$o\text{-}(HK)\int_a^b f(x)dx = I.$$

Proof. Since by [15, Lemma 4.1] without loss of generality it may be assumed $f_n(x) \uparrow f(x)$ for every $x \in \langle a|b\rangle$, it is clear by Theorem 3.1. $\square$

Lemma 3.3. *Let $X = \mathbf{R}$, Y a complete vector lattice and $a, b \in D \in \mathcal{CO}_X$. Suppose that $\mathcal{U}^s_Y(\mathcal{K}_X, \geq)$ satisfies* (U5).

If $f_1, f_2, g, h \in \mathbf{(HK)}(\langle a|b\rangle, Y)$ satisfy that $g(x) \leq f_1(x) \leq h(x)$ and $g(x) \leq f_2(x) \leq h(x)$ for any $x \in \langle a|b\rangle$, then $f_1 \vee f_2, f_1 \wedge f_2 \in \mathbf{(HK)}(\langle a|b\rangle, Y)$.

Proof. Since $f_1 \wedge f_2 = -((-f_1) \vee (-f_2))$, we show only $f_1 \vee f_2 \in \mathbf{(HK)}\langle a|b\rangle, Y)$. Since X satisfies the principal projection property, we consider in the case of $a < b$.

Let F_1 and F_2 be primitives of f_1 and f_2 on $\langle a|b\rangle$, respectively. For $u, v \in \langle a|b\rangle$ let $F^*(u,v) = (F_1(v) - F_1(u)) \vee (F_2(v) - F_2(u))$. Note that $(a+b) \vee (c+d) \leq a \vee c + b \vee d$ for any a, b, c, d in a vector lattice. Then $F^*(u,w) \leq F^*(u,v) + F^*(v,w)$ for any $u, v, w \in \langle a|b\rangle$. Let $\Delta : a = x_0 < \cdots < x_K = b, x_k \in \langle a|b\rangle$. Since by Lemma 3.1 and [15, Theorem 2.3]

$$\begin{aligned}\sum_{k=1}^{K} F^*(x_{k-1}, x_k) &\leq \sum_{k=1}^{K} o\text{-}(HK)\int_{x_{k-1}}^{x_k} h(x)dx \\ &= o\text{-}(HK)\int_a^b h(x)dx\end{aligned}$$

and Y is complete, there exists

$$I = \bigvee_{\Delta} \sum_{k=1}^{K} F^*(x_{k-1}, x_k).$$

Let Λ be the class of divisions of $\langle a|b\rangle$. It is an upward directed set with respect to inclusive relation of division points. Then there exists $\{v_\Delta\} \in \mathcal{U}_Y(\Lambda)$ such that for any $\Delta \in \Lambda$

$$0 \leq I - \sum_{k=1}^{K} F^*(x_{k-1}, x_k) \leq v_\Delta.$$

Since $\mathcal{U}_Y^s(\mathcal{K}_X, \geq)$ satisfies (U5), there exists $\{v_{0,e}\} \in \mathcal{U}_Y^s(\mathcal{K}_X, \geq)$ such that for any $e \in \mathcal{K}_X$ there exists a gauge δ_1 such that $v_\Delta \leq v_{0,e}$ for any δ_0-fine division $\{(\langle x_{k-1}|x_k\rangle, \xi_k) \mid k = 1, \ldots, K\}$ and for $\Delta : a = x_0 < \cdots < x_K = b$. Moreover by [15, Theorem 3.1] there exists $\{v_{1,e}\} \in \mathcal{U}_Y^s(\mathcal{K}_X, \geq)$ such that for any $e \in \mathcal{K}_X$ there exists a gauge δ_1 such that for any δ_1-fine division $\{(\langle x_{k-1}|x_k\rangle, \xi_k) \mid k = 1, \ldots, K\}$

$$\sum_{k=1}^{K} |f_1(\xi_k)(x_k - x_{k-1}) - (F_1(x_k) - F_1(x_{k-1}))| \quad \leq \quad 4v_{1,e}$$

and there exists $\{v_{2,e}\} \in \mathcal{U}_Y^s(\mathcal{K}_X, \geq)$ such that for any $e \in \mathcal{K}_X$ there exists a gauge δ_2 such that for any δ_2-fine division $\{(\langle x_{k-1}|x_k\rangle, \xi_k) \mid k = 1, \ldots, K\}$

$$\sum_{k=1}^{K} |f_2(\xi_k)(x_k - x_{k-1}) - (F_2(x_k) - F_2(x_{k-1}))| \quad \leq \quad 4v_{2,e}.$$

Let $\delta(\cdot, \cdot) = \delta_0(\cdot, \cdot) \wedge \delta_1(\cdot, \cdot) \wedge \delta_2(\cdot, \cdot)$. Then the above three inequalities are satisfied for any δ-fine division. Since $X = \mathbf{R}$, it holds that $(l_1 \vee l_2)(x) = l_1(x) \vee l_2(x)$ for any $l_1, l_2 \in \mathcal{L}(X, Y)$ and for any $x \in X$ with $x > 0$. Note that $|a \vee b - c \vee d| \leq |a - c| + |b - d|$ for any a, b, c, d in a vector lattice. Then

$$\begin{aligned}
&\left|\sum_{k=1}^{K}((f_1 \vee f_2)(\xi_k)(x_k - x_{k-1}) - F^*(x_{k-1}, x_k))\right| \\
&\quad \leq \sum_{k=1}^{K} |f_1(\xi_k)(x_k - x_{k-1}) \vee f_2(\xi_k)(x_k - x_{k-1}) - F^*(x_{k-1}, x_k)| \\
&\quad \leq \sum_{k=1}^{K}(|f_1(\xi_k)(x_k - x_{k-1}) - (F_1(x_k) - F_1(x_{k-1}))| \\
&\qquad\quad + |f_2(\xi_k)(x_k - x_{k-1}) - (F_2(x_k) - F_2(x_{k-1}))|) \\
&\quad \leq 4v_{1,e} + 4v_{2,e}.
\end{aligned}$$

Therefore

$$\begin{aligned}
&\left|\sum_{k=1}^{K}(f_1 \vee f_2)(\xi_k)(x_k - x_{k-1}) - I\right| \\
&\quad \leq \left|\sum_{k=1}^{K}((f_1 \vee f_2)(\xi_k)(x_k - x_{k-1}) - F^*(x_{k-1}, x_k))\right| \\
&\qquad + \left|\sum_{k=1}^{K} F^*(x_{k-1}, x_k) - I\right| \\
&\quad \leq 4v_{1,e} + 4v_{2,e} + v_{0,e}
\end{aligned}$$

and hence $f_1 \vee f_2 \in \mathbf{(HK)}(\langle a|b\rangle, Y)$ and its integral is equal to I. □

Lemma 3.4. *Let $X = \mathbf{R}$, Y a complete vector lattice and $a, b \in D \in \mathcal{CO}_X$. Suppose that $\mathcal{L}(X, Y)$ satisfies* (CB) *and $\mathcal{U}_Y^s(\mathcal{K}_X, \geq)$ satisfies* (U5).

If $f_1, f_2, g, h \in \mathbf{(HK)}(\langle a|b\rangle, Y)$ satisfy that $g(x) \leq f_1(x) \leq h(x)$ and $g(x) \leq f_2(x) \leq h(x)$ for almost every $x \in \langle a|b\rangle$, then $f_1 \vee f_2, f_1 \wedge f_2 \in \mathbf{(HK)}(\langle a|b\rangle, Y)$.

Proof. Since by [15, Lemma 4.1] without loss of generality it may be assumed that $g(x) \leq f_1(x) \leq h(x)$ and $g(x) \leq f_2(x) \leq h(x)$ for any x, it is clear by Lemma 3.3. $\square$

Definition 3.2. Let X be a vector lattice with unit, Y a complete vector lattice, $a, b \in D \in \mathcal{CO}_X$ with $a < b$ and F a mapping from D into Y.

The following value

$$V(F; a, b) = \bigvee_{\varepsilon \in \mathcal{K}_{\mathbf{R}}} \bigvee_{\substack{a = x_0 < \cdots < x_K = b \\ x_k \in \langle a|b\rangle \\ x_k - x_{k-1} \leq \varepsilon(b-a)}} \sum_{k=1}^{K} |F(x_k) - F(x_{k-1})|$$

is said to be the total variation of F on $\langle a|b\rangle$. F is said to be of bounded variation if there exists the above value.

Remark 3.1. When $X = \mathbf{R}$ and F is absolutely continuous, then it is clear that F is of bounded variation.

Lemma 3.5. *Let $X = \mathbf{R}$, Y a complete vector lattice and $a, b \in D \in \mathcal{CO}_X$. Suppose that $\mathcal{U}_Y^s(\mathcal{K}_X, \geq)$ satisfies* (U5).

*Let F_1, F_2 be primitives of $f_1, f_2 \in$ **(HK)**$(\langle a|b\rangle, Y)$, respectively. If both F_1 and F_2 are of bounded variation, then $f_1 \vee f_2, f_1 \wedge f_2 \in$ **(HK)**$(\langle a|b\rangle, Y)$.*

Proof. Let F^* be the same as in the proof of Lemma 3.3. Note that $|a \vee b| \leq |a| + |b|$ for any a, b in a vector lattice. Then

$$\begin{aligned} \sum_{k=1}^{K} F^*(x_{k-1}, x_k) &\leq \sum_{k=1}^{K} (|F_1(x_k) - F_1(x_{k-1})| + |F_2(x_k) - F_2(x_{k-1})|) \\ &\leq V(F_1; a, b) + V(F_2; a, b). \end{aligned}$$

Since Y is complete, there exists

$$I = \bigvee_{a = x_0 < \cdots < x_K = b} \sum_{k=1}^{K} F^*(x_{k-1}, x_k).$$

The rest of the proof is same in Lemma 3.3. $\square$

Theorem 3.3. *Let $X = \mathbf{R}$, Y a complete vector lattice and $a, b \in D \in \mathcal{CO}_X$. Suppose that $\mathcal{U}_{\mathcal{L}(X,Y)}^s(\mathcal{K}_X, \geq) \cong \mathcal{U}_Y^s(\mathcal{K}_X, \geq)$ satisfies* (U4) *and* (U5).

*If $f_n, g, h \in$ **(HK)**$(\langle a|b\rangle, Y)$, $g(x) \leq f_n(x) \leq h(x)$ and $f_n(x) \to f(x)$ for any $x \in D$, then $f \in$ **(HK)**$(\langle a|b\rangle, Y)$ and*

$$o\text{-}(HK) \int_a^b f_n(x)dx \to o\text{-}(HK) \int_a^b f(x)dx.$$

Proof. We show in the case of $a < b$. It can be proved similarly in the case of $a > b$. By Lemma 3.3 it holds that $\bigwedge_{n=i}^{j} f_n \in$ **(HK)**$(\langle a|b\rangle, Y)$ for any natural numbers i, j with $i \leq j$. The sequence

$$\left\{ -\bigwedge_{n=i}^{i} f_n, -\bigwedge_{n=i}^{i+1} f_n, \ldots \right\}$$

is monotone increasing and its upper bound is $-g$. By Theorem 3.1 it holds that $-\bigwedge_{n=i}^{\infty} f_n \in$ **(HK)**$(\langle a|b\rangle, Y)$. Therefore $\bigwedge_{n=i}^{\infty} f_n \in$ **(HK)**$(\langle a|b\rangle, Y)$. In the same way $\bigvee_{n=i}^{\infty} f_n \in$ **(HK)**$(\langle a|b\rangle, Y)$. Moreover the sequence

$$\left\{\bigwedge_{n=1}^{\infty} f_n, \bigwedge_{n=2}^{\infty} f_n, \dots\right\}$$

is monotone increasing and its upper bound is h. By Theorem 3.1 it holds that $\bigvee_{i=1}^{\infty} \bigwedge_{n=i}^{\infty} f_n = f \in$ **(HK)**$(\langle a|b\rangle, Y)$. Therefore

$$o\text{-}(HK)\int_a^b \left(\bigwedge_{n=i}^{\infty} f_n\right)(x)dx \uparrow o\text{-}(HK)\int_a^b f(x)dx.$$

In the same way

$$o\text{-}(HK)\int_a^b \left(\bigvee_{n=i}^{\infty} f_n\right)(x)dx \downarrow o\text{-}(HK)\int_a^b f(x)dx.$$

Since by Lemma 3.1

$$\begin{aligned} o\text{-}(HK)\int_a^b \left(\bigwedge_{n=i}^{\infty} f_n\right)(x)dx &\le \bigwedge_{n=i}^{\infty} o\text{-}(HK)\int_a^b f_n(x)dx \\ &\le \bigvee_{n=i}^{\infty} o\text{-}(HK)\int_a^b f_n(x)dx \le o\text{-}(HK)\int_a^b \left(\bigvee_{n=i}^{\infty} f_n\right)(x)dx, \end{aligned}$$

it holds that

$$o\text{-}(HK)\int_a^b f_n(x)dx \to o\text{-}(HK)\int_a^b f(x)dx.$$

□

Theorem 3.4. *Let $X = \mathbf{R}$, Y a complete vector lattice and $a, b \in D \in \mathcal{CO}_X$. Suppose that $\mathcal{L}(X,Y)$ satisfies* (CB) *and $\mathcal{U}^s_{\mathcal{L}(X,Y)}(\mathcal{K}_X, \ge) \cong \mathcal{U}^s_Y(\mathcal{K}_X, \ge)$ satisfies* (U4) *and* (U5).

If $f_n, g, h \in$ **(HK)**$(\langle a|b\rangle, Y)$*, $g(x) \le f_n(x) \le h(x)$ and $f_n(x) \to f(x)$ for almost every $x \in D$, then $f \in$* **(HK)**$(\langle a|b\rangle, Y)$ *and*

$$o\text{-}(HK)\int_a^b f_n(x)dx \to o\text{-}(HK)\int_a^b f(x)dx.$$

Proof. Since by [15, Lemma 4.1] without loss of generality it may be assumed that $g(x) \le f_n(x) \le h(x)$ and $f_n(x) \to f(x)$ for any x, it is clear by Theorem 3.3. □

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