RELATIVE OPERATOR ENTROPY, OPERATOR DIVERGENCE AND SHANNON INEQUALITY

Dedicated to the memory of Professor Hisaharu Umegaki

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Abstract. Let A and B be positive operators and $A \natural_r B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r A^{\frac{1}{2}}$ is a path going through A and B. The tangent of $A \natural_r B$ at r is given by $S_r(A|B) = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}$ and especially the case $r=0$ is the relative operator entropy. We can find the behavior of $S_r(A|B)$, for $r \in [n,\ n+1]$, is similar to the case $r \in [0,\ 1]$. So we can extend several relations known for $r \in [0,\ 1]$ to $r \in [n,\ n+1]$.

1 Introduction. In [11], Umegaki introduced the relative entropy as a noncommutative version of the Kullback-Leibler entropy and Nakamura-Umegaki defined the operator entropy in [9] as an extension of the entropy formulated by von Neumann. Our discssions are based on their achievements.

Throughtout this paper, an operator means a bounded linear operator on a Hilbert space H. A bounded operator T on H is said to be positive if $(Tx, x) \geq 0$ for all $x \in H$ and denote $T \geq 0$, and if T is invertible and positive, we denote $T > 0$ and call it a strictly positive.

For fixed positive invertible operators A and B, we consider a path

$$A \natural_r B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r A^{\frac{1}{2}},\ r \in \mathbf{R},$$

which is going through $A = A \natural_0 B$ and $B = A \natural_1 B$ ([2], [3], [7] etc.). If $0 < r < 1$, then we denote this by $A \sharp_r B$, the generalized operator geometric mean, the operator mean is axiomatically given by Kubo-Ando [8]. Then we can give a tangent at r of this path by

$$S_r(A|B) = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}.$$

If $r = 0$, then $S_0(A|B) = A^{\frac{1}{2}}(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} = S(A|B)$, the relative operator entropy which we introduced in [1] as a relative version of the operator entropy given by Nakamura-Umegaki [9]. Furuta introduced $S_r(A|B)$ in [4] and Yanagi, Kuriyama and Furuichi called this the generalized relative operator entropy [12].

In section 2, we show several relations $S(A|B)$ and $S_r(A|B)$, for example, if $r = 2n,\ n$ is an integer, then $S_{2n}(A|B) = (BA^{-1})^n S(A|B)(A^{-1}B)^n$, etc..

The Tsallis relative operator entropy introduced by Furuichi-Yanagi-Kuriyama [12] is given by

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$$T_r(A|B) = \frac{A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r A^{\frac{1}{2}} - A}{r}, \quad \text{for} \quad 0 < r \leq 1,$$

that is,

$$T_r(A|B) = \frac{A \sharp_r B - A}{r}, \text{ for } r \in \mathbf{R}, \quad \text{and} \quad \lim_{r \to 0} T_r(A|B) = S(A|B).$$

In section 3, we show the following essential relation for $0 < r < 1$;

$$(*) \qquad S(A|B) \leq T_r(A|B) \leq S_r(A|B) \leq -T_{1-r}(B|A) \leq -S(B|A) = S_1(A|B)$$

and similar phenomena to $(*)$ can be observed for $n \leq r \leq n+1$, for an integer n.

In section 4, we try to extend the Bregman operator divergence

$$(**) \qquad D_{FK}(A|B) = B - A - S(A|B),$$

which is given by Petz [10]. Our proposal is to extend $(**)$ to

$$D_r(A|B) = B \natural_{-r} A - A \natural_r B - S_r(A|B).$$

Finally, we inspect the operator version of the Shannon inequality introduced by Furuta [4].

$$0 \geq \sum_{i=1}^{n} S(A_i|B_i)$$

Moreover Yanagi, Kuriyama and Furuichi [12] improved it to

$$0 \geq \sum_{i=1}^{n} T_r(A_i|B_i) \geq \sum_{i=1}^{n} S(A_i|B_i),$$

for $A_i,\ B_i > 0$ with $\sum_{i=1}^{n} A_i = \sum_{i=1}^{n} B_i = I$. Related to this, we show

$$\sum_{i=1}^{n} S(A_i|B_i) \leq \sum_{i=1}^{n} T_r(A_i|B_i) \leq \sum_{i=1}^{n} S_r(A_i|B_i) \leq -\sum_{i=1}^{n} T_{1-r}(B_i|A_i) \leq -\sum_{i=1}^{n} S(B_i|A_i).$$

2 Derivative of the path $A \natural_r B$. We introduced a path $A \natural_r B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r A^{\frac{1}{2}}$ for $r \in \mathbf{R}$, which is going through $A = A \natural_0 B$ and $B = A \natural_1 B$ and if $0 < r < 1$ we usually denote by $A \sharp_r B$, the power operator mean or generalized geometric operator mean. The relative operator entropy $S(A|B)$, we introduced in [1], is given by the derivative of $A \natural_r B$ at $r = 0$. In [4], Furuta introduced the following $S_r(A|B),\ r \in \mathbf{R}$, as a generalized form of $S(A|B)$.

Definition 1. For $A > 0,\ B > 0$ and $r \in \mathbf{R}$, we give $S_r(A|B)$ as follows:

$$S_r(A|B) = \lim_{\epsilon \to 0} \frac{A \natural_{r+\epsilon} B - A \natural_r B}{\epsilon} = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r (\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}}) A^{\frac{1}{2}},$$

where $A \natural_r B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r A^{\frac{1}{2}},\ \ r \in \mathbf{R}$, and if $0 \leq r \leq 1,\ A \natural_r B = A \sharp_r B$.

As a special case, $S_0(A|B) = S(A|B)$ and $S(A|I) = -A \log A$, the operator entropy [9].

Yanagi, Kuriyama and Furuichi [12] called $S_r(A|B)$ the generalized relative operator entropy. We have to note that $F_r(x) = x^r \log x$ is not operator concave function except $r = 0$.

For given positive operators $A,\ B$, if we put $\Phi(t) = A \natural_t B$, then the convexity of this function is known, so the following theorem is natural and fundamental in our discussion.

Theorem 1. *For $A > 0,\ B > 0$, $S_r(A|B)$ is monotone increasing for $r \in \mathbf{R}$, and the following holds.*

$$S_r(A|B) \le \frac{A \natural_q B - A \natural_r B}{q-r} \le S_q(A|B) \quad \text{for } q,\ r \in \mathbf{R},\ q > r. \tag{1}$$

Especially, in the case $r = 0$ and $0 < q < 1$, (1) *is expressed as follows:*

$$S(A|B) \le \frac{A \sharp_q B - A}{q} = T_q(A|B) \le S_q(A|B). \tag{2}$$

To prove Theorem 1, we need the next Lemma.

Lemma 2. *Let $a > 0$. Then the following holds for $q,\ r \in \mathbf{R}$.*

$$a^r \log a \le \frac{a^q - a^r}{q-r} \le a^q \log a, \quad \text{for} \quad q > r.$$

Since a^t is convex function, this is easily given, but we give an elementary proof.

Proof. We show this inequality as follows:

$$\frac{a^q}{a^r}\log\frac{a^q}{a^r} = -\frac{a^q}{a^r}\log\frac{a^r}{a^q} \ge -\frac{a^q}{a^r}\Big(\frac{a^r}{a^q} - 1\Big) = \frac{a^q}{a^r} - 1 \ge \log\frac{a^q}{a^r},$$

that is,

$$a^q(\log a^q - \log a^r) \ge a^q - a^r \ge a^r(\log a^q - \log a^r).$$

So we have

$$(q-r)a^q \log a \ge a^q - a^r \ge (q-r)a^r \log a.$$ □

Proof of Thorem 1. In Lemma 2, we can easily draw (1) replacing a by $A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$ and multiplying $A^{\frac{1}{2}}$ to both sides, and (2) is a special case of (1). □

Next, we prepare several properties of $S_r(A|B)$ to show the results in the following section, some of them are already shown in [4], [12] .

Lemma 3. *$A > 0,\ B > 0$ and $r \in R$, n is an integer. Then $S_r(A|B)$ has the following properties:*

$$S_r(A|B) = -S_{1-r}(B|A) = BS_{r-1}(B^{-1}|A^{-1})B = -AS_{-r}(A^{-1}|B^{-1})A, \tag{1}$$

$$S_n(A|B) = (BA^{-1})^n S(A|B) = S(A|B)(A^{-1}B)^n, \tag{2}$$

$$S_{2n}(A|B) = (BA^{-1})^n S(A|B)(A^{-1}B)^n, \tag{3}$$

$$S_{2n+1}(A|B) = (BA^{-1})^n S_1(A|B)(A^{-1}B)^n. \tag{4}$$

Proof. (1) is given as follows:

$$\begin{aligned} S_r(A|B) &= A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} \\ &= -B^{\frac{1}{2}}B^{-\frac{1}{2}}A^{\frac{1}{2}}(A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}})^{-r}(\log A^{\frac{1}{2}}B^{-1}A^{\frac{1}{2}})A^{\frac{1}{2}}B^{-\frac{1}{2}}B^{\frac{1}{2}} \\ &= -B^{\frac{1}{2}}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{-r}(\log B^{-\frac{1}{2}}AB^{-\frac{1}{2}})(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})B^{\frac{1}{2}} \\ &= -B^{\frac{1}{2}}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{-r+1}(\log B^{-\frac{1}{2}}AB^{-\frac{1}{2}})B^{\frac{1}{2}} = -S_{-r+1}(B|A), \\ &\ \textit{or} \\ &= B^{\frac{1}{2}}(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}})^{r-1}(\log B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}})B^{\frac{1}{2}} \\ &= BB^{-\frac{1}{2}}(B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}})^{r-1}(\log B^{\frac{1}{2}}A^{-1}B^{\frac{1}{2}})B^{-\frac{1}{2}}B = BS_{r-1}(B^{-1}|A^{-1})B. \end{aligned}$$

The last equation is shown by the similar way.

(2) is shown as follows:

$$S_n(A|B) = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^n(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}$$
$$= (BA^{-1})^n A^{\frac{1}{2}}(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} = (BA^{-1})^n S(A|B)$$

and

$$S_n(A|B) = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^n(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}$$
$$= A^{\frac{1}{2}}(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^n A^{\frac{1}{2}} = S(A|B)(A^{-1}B)^n.$$

We show (3) and (4) as follows:

$$\begin{aligned} S_{2n}(A|B) &= A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{2n}(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} \\ &= A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{n}(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^n A^{\frac{1}{2}} \\ &= (BA^{-1})^n A^{\frac{1}{2}}(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}(A^{-1}B)^n = (BA^{-1})^n S(A|B)(A^{-1}B)^n. \\ S_{2n+1}(A|B) &= A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{2n+1}(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} \\ &= A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{n}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^n A^{\frac{1}{2}} \\ &= (BA^{-1})^n A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}}(A^{-1}B)^n \\ &= (BA^{-1})^n S_1(A|B)(A^{-1}B)^n. \end{aligned}$$

□

Remark 1. We list up some special cases of Lemma 3;

(1) $$S_1(A|B) = -S(B|A) = (BA^{-1})S(A|B) = S(A|B)(A^{-1}B) = BS(B^{-1}|A^{-1})B,$$

(2) $$S_2(A|B) = BA^{-1}S(A|B)A^{-1}B,$$

(3) $$S_3(A|B) = BA^{-1}S_1(A|B)A^{-1}B,$$

(4) $$S_{-1}(A|B) = BS_{-2}(B^{-1}|A^{-1})B.$$

3 Tsallis relative operator entropy and $S_r(A|B)$.

First, we exhibit fundamental relations which are essential in our following discussions.

Theorem 4. *Let $A > 0,\ B > 0$. Then the following hold;*

(1) *for* $0 < r < 1$,

(∗) $$S(A|B) \le T_r(A|B) \le S_r(A|B) \le -T_{1-r}(B|A) \le -S(B|A) = S_1(A|B).$$

(2) *for* $1 < r < 2$,

$$S_1(A|B) \le \frac{A \natural_r B - B}{r-1} \le S_r(A|B) \le \frac{A \natural_2 B - A \natural_r B}{2-r} \le S_2(A|B).$$

or equivalently,

(2')

$$S(B^{-1}|A^{-1}) \le T_{r-1}(B^{-1}|A^{-1}) \le S_{r-1}(B^{-1}|A^{-1}) \le -T_{2-r}(A^{-1}|B^{-1}) \le -S(A^{-1}|B^{-1}).$$

Proof of Theorem 4. (1) and (2) are easy results of Theorem 1, so we show (2'). By (1) in Lemma 3, we have

$$S_1(A|B) = BS(B^{-1}|A^{-1})B,\ S_r(A|B) = BS_{r-1}(B^{-1}|A^{-1})B,\ S_2(A|B) = BS_1(B^{-1}|A^{-1})B$$

$$\text{and}\quad A \natural_r B = B \natural_{1-r} A = B^{\frac{1}{2}}(B^{-\frac{1}{2}}AB^{-\frac{1}{2}})^{1-r}B^{\frac{1}{2}} = B(B^{-1} \sharp_{r-1} A^{-1})B.$$

So we obtaine (2'). □

General cases are given by the use of (2) in Lemma 3 as follows:

Theorem 5. *Let $A > 0,\ B > 0$ and $n < r < n+1$ for an integer n. Then the following hold and they are equivalent:*

(1) $$S_n(A|B) \le \frac{A \natural_r B - A \natural_n B}{r-n} \le S_r(A|B) \le \frac{A \natural_{n+1} B - A \natural_r B}{n+1-r} \le S_{n+1}(A|B),$$

(2) $$\begin{aligned}(BA^{-1})^n S(A|B) &\le (BA^{-1})^n T_{r-n}(A|B) \le (BA^{-1})^n S_{r-n}(A|B)\\ &\le -(BA^{-1})^n T_{n+1-r}(B|A) \le -(BA^{-1})^n S(B|A) = (BA^{-1})^n S_1(A|B),\end{aligned}$$

(3) $$\begin{aligned}S(A|B)(A^{-1}B)^n &\le T_{r-n}(A|B)(A^{-1}B)^n \le S_{r-n}(A|B)(A^{-1}B)^n\\ &\le -T_{n+1-r}(B|A)(A^{-1}B)^n \le -S(B|A)(A^{-1}B)^n = S_1(A|B)(A^{-1}B)^n.\end{aligned}$$

To prove this theorem, we prepare the next lemma concerning to $T_r(A|B)$.

Lemma 6. *For $A > 0,\ B > 0,\ r \in \mathbf{R}$ and an integer n,*

(1) $$\frac{A \natural_r B - A \natural_n B}{r-n} = (BA^{-1})^n T_{r-n}(A|B) = T_{r-n}(A|B)(A^{-1}B)^n,$$

(2) $$\frac{A \natural_{n+1} B - A \natural_r B}{n+1-r} = -(BA^{-1})^n T_{n+1-r}(B|A) = -T_{n+1-r}(B|A)(A^{-1}B)^n,$$

(3) $$\frac{A \natural_r B - A \natural_{2n} B}{r-2n} = (BA^{-1})^n T_{r-2n}(A|B)(A^{-1}B)^n,$$

(4) $$\frac{A \natural_{2n+1} B - A \natural_r B}{2n+1-r} = -(BA^{-1})^n T_{2n+1-r}(B|A)(A^{-1}B)^n,$$

(5) $$S_r(A|B) = (BA^{-1})^n S_{r-n}(A|B) = S_{r-n}(A|B)(A^{-1}B)^n,$$

(6) $$S_r(A|B) = (BA^{-1})^n S_{r-2n}(A|B)(A^{-1}B)^n.$$

Proof. (1) and (2) are shown as follows:

$$\begin{aligned}\frac{A \natural_r B - A \natural_n B}{r-n} &= \frac{A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^n\{(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{r-n} - I\}A^{\frac{1}{2}}}{r-n}\\ &= \frac{(BA^{-1})^n A^{\frac{1}{2}}\{(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{r-n} - I\}A^{\frac{1}{2}}}{r-n}\\ &= \frac{(BA^{-1})^n(A \sharp_{r-n} B - A)}{r-n} = (BA^{-1})^n T_{r-n}(A|B),\end{aligned}$$

and

$$\begin{aligned}\frac{A \natural_{n+1} B - A \natural_r B}{n+1-r} &= \frac{A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{n+1}A^{\frac{1}{2}} - A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^r A^{\frac{1}{2}}}{n+1-r} \\ &= \frac{A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^n\{A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{r-n}\}A^{\frac{1}{2}}}{n+1-r} \\ &= \frac{(BA^{-1})^n A^{\frac{1}{2}}\{A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{r-n}\}A^{\frac{1}{2}}}{n+1-r} \\ &= \frac{(BA^{-1})^n(B - A \sharp_{r-n} B)}{n+1-r} = \frac{(BA^{-1})^n(B - B \sharp_{n+1-r} A)}{n+1-r} \\ &= -(BA^{-1})^n T_{n+1-r}(B|A)\end{aligned}$$

The rest can be obtained by a similar method to the proof of Lemma 3. □

Proof of Theorem 5. The first half inequalities of (1) are obtained by replacing $r = n$ and $q = r$ in Theorem 1 (1), and the second ones are the case $q = n+1$.

Equivalence among (1), (2) and (3) is obtained by Lemma 3 and Lemma 6. □

Theorem 5 says that the same form as that of Theorem 4 comes over and over again like waves, so we want to call it a waving property. More precisely, we have the following:

Theorem 7. *Let $A > 0,\ B > 0$. Then the following hold.*

(1) *In the case where $2n \le r \le 2n+1$,*

$$S_{2n}(A|B) \le \frac{A \natural_r B - A \natural_{2n} B}{r-2n} \le S_r(A|B) \le \frac{A \natural_{2n+1} B - A \natural_r B}{2n+1-r} \le S_{2n+1}(A|B),$$

or equivalently,

$$\begin{aligned}(BA^{-1})^n S(A|B)(A^{-1}B)^n &\le (BA^{-1})^n T_{r-2n}(A|B)(A^{-1}B)^n \le (BA^{-1})^n S_{r-2n}(A|B)(A^{-1}B)^n \\ &\le -(BA^{-1})^n T_{2n+1-r}(B|A)(A^{-1}B)^n \le (BA^{-1})^n S_1(A|B)(A^{-1}B)^n.\end{aligned}$$

(2) *In the case where $2n+1 \le r \le 2(n+1)$,*

$$S_{2n+1}(A|B) \le \frac{A \natural_r B - A \natural_{2n+1} B}{r-(2n+1)} \le S_r(A|B) \le \frac{A \natural_{2(n+1)} B - A \natural_r B}{2(n+1)-r} \le S_{2(n+1)}(A|B),$$

or equivalently,

$$\begin{aligned}&(BA^{-1})^n S_1(A|B)(A^{-1}B)^n \le \frac{(BA^{-1})^n(A \natural_{r-2n} B - B)(A^{-1}B)^n}{r-(2n+1)} \\ \le\ & (BA^{-1})^n S_{r-2n}(A|B)(A^{-1}B)^n \\ \le\ & \frac{(BA^{-1})^n(A \natural_2 B - A \natural_{r-2n} B)(A^{-1}B)^n}{2(n+1)-r} \le (BA^{-1})^n S_2(A|B)(A^{-1}B)^n.\end{aligned}$$

This is also equivalent to the following form:

$$\begin{aligned}&(BA^{-1})^n BS(B^{-1}|A^{-1})B(A^{-1}B)^n \le (BA^{-1})^n BT_{r-(2n+1)}(B^{-1}|A^{-1})B(A^{-1}B)^n \\ \le\ & (BA^{-1})^n BS_{r-(2n+1)}(B^{-1}|A^{-1})B(A^{-1}B)^n \\ \le\ & -(BA^{-1})^n BT_{2(n+1)-r}(A^{-1}|B^{-1})B(A^{-1}B)^n \le (BA^{-1})^n BS_1(B^{-1}|A^{-1})B(A^{-1}B)^n.\end{aligned}$$

Proof. We obtain Theorem 7 by using Theorem 5, Lemma 6 and the following equations.

$$A \natural_r B - A \natural_{2n+1} B = (BA^{-1})^n(A \natural_{r-2n} B - B)(A^{-1}B)^n,$$

$$A \natural_{2(n+1)} B - A \natural_r B = (BA^{-1})^n(A \natural_2 B - A \natural_{r-2n} A)(A^{-1}B)^n,$$

$$A \natural_{r-2n} B - B = B(B^{-1} \natural_{r-(2n+1)} A^{-1} - B^{-1})B,$$

$$A \natural_2 B - A \natural_{r-2n} B = -B(A^{-1} \natural_{2(n+1)-r} B^{-1} - A^{-1})B.$$

□

4 Operator divergence. Petz introduced the Bregman operator divergence [10] : For an operator convex function F and positive (invertible) operators A and B,

$$\begin{aligned} D_{[F]}(A|B) &= F(A) - F(B) - \lim_{t\to+0} \frac{F(B + t(A - B)) - F(B)}{t} \\ &= \lim_{t\to+0} \frac{tF(A) + (1 - t)F(B) - F(tA + (1 - t)B)}{t} \geq 0. \end{aligned}$$

By hard calculation, he gave a nice representation of $D_{[F]}$. For $F(x) = x \log x$ and density matrices A and B,

$$TrD_{[x \log x]}(A|B) = TrA(\log A - \log B) = s(A|B),$$

the Umegaki relative entropy [11]. As a slightly modified form of $S(A|B)$, Petz gives also an operator divergence

$$D_{FK}(A,\ B) = B - A - S(A|B),$$

whose non negativity is assured by

$$S(A|B) = A^{\frac{1}{2}}(\log A^{-\frac{1}{2}}BA^{-\frac{1}{2}})A^{\frac{1}{2}} \leq A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}} - I)A^{\frac{1}{2}} = B - A.$$

We may generalize $D_{FK}(A, B)$ as follows:

$$D_r(A,\ B) = A \natural_{r+1} B - A \natural_r B - S_r(A|B) = B \natural_{-r} A - A \natural_r B - S_r(A|B),$$

particularly $D_{FK}(A,\ B) = D_0(A,\ B)$. The following property holds by Theorem 1 and Lemma 6.

Theorem 8. *Let A and B be positive invertible operators and $r \in \mathbf{R}$. Then*

$$D_r(A,\ B) \geq 0.$$

Corollary 9. *Let n be an integer. Then*

(1) $$D_n(A,\ B) = (BA^{-1})^n D_0(A,\ B) = D_0(A,\ B)(A^{-1}B)^n \geq 0,$$

(2) $$D_{2n}(A,\ B) = (BA^{-1})^n D_0(A,\ B)(A^{-1}B)^n \geq 0,$$

(3) $$D_{2n+1}(A,\ B) = (BA^{-1})^n BD_0(B^{-1},\ A^{-1})B(A^{-1}B)^n \geq 0.$$

5 Shannon inequality. Shannon inequality is given as follows:

$$0 \geq \sum_{i=1}^{n} a_i \log \frac{b_i}{a_i}$$

for $a_i,\ b_i > 0$ with $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i = 1$. Furuta [4] introduced an operator version for the Shannon inequality, that is,

$$0 \geq \sum_{i=1}^{n} S(A_i|B_i)$$

for $A_i,\ B_i > 0$ with $\sum_{i=1}^{n} A_i = \sum_{i=1}^{n} B_i = I$.

Definition 2. Let $\{A_1, \cdots, A_n\}$ and $\{B_1, \cdots, B_n\}$ be sequences of strictly positive operators with $\sum_{i=1}^{n} A_i = \sum_{i=1}^{n} B_i = I$. We give the operator versions of relative entropy, Rényi's relative entropy, Tsallis relative entropy and $S_r((A_i),\ (B_i))$ as follows:

$$S((A_i),(B_i)) = \sum_{i=1}^{n} S(A_i|B_i),$$

$$I_r((A_i),(B_i)) = \frac{1}{r} \log \sum_{i=1}^{n}\ A_i \ \natural_r \ B_i,$$

$$T_r((A_i),(B_i)) = \sum_{i=1}^{n} \frac{A_i \ \natural_r \ B_i - A_i}{r}$$

and

$$S_r((A_i),\ (B_i)) = \sum_{i=1}^{n} S_r(A_i|B_i).$$

Among these quantities, the following inequalities hold.

Theorem 10. *For sequences of positive operators $\{A_1, \cdots, A_n\}$ and $\{B_1, \cdots, B_n\}$ with $\sum_{i=1}^{n} A_i = \sum_{i=1}^{n} B_i = I$,*

$$0 \geq T_r((A_i),(B_i)) \geq I_r((A_i),(B_i)) \geq S((A_i),(B_i)),$$

$$0 \leq -T_{1-r}((A_i),(B_i)) \leq -I_{1-r}((A_i),(B_i)) \leq S_1((A_i),(B_i))$$

and

$$T_r((A_i),\ (B_i)) \leq S_r((A_i),\ (B_i)) \leq -T_{1-r}((B_i),\ (A_i))$$

hold for $0 < r < 1$.

To prove Theorem 10, we use the next;

$$\frac{x^r - 1}{r} \leq x - 1, \quad \text{for} \quad 0 < r < 1,$$

and the following Jensen's operator inequality [6].

Theorem 11 (Jensen's operator inequality(cf. [4], [5], [6]).)**.** *Let $f(x)$ be operator concave function and $\{C_j\}_{j=1}^{n}$ be operators with $\sum_{j=1}^{n} C_j^* C_j = I$, then*

$$f(\sum_{i=1}^{n} C_j^* A_j C_j) \geq \sum_{i=1}^{n} C_j^* f(A_j) C_j.$$

Proof of Theorem 10.

$$\begin{aligned} I_r((A_i),(B_i)) &= \frac{1}{r}\log\sum_{i=1}^{n} A_i \sharp_r B_i = \frac{1}{r}\log\sum_{i=1}^{n} A_i^{\frac{1}{2}}(A_i^{-\frac{1}{2}}B_iA_i^{-\frac{1}{2}})^r A_i^{\frac{1}{2}} \\ &\geq \frac{1}{r}\sum_{i=1}^{n} A_i^{\frac{1}{2}}\log(A_i^{-\frac{1}{2}}B_iA_i^{-\frac{1}{2}})^r A_i^{\frac{1}{2}} = \sum_{i=1}^{n} A_i^{\frac{1}{2}}\log(A_i^{-\frac{1}{2}}B_iA_i^{-\frac{1}{2}})A_i^{\frac{1}{2}} \\ &= S((A_i),(B_i)). \end{aligned}$$

And

$$\begin{aligned} I_r((A_i),(B_i)) &= \frac{1}{r}\log\sum_{i=1}^{n} A_i \sharp_r B_i \leq \frac{1}{r}(\sum_{i=1}^{n} A_i \sharp_r B_i - I) \\ &= \frac{1}{r}\sum_{i=1}^{n}(A_i \sharp_r B_i - A_i) = \sum_{i=1}^{n}\frac{A_i \sharp_r B_i - A_i}{r} = T_r((A_i),(B_i)). \end{aligned}$$

$$\begin{aligned} T_r((A_i),(B_i)) &= \sum_{i=1}^{n}\frac{A_i \sharp_r B_i - A_i}{r} = \sum_{i=1}^{n}\frac{A_i^{\frac{1}{2}}\{(A_i^{-\frac{1}{2}}B_iA_i^{-\frac{1}{2}})^r - I\}A_i^{\frac{1}{2}}}{r} \\ &\leq \sum_{i=1}^{n} A_i^{\frac{1}{2}}(A_i^{-\frac{1}{2}}B_iA_i^{-\frac{1}{2}} - I)A_i^{\frac{1}{2}} = \sum_{i=1}^{n}(B_i - A_i) = 0. \end{aligned}$$

The second relation is shown by similar methods to the above. By Theorem 4, we can obtain the final inequality. □

References

[1] J.I.Fujii and E.Kamei, Relative operator entropy in noncommutative information theory, Math. Japon., **34**(1989), 341–348.

[2] J.I.Fujii and E.Kamei, Path of Bregman-Petz operator divergence, Sci. Math. Japon., **70**(2009), 329–333.

[3] J.I.Fujii and E.Kamei, Interpolational paths and their derivatives, Math. Japon., **39**(1994), 557–560.

[4] T.Furuta, Parametric extensions of Shannon inequality and its reverse one in Hilbert space operators, Linear Algebra Appl., **381**(2004), 219–235.

[5] F.Hansen, An operator inequality, Math. Ann., **246**(1980), 249–256.

[6] F.Hansen and G.K.Pedersen, Jensen's operator inequality, Bull. London Math. Soc., **35**(2003), 553–564.

[7] E.Kamei, Paths of operators parametrized by operator means, Math. Japon., **39**(1994), 395–400.

[8] F.Kubo and T.Ando, Means of positive linear operators, Math. Ann., **248**(1980), 205–224.

[9] M.Nakamura and H.Umegaki, A note on the entropy for operator algebra, Proc. Japan Acad., **37**(1961), 149–154.

[10] D.Petz, Bregman divergence as relative operator entropy, Acta Math. Hungar., **116**(2007), 127–131.

[11] H.Umegaki, Conditional expectation in operator algebra IV, Kodai Math. Sem. Rep., **14**(1962), 59–85.

[12] K.Yanagi, K.Kuriyama, S.Furuichi, Generalized Shannon inequalities based on Tsallis relative operator entropy, Linear Algebra Appl., **394**(2005), 109–118.

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