Scientiae Mathematicae Japonicae Online, **e-2012**, 363–369

THE ARITHMETIC-GEOMETRIC MEAN INEQUALITY IN AN EXTERNAL FORMULA

Yuki Seo

Received October 29, 2012

Abstract. The classical Jensen inequality and its reverse are discussed by means of internally dividing points. J.I. Fujii pointed out that the concavity is also expressed by externally dividing points. In this paper, we shall discuss an external version of the arithmetic-geometric mean inequality: For positive real numbers $x_i, y_i \geq 0$ for $i = 1, 2, \cdots, n$ and $r \geq 0$

$$(1+r)\cdot\frac{x_1+x_2+\cdots+x_n}{n} - r\cdot\frac{y_1+y_2+\cdots+y_n}{n} \leq \left(\frac{x_1+x_2+\cdots+x_n}{n}\right)^{1+r}\left(\sqrt[n]{y_1y_2\cdots y_n}\right)^{-r}.$$

1 Introduction. The Jensen inquality for concave functions is one of the most important inequalities in the functional analysis. Let f be a real valued function on an interval J. The classical Jensen inequality is expressed by internally dividing points: If f is concave on J, then

$$\sum_{i=1}^n \alpha_i f(x_i) \leq f\left(\sum_{i=1}^n \alpha_i x_i\right) \tag{1.1}$$

for all $x_i \in J$ and all $\alpha_i \geq 0$ $(i = 1, 2, \cdots, n)$ such that $\sum_{i=1}^n \alpha_i = 1$, see [3, Theorem 1.1]. Mond-Pečarić [4] showed the following operator version of (1.1): If A is a selfadjoint operator on a Hilbert space H with the spectrum in J, then

$$\langle f(A)x, x\rangle \leq f(\langle Ax, x\rangle) \tag{1.2}$$

for every unit vector $x \in H$.

In [1], J.I. Fujii pointed out that the concavity is also expressed by externally dividing points: f is concave on J if and only if

$$f((1+r)x - ry) \leq (1+r)f(x) - rf(y) \tag{1.3}$$

for all $x, y \in J$ and $r > 0$ with $(1+r)x - ry \in J$. Thus, an external version of the classical Jensen inequality is as follows: f is concave on J if and only if

$$f\left(\sum_{i=1}^n \alpha_i x_i - \sum_{j=1}^k \beta_j y_j\right) \leq (\sum_{i=1}^n \alpha_i) f\left(\frac{1}{\sum_{i=1}^n \alpha_i}\sum_{i=1}^n \alpha_i x_i\right) - \sum_{j=1}^k \beta_j f(y_j)$$

for all $x_i, y_j \in J$ and $\alpha_i, \beta_j \geq 0$ $(i = 1, \cdots, n$ and $j = 1, \cdots, k)$ such that $\sum_{i=1}^n \alpha_i - \sum_{j=1}^k \beta_j = 1$, and $\sum_{i=1}^n \alpha_i x_i - \sum_{j=1}^k \beta_j y_j \in J$, also see [5, p83].

2000 *Mathematics Subject Classification.* 47A63, 47A64.

Key words and phrases. Concave function, Jensen inequality, Reverse inequalty, Positive operator, Arithmetic-Geometric mean inequality.

In this paper, by virture of an external formula, we shall discuss the arithmetic-geometric mean inequality. Moreover, we show reverses of the Jensen operator inequality by means of externally dividing points.

2 Arithmetic-Geometric mean inequality.

The arithmetic-geometric mean inequality says that for non-negative real numbers $x_1, x_2, \cdots, x_n$

$$\sqrt[n]{x_1 x_2 \cdots x_n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}. \tag{2.1}$$

By virture of an external formula (1.3), the inequality (2.1) is regarded as one by means of internally dividing points.

In [6], Specht estimated the upper boundary of the arithmetic mean by the geometric one for positive numbers: For $x_1, x_2, \cdots, x_n \in [m, M]$ with $0 < m \leq M$,

$$\frac{x_1 + x_2 + \cdots + x_n}{n} \leq S(h) \sqrt[n]{x_1 x_2 \cdots x_n}, \tag{2.2}$$

where $h = \frac{M}{m}$ and the Specht ratio $S(h)$ is defined by

$$S(h) = \frac{(h-1)h^{\frac{1}{h-1}}}{e \log h} \ (h \neq 1, h > 0) \quad \text{and} \quad S(1) = 1. \tag{2.3}$$

We call (2.2) the Specht theorem, see [3, Theorem 2.49]. We also have the weighted Specht theorem: For $x_i \geq 0$ and $\omega_i \geq 0$ for $i = 1, 2, \cdots, n$ with $\sum_{i=1}^n \omega_i = 1$

$$\omega_1 x_1 + \omega_2 x_2 + \cdots + \omega_n x_n \leq S(h) x_1^{\omega_1} x_2^{\omega_2} \cdots x_n^{\omega_n}. \tag{2.4}$$

First of all, we consider the arithmetic-geometric mean inequality by virture of an external formula.

Theorem 1. *For positive real numbers x_i, y_i for $i = 1, 2, \cdots, n$ and $r \geq 0$*

$$\begin{aligned} (1+r) \cdot \frac{x_1 + x_2 + \cdots + x_n}{n} - r \cdot \frac{y_1 + y_2 + \cdots + y_n}{n} \\ \leq \left(\frac{x_1 + x_2 + \cdots + x_n}{n} \right)^{1+r} \left(\sqrt[n]{y_1 y_2 \cdots y_n} \right)^{-r}. \end{aligned} \tag{2.5}$$

Proof. We may assume that $(1+r) \cdot \frac{x_1+x_2+\cdots+x_n}{n} - r \cdot \frac{y_1+y_2+\cdots+y_n}{n} > 0$. Since the logrithm function $\log t$ is concave, it follows that

$$\begin{aligned} &(1+r) \log \left(\frac{x_1 + x_2 + \cdots + x_n}{n} \right) \\ &= (1+r) \log \left(\frac{1}{1+r} \left((1+r) \frac{x_1 + \cdots + x_n}{n} - r \frac{y_1 + \cdots + y_n}{n} \right) + \frac{r}{1+r} \frac{y_1 + \cdots + y_n}{n} \right) \\ &\geq (1+r) \left(\frac{1}{1+r} \log \left((1+r) \frac{x_1 + \cdots + x_n}{n} - r \frac{y_1 + \cdots + y_n}{n} \right) + \frac{r}{1+r} \log \frac{y_1 + \cdots + y_n}{n} \right) \\ &= \log \left((1+r) \frac{x_1 + \cdots + x_n}{n} - r \frac{y_1 + \cdots + y_n}{n} \right) + r \log \frac{y_1 + \cdots + y_n}{n} \\ &\geq \log \left((1+r) \frac{x_1 + \cdots + x_n}{n} - r \frac{y_1 + \cdots + y_n}{n} \right) + \frac{r}{n} (\log y_1 + \cdots + \log y_n) \end{aligned}$$

and this implies

$$\log\left(\frac{x_1+\cdots+x_n}{n}\right)^{1+r} - \log(\sqrt[n]{y_1\cdots y_n})^{\,r} \geq \log\left((1+r)\frac{x_1+\cdots+x_n}{n} - r\frac{y_1+\cdots+y_n}{n}\right).$$

By taking the exponent of both sides, we have the desired inequality (2.5). □

Remark 2. Theorem 1 implies the arithmetic-geometric mean inequality (2.1). In fact, if we put $r = 1$ and $x_i = y_i$ for $i = 1, \cdots, n$ in (2.5) of Theorem 1, then we have (2.1).

Similarly we have an external version of the weighted arithmetic-geometric mean inequality.

Theorem 3. *For positive real numbers x_i, y_i for $i = 1, 2, \cdots, n$ and $\omega_1 \geq 0$ for $i = 1, \cdots, n$ such that $\sum_{i=1}^{n} \omega_i = 1$,*

$$(2.6) \qquad (1+r)\sum_{i=1}^{n}\omega_i x_i - r\sum_{i=1}^{n}\omega_i y_i \leq \left(\sum_{i=1}^{n}\omega_i x_i\right)^{1+r}\left(\prod_{i=1}^{n} y_i^{\omega_i}\right)^{-r}.$$

By virture of an external formula (1.3), one might expect that

$$(2.7) \qquad (1+r)\cdot\frac{x_1+x_2+\cdots+x_n}{n} - r\cdot\frac{y_1+y_2+\cdots+y_n}{n} \leq (\sqrt[n]{x_1x_2\cdots x_n})^{1+r}(\sqrt[n]{y_1y_2\cdots y_n})^{-r}.$$

However, the inequality (2.7) does not hold in general. By the Specht theorem, we have the following complementary inequality to the arithmetic-geometric mean inequality in an external formula.

Theorem 4. *Let x_i and y_i be positive real numbers such that $x_i, y_i \in [m, M]$ with $0 < m \leq M$ for $i = 1, 2, \cdots, n$, and $r \geq 0$. If $(1+r)\cdot\frac{x_1+x_2+\cdots+x_n}{n} - r\cdot\frac{y_1+y_2+\cdots+y_n}{n} > 0$, then*

$$\begin{aligned} S(h)^{-2r-1}(\sqrt[n]{x_1x_2\cdots x_n})^{1+r}(\sqrt[n]{y_1y_2\cdots y_n})^{-r} &\leq (1+r)\cdot\frac{x_1+x_2+\cdots+x_n}{n} - r\cdot\frac{y_1+y_2+\cdots+y_n}{n} \\ &\leq S(h)^{1+r}(\sqrt[n]{x_1x_2\cdots x_n})^{1+r}(\sqrt[n]{y_1y_2\cdots y_n})^{-r}, \end{aligned}$$

where $h = \frac{M}{m}$ and the Specht ratio $S(h)$ is defined as (2.3).

Proof. By the Specht theorem (2.2) and Theorem 1, it follows that

$$\begin{aligned} (1+r)\cdot\frac{x_1+x_2+\cdots+x_n}{n} - r\cdot\frac{y_1+y_2+\cdots+y_n}{n} &\leq \left(\frac{x_1+x_2+\cdots+x_n}{n}\right)^{1+r}(\sqrt[n]{y_1y_2\cdots y_n})^{-r} \\ &\leq S(h)^{1+r}(\sqrt[n]{x_1x_2\cdots x_n})^{1+r}(\sqrt[n]{y_1y_2\cdots y_n})^{-r} \end{aligned}$$

and this implies the second part of Theorem 4.

Since the weighted version of the Specht theorem (2.4) holds, we have

$$
\begin{aligned}
&(1+r)\log \sqrt[n]{x_1x_2\cdots x_n} \le (1+r)\log \frac{x_1+x_2+\cdots+x_n}{n}\\
&= (1+r)\log\left(\frac{1}{1+r}\left((1+r)\frac{x_1+\cdots+x_n}{n} - r\frac{y_1+\cdots+y_n}{n}\right) + \frac{r}{1+r}\frac{y_1+\cdots+y_n}{n}\right)\\
&\le (1+r)\left(\log S(h) + \frac{1}{1+r}\log\left((1+r)\frac{x_1+\cdots+x_n}{n} - r\frac{y_1+\cdots+y_n}{n}\right)\right.\\
&\qquad \left. + \frac{r}{1+r}\log\frac{y_1+\cdots+y_n}{n}\right)\\
&= \log S(h)^{1+r} + \log\left((1+r)\frac{x_1+\cdots+x_n}{n} - r\frac{y_1+\cdots+y_n}{n}\right) + r\log\frac{y_1+\cdots+y_n}{n}\\
&\le \log S(h)^{1+r} + \log\left((1+r)\frac{x_1+\cdots+x_n}{n} - r\frac{y_1+\cdots+y_n}{n}\right)\\
&\qquad + r\left(\log S(h) + \frac{\log y_1+\cdots \log y_n}{n}\right)
\end{aligned}
$$

and this implies the first part of Theorem 4. □

Remark 5. If $r=0$ and $x_i=y_i$ for $i=1,\cdots,n$ in Theorem 4, then we have the Specht theorem (2.2).

Similarly we have the following external version of Theorem 4.

Theorem 6. *For positive real numbers $x_i, y_i \in [m,M]$ for $i=1,2,\cdots,n$ with $0<m\le M$ and $\omega_1 \ge 0$ for $i=1,\cdots,n$ such that $\sum_{i=1}^n \omega_i = 1$,*

$$
\begin{aligned}
S(h)^{-2r-1}\left(\prod_{i=1}^n x_i^{\omega_i}\right)^{1+r}\left(\prod_{i=1}^n y_i^{\omega_i}\right)^{-r} &\le (1+r)\sum_{i=1}^n \omega_i x_i - r\sum_{i=1}^n \omega_i y_i\\
&\le S(h)^{1+r}\left(\prod_{i=1}^n x_i^{\omega_i}\right)^{1+r}\left(\prod_{i=1}^n y_i^{\omega_i}\right)^{-r},
\end{aligned}
$$

where where $h=\frac{M}{m}$ and the Specht ratio $S(h)$ is defined as (2.3).

3 Jensen operator inequality. By virture of an external formula, the Jensen operator inequality (1.2)

$$\langle f(A)x,x\rangle \le f(\langle Ax,x\rangle)$$

is regarded as an inequality in an internal formula. In [2], we showed the following external version of the Jensen operator inequality.

Theorem A. *Let f be a real valued function on an interval J. Then f is concave on J if and only if*

$$f\left(\langle Ax,x\rangle - \langle By,y\rangle\right) \le \| x \|^2\, f(\langle A\frac{x}{\| x \|}, \frac{x}{\| x \|}\rangle) - \langle f(B)y,y\rangle$$

for all $x,y\in H$ such that $\| x \|^2 - \| y \|^2 = 1$ and for all selfadjoint operators A and B with the spectra in J such that $\langle Ax,x\rangle - \langle By,y\rangle \in J$.

To show the fluctuation of concavity in an external formula, we need the following well-known lemma which is regarded as a reverse of the Jensen operator inequality in an internal formula, also see [3, Remark 2.7].

Lemma 7. *Let A be a selfadjoint operator on H with $mI \leq A \leq MI$ for some scalars $m \leq M$. If f is concave on $[m, M]$, then*

$$\| x \|^2 f\left(\langle A\frac{x}{\| x \|}, \frac{x}{\| x \|}\rangle\right) \leq \langle f(A)x, x\rangle + \mu(m, M, f) \| x \|^2 \tag{3.1}$$

for all nonzero vectors $x \in H$ where the bound $\mu(m, M, f)$ of concavity is defined by

$$\mu(m, M, f) = \max\left\{f(t) - \frac{f(M) - f(m)}{M - m}(t - m) + f(m) : t \in [m, M]\right\}.$$

Proof. For readers' convinence, we give a proof of Lemma 7. Put $y = x/ \| x \|$. Since f is concave, it follows that $\frac{f(M)-f(m)}{M-m}A + \frac{Mf(m)-mf(M)}{M-m}I \leq f(A)$. Therefore we have

$$\begin{aligned} f(\langle Ay, y\rangle) &\leq \frac{f(M) - f(m)}{M - m}\langle Ay, y\rangle + \frac{Mf(m) - mf(M)}{M - m} + \mu(m, M, f) \\ &= \langle\left(\frac{f(M) - f(m)}{M - m}A + \frac{Mf(m) - mf(M)}{M - m}\right)y, y\rangle + \mu(m, M, f) \\ &\leq \langle f(A)y, y\rangle + \mu(m, M, f). \end{aligned}$$

If we replace y by $x/ \| x \|$, then we have the desired inequality (3.1). □

Though the inequality $f(\langle Ax, x\rangle - \langle By, y\rangle) \leq \langle f(A)x, x\rangle - \langle f(B)y, y\rangle$ does not hold in general for $\| x \|^2 - \| y \|^2 = 1$, we show the fluctuation of concavity in an external formula by using the bound $\mu(m, M, f)$ of concavity.

Theorem 8. *If f is concave on $[m, M]$, then*

$$\begin{aligned} -\mu(m, M, f)(\| x \|^2 + \| y \|^2) &\leq f(\langle Ax, x\rangle - \langle By, y\rangle) - (\langle f(A)x, x\rangle - \langle f(B)y, y\rangle) \\ &\leq \mu(m, M, f) \| x \|^2 \end{aligned}$$

for all $x, y \in H$ such that $\| x \|^2 - \| y \|^2 = 1$ and for all selfadjoint operators A and B with the spectra in J such that $\langle Ax, x\rangle - \langle By, y\rangle \in J$.

Proof. By Theorem A and Lemma 7 it follows that

$$\begin{aligned} f(\langle Ax, x\rangle - \langle By, y\rangle) &\leq \| x \|^2 f(\langle A\frac{x}{\| x \|}, \frac{x}{\| x \|}\rangle) - \langle f(B)y, y\rangle \\ &\leq \langle f(A)x, x\rangle + \mu(m, M, f) \| x \|^2 - \langle f(B)y, y\rangle \end{aligned}$$

and this implies the second part of Theorem 8.

For the first part of Theorem 8, it follows from Lemma 7 that

$$\begin{aligned} &\langle f(A)x, x\rangle \leq \| x \|^2 f(\langle A\frac{x}{\| x \|}, \frac{x}{\| x \|}\rangle) \\ &\leq \| x \|^2 f\left(\frac{1}{1+ \| y \|^2}(\langle Ax, x\rangle - \langle By, y\rangle) + \frac{\| y \|^2}{1+ \| y \|^2}\langle B\frac{y}{\| y \|}, \frac{y}{\| y \|}\rangle\right) \\ &\leq \| x \|^2 \left(\frac{1}{1+ \| y \|^2}f(\langle Ax, x\rangle - \langle By, y\rangle) + \frac{\| y \|^2}{1+ \| y \|^2}f(\langle B\frac{y}{\| y \|}, \frac{y}{\| y \|}\rangle) + \mu(m, M, f)\right) \\ &= f(\langle Ax, x\rangle - \langle By, y\rangle) + \| y \|^2 f(\langle B\frac{y}{\| y \|}, \frac{y}{\| y \|}\rangle) + \mu(m, M, f) \| x \|^2 \\ &\leq f(\langle Ax, x\rangle - \langle By, y\rangle) + \langle f(B)y, y\rangle + \mu(m, M, f) \| y \|^2 + \mu(m, M, f) \| x \|^2 \end{aligned}$$

as desired. □

Remark 9. If $y = 0$ in Theorem 8, then we have Lemma 7. Hence Theorem 8 is an extension of (3.1).

As an application of Theorem 8, we show the fluctuation of the logarithm function in an external formula by means of the Specht ratio.

Corollary 10. *If $x_i, y_j \in [m, M]$ with $0 < m \leq M$ and $a_i, b_j \geq 0$ for $i = 1, \cdots, n$ and $j = 1, \cdots, k$ such that $\sum_{i=1}^n a_i - \sum_{j=1}^k b_j = 1$ and $\sum_{i=1}^n a_i x_i - \sum_{j=1}^k b_j y_j > 0$, then*

$$-\log S(h)\left(\sum_{i=1}^n a_i + \sum_{j=1}^k b_j\right) \leq \log\left(\sum_{i=1}^n a_i x_i - \sum_{j=1}^k b_j y_j\right) - \left(\sum_{i=1}^n a_i \log x_i - \sum_{j=1}^k b_j \log y_j\right)$$
$$\leq \log S(h) \sum_{i=1}^n a_i$$

where $h = \frac{M}{m}$ and the Specht ratio $S(h)$ is defined by (2.3).

4 Arithmetic-Geometric mean operator inequality.

Let A and B be positive operators on H. For each $\alpha \in [0,1]$, the weighted arithmetic meaan $A \nabla_\alpha B$ is defined as $A \nabla_\alpha B = (1-\alpha)A + \alpha B$ and the weighted geometric mean $A \sharp_\alpha B$ is defined as $A \sharp_\alpha B = A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^\alpha A^{\frac{1}{2}}$. Then the following arithmetic-geometric mean operator inequality holds:

$$A \sharp_\alpha B \leq A \nabla_\alpha B \tag{4.1}$$

for all positive operators A and B, and $\alpha \in [0,1]$. The inequality (4.1) is regarded as one in an internal formula. In the following theorem we propose the arithmetic-geometric mean operator inequality in the external formula.

Theorem 11. *Let A, B, C and D be positive operators on a Hilbert space H such that C, D are invertible. Then for each $\alpha \in [0,1]$*

$$2(A \nabla_\alpha B) - C \nabla_\alpha D \leq (A \nabla_\alpha B)(C \sharp_\alpha D)^{-1}(A \nabla_\alpha B).$$

Proof. Since the arithmetic-geometric mean operator inequality $C \sharp_\alpha D \leq C \nabla_\alpha D$ holds for each $\alpha \in [0,1]$, it follows that

$$\begin{aligned}
&(A \nabla_\alpha B)(C \sharp_\alpha D)^{-1}(A \nabla_\alpha B) - 2(A \nabla_\alpha B) + (C \nabla_\alpha D) \\
&= \left((C \sharp_\alpha D)^{-\frac{1}{2}}(A \nabla_\alpha B) - (C \sharp_\alpha D)^{\frac{1}{2}}\right)^* \left((C \sharp_\alpha D)^{-\frac{1}{2}}(A \nabla_\alpha B) - (C \sharp_\alpha D)^{\frac{1}{2}}\right) \\
&\qquad - (C \sharp_\alpha D) + (C \nabla_\alpha D) \\
&\geq 0
\end{aligned}$$

as desired. □

Remark 12. Theorem 11 is an extension of the arithmetic-geometric mean operator inequality. In fact, if we put $C = A$ and $D = B$ in Theorem 11, then we have $A \sharp_\alpha B \leq A \nabla_\alpha B$.

The inequality $2(A \nabla_\alpha B) - (C \nabla_\alpha D) \leq (A \sharp_\alpha B)(C \sharp_\alpha D)^{-1}(A \sharp_\alpha B)$ does not hold in general. However, we have the following theorem by virture of the Specht theorem.

Theorem 13. *Let A, B, C and D be positive invertible operators on H such that $mI \leq A, B \leq MI$ for some scalars $0 < m \leq M$. Then for each $\alpha \in [0,1]$*

$$2(A \nabla_\alpha B) - (C \nabla_\alpha D) \leq S(h)^2 (A \sharp_\alpha B)(C \sharp_\alpha D)^{-1}(A \sharp_\alpha B),$$

where $h = \frac{M}{m}$ and the Specht ratio $S(h)$ is defined as (2.3).

Proof. Since it follows from [7] that the Specht theorem $A\nabla_\alpha B \leq S(h)\ A\sharp_\alpha B$ holds for each $\alpha \in [0,1]$, we have

$$\begin{aligned}
&S(h)^2(A \sharp_\alpha B)(C \sharp_\alpha D)^{-1}(A \sharp_\alpha B) - 2(A \nabla_\alpha B) + (C \nabla_\alpha D) \\
&= \left(S(h)(C \sharp_\alpha D)^{-\frac{1}{2}}(A \sharp_\alpha B) - (C \sharp_\alpha D)^{\frac{1}{2}}\right)^* \left(S(h)(C \sharp_\alpha D)^{-\frac{1}{2}}(A \sharp_\alpha B) - (C \sharp_\alpha D)^{\frac{1}{2}}\right) \\
&\qquad + 2S(h)(A \sharp_\alpha B) - 2(A \nabla_\alpha B) - (C \sharp_\alpha D) + (C \nabla_\alpha D) \\
&\geq 0
\end{aligned}$$

as desired. □

References

[1] J.I. Fujii, *An external version of the Jensen operator inequality*, Sci. Math. Japon., Online, e-2011, 59–62.

[2] J.I. Fujii, J. Pečarić and Y. Seo, *The Jensen inequality in an external formula*, to appear in J. Math. Inequal.

[3] T. Furuta, J. Mićić Hot, J. Pečarić and Y. Seo, *Mond-Pečarić Method in Operator Inequalities*, Monographs in Inequalities 1, Element, Zagreb, 2005.

[4] B. Mond and J. Pečarić, *Convex inequalities in Hilbert space*, Houston J. Math., **19** (1993), 405–420.

[5] J. Pečarić, F. Proschan and Y.L. Tong, *Convex functions, Partial Orderings, and Statistical Applications*, Academic Press, Inc. 1992.

[6] W. Specht, *Zur Theorie der elementaren Mittel*, Math. Z. **74** (1960), 91–98.

[7] M. Tominaga, *Specht's ratio in the Young inequality*, Sci. Math. Japon., **55** (2002), 585–588.

Communicated by *Masatoshi Fujii*

Department of Mathematics Education, Osaka Kyoiku University, 4-698-1 Asahigaoka Kashiwara Osaka 582-8582 Japan.
E-mail address : yukis@cc.osaka-kyoiku.ac.jp