ON SOME TIME-NON-HOMOGENEOUS LINEAR DIFFUSION PROCESSES AND RELATED BRIDGES

dedicated to the memory of our "Maestro" Professor Luigi M. Ricciardi

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Abstract. In this paper we analyze the dynamics of the diffusion bridges (or tied-down diffusion processes), derived from time-non-homogeneous linear diffusion processes. For the Ornstein-Uhlenbeck and the Feller-type diffusion bridges, the distribution of first passage time through particular boundaries is determined.

1 Introduction One-dimensional diffusion processes are widely used for modeling the time evolution of dynamical systems in economics, finance, biology, genetics, physics, engineering, neuroscience, queueing and other fields (cf., for instance, [1]÷[39]). For many applications, it is often useful to consider a time-non-homogeneous linear diffusion process $\{X(t), t \geq 0\}$ obeying the stochastic differential equation:

$$dX(t) = A_1[X(t), t]\, dt + \sqrt{A_2[X(t), t]}\, dB(t), \qquad t \geq 0, \tag{1}$$

where $\{B(t), t \geq 0\}$ denotes the standard Wiener process and where the drift $A_1(x, t)$ and the infinitesimal variance $A_2(x, t) > 0$ are continuous functions, linear in the state variable x. The class of linear diffusion processes incorporates the Ornstein-Uhlenbeck process:

$$A_1(x, t) = \alpha(t)\, x + \beta(t) \qquad A_2(x, t) = \sigma^2(t), \qquad x \in \mathbb{R},\ t \geq 0, \tag{2}$$

where $\alpha(t) : \mathbb{R}^+ \to \mathbb{R}$, $\beta(t) : \mathbb{R}^+ \to \mathbb{R}$ and $\sigma(t) : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous functions. This class also includes the Feller-type diffusion process:

$$A_1(x, t) = \alpha(t)\, x + \beta(t), \qquad A_2(x, t) = 2\, \xi\, [\beta(t) + \nu\, \alpha(t)]\, (x - \nu), \qquad x > \nu,\ t \geq 0, \tag{3}$$

where $\alpha(t) : \mathbb{R}^+ \to \mathbb{R}$ and $\beta(t) : \mathbb{R}^+ \to \mathbb{R}$ are continuous functions, with $\xi > 0$, $\nu \in \mathbb{R}$ and $\beta(t) + \nu\, \alpha(t) > 0$.

Diffusion processes of type (2) and (3) play an important role in the description of input-output behavior of single neurons subject to a diffusion-like dynamics. In particular, in the Ornstein-Uhlenbeck neuronal model, the membrane potential is modeled by a time-homogeneous one-dimensional diffusion process, characterized by the following infinitesimal moments $A_1(x) = -(x - \varrho)/\vartheta$ and $A_2 = \sigma^2$ ($\varrho \in \mathbb{R}, \sigma > 0, \vartheta > 0$). In this model, in the absence of inputs, the membrane potential exponentially decays to the resting potential ϱ with a time constant ϑ (cf., for instance, [13], [37], [38]). Although the Ornstein-Uhlenbeck neuronal model rests on several assumptions that can be justified on neurophysiological grounds, the state space for the underlying stochastic process is identified with the entire real axis, implying that arbitrarily large hyperpolarizations are possible. Some authors have

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thus suggested alternative models in which the changes in the depolarization of a nerve cell are state-dependent in a way to restrict the magnitude of the membrane potential to a finite interval. The essential point here is to assume the existence of a "reversal potential", so that the magnitude of postsynaptic potentials increases as the membrane potential departs from a preassigned fixed value that is taken as the left end point of the range in which the membrane potential is allowed to vary. The Feller neuronal model is a time-homogeneous diffusion process defined on $(\nu, +\infty)$ and characterized by the infinitesimal moments $A_1(x) = -(x-\rho)/\vartheta$, $A_2(x) = 2\xi(x-\nu)$ $(\rho, \nu \in \mathbb{R}, \varrho > \nu, \vartheta > 0, \xi > 0)$ (cf., for instance, [25]). Boundary $x = \nu$ is regular if $\varrho - \nu < \xi\,\vartheta$ and entrance if $\varrho - \nu \geq \xi\vartheta$, whereas $x = +\infty$ is natural. Ornstein-Uhlenbeck and Feller models have identical drifts involving parameters ϑ and ϱ. Infinitesimal variances are instead dramatically different from one another, both functionally and in terms of the involved parameters. In some cases it is necessary to include non-stationary effects in the neuronal models in order to be able to take in account the circumstance that the firing frequency is subject to some kind of modulation. For instance, the non-homogeneous neuronal process $A_1(x,t) = -(x-\rho)/\vartheta + L(t)$ and $A_2(x) = \sigma^2$ includes a time-dependent extra-effect induced by some kind of external stimulation acting on the neuron (cf., for instance, [11], [12], [18]).

Diffusion processes of type (2) and (3) are also used in the description of queueing systems. For instance, in [24] the Ornstein-Uhlenbeck diffusion process characterized by drift and infinitesimal variance $A_1(x,t) = (\lambda - \mu)k(t)\,x$ and $A_2(x,t) = \sigma^2 k(t)$, restricted to the interval $[0, +\infty)$ with a reflecting boundary at 0, is taken in account, whereas in [17] the Feller-type diffusion process $A_1(x,t) = \beta(t)x + \gamma\alpha(t)$ and $A_2(x,t) = 2\alpha(t)x$, with $\gamma > 0$ and $\beta(t) > 0$ is analyzed.

Linear diffusion processes are also widely used in mathematical finance for modelling asset prices, market indices, interest rates and stochastic volatility (cf., for instance, [6], [9], [16], [27], [31], [39]).

The first-passage time (FPT) problem plays a relevant role in a wide range of applications in mathematics, physics, biology and finance. Mathematically, such a problem can be reduced to estimate the probability that a stochastic process reaches a critical level or threshold for the first time. For instance, in neuronal models one is mainly interested to determination of the firing probability density function (pdf), i.e. the FPT pdf through the threshold potential, denoted by $S(t)$, that is customarily assumed to be a deterministic function of time. In queueing system, one would like to have information on the busy period, i.e. on the first hitting time to state 0. Further, many problems in mathematical finance require some informations on FPT of a diffusion process, as the triggering of stock options. However, apart from a few special cases, no closed form expressions are available in the literature to determine FPT densities for time-dependent boundaries.

In many instances, it is often useful to analyze the dynamics of a stochastic bridge $\{Y(t), t \geq 0\}$, derived from a diffusion process $X(t)$, by conditioning it not only on its initial point x_0 at time t_0, but also on its ending point x_1 at time t_1, where $0 \leq t_0 < t < t_1$. Important examples are provided by Wiener bridges and Bessel bridges, which have been extensively studied and applied in mathematical finance, neurobiology, simulation of markovian processes, and in various other applied fields (cf., for instance, [3], [4], [7], [10], [21], [29], [34]). In particular, in [3] and [4], the authors derive bridges from general multidimensional linear non time-homogeneous Ornstein-Uhlenbeck processes using only the transition densities of the original process and specialize their results to the one-dimensional case. Instead, in [32] a time-homogeneous squared Bessel process is considered and its transition density is determined. Furthermore, in [26], a Monte Carlo method relying on the estimation of the tied-down crossing probabilities is proposed and some examples of applications concerning the evaluation of FPT pdf for time-homogeneous Ornstein-Uhlenbeck and Feller diffusion processes are given. Moreover, the simulation of diffusion bridges plays a fundamental role in likelihood and Bayesian inference for diffusion-type processes (cf., for instance, [8], [22]).

The purpose of this paper is to analyze the dynamics of diffusion bridge $\{Y(t), t \geq 0\}$, derived from a time-non-homogeneous linear diffusion process $X(t)$, by conditioning $X(t)$ to start from x_0 at time t_0 and arrive at x_1 at time t_1, where $0 \leq t_0 < t < t_1$.

Sections 2 and 4 are devoted to explore some properties of Gauss-Markov bridge and of time-non-homogeneous diffusion bridge, respectively. Particular attention is dedicated to FPT problem and some relations between the FPT densities of $X(t)$ and those of the derived stochastic bridge $Y(t)$ are explicitly given. Furthermore, in Section 3 the time-non-homogeneous Ornstein-Uhlenbeck diffusion process (2) is considered, whereas in Section 5 we examine the Feller-type diffusion process (3). Closed form expressions for FPT densities are explicitly obtained; they provide a tool to test the accuracy of numerical or simulation procedures.

We want to dedicate the remainder of this paper to the memory of our late mentor, colleague and unforgettable friend, Luigi M. Ricciardi.

2 FPT for Gauss-Markov Bridge In this section, starting from Gauss-Markov processes, we derive stochastic bridges and for them we analyze the FPT problem.

Let $\{X(t), t \in T\}$, where T is a continuous parameter set, be a real continuous Gauss-Markov process with the following properties (cf. [1], [33]):

(i) $m_X(t) := \mathrm{E}[X(t)]$ is continuous in T;

(ii) the covariance $c_X(s,t) := \mathrm{E}\{[X(s) - m_X(s)]\,[X(t) - m_X(t)]\}$ is continuous in $T \times T$ and for $s \leq t$, one has $c_X(s,t) = h_1(s)h_2(t)$, where $h_1(t)$ and $h_2(t)$ are continuous functions in T;

(iii) $X(t)$ is non-singular for each $t \in T$, except possibly singular on ∂T; for instance, if $T = [a, +\infty[$, one could have $X(a) = m_X(a)$ with probability 1.

Any Gauss-Markov process can be represented in terms of the standard Wiener process $\{W(t), t \geq 0\}$ as

$$X(t) = m_X(t) + h_2(t)\, W\big[r(t)\big], \tag{4}$$

where

$$r(t) = \frac{h_1(t)}{h_2(t)} \tag{5}$$

is a monotonically increasing function. Furthermore, the transition pdf $f_X(x,t|y,\tau)$ of the Gauss-Markov process $\{X(t), t \in T\}$ is a normal density, characterized respectively by mean and variance:

$$\begin{aligned} &\mathrm{E}[X(t)|X(\tau)=y] = m_X(t) + \frac{h_2(t)}{h_2(\tau)}\,\big[y - m_X(\tau)\big], \\ &\mathrm{Var}[X(t)|X(\tau)=y] = h_2(t)\Big[h_1(t) - \frac{h_2(t)}{h_2(\tau)}h_1(\tau)\Big]. \end{aligned} \qquad (t,\tau \in T, \tau < t) \tag{6}$$

Let x, y be admissible states of $\{X(t), t \in T\}$ and $\tau < t$, with $\tau, t \in T$. We assume that $m_X(t), h_1(t), h_2(t) \in C^1(T)$. Then, the transition pdf $f_X(x,t|y,\tau)$ satisfies the Fokker-Planck equation and the associated initial condition

$$
\begin{aligned}
&\frac{\partial f_X(x,t|y,\tau)}{\partial t} = -\frac{\partial}{\partial x}\,[A_1(x,t)\, f_X(x,t|y,\tau)] + \frac{1}{2}\,\frac{\partial^2}{\partial x^2}\,[A_2(t)\, f_X(x,t|y,\tau)],\\
&\lim_{\tau \uparrow t} f(x,t|y,\tau) = \delta(x-y),
\end{aligned}
\tag{7}
$$

with $A_1(x,t)$ and $A_2(t)$ given by

$$
A_1(x,t) = m'_X(t) + \big[x - m_X(t)\big]\,\frac{h'_2(t)}{h_2(t)}\,, \qquad A_2(t) = h_2^2(t)\, r'(t), \tag{8}
$$

the prime denoting derivative with respect to the argument.

We now consider the FPT of the Gauss-Markov process $X(t)$ from $X(\tau) = y$ to the continuous boundary $S(t)$ $(t \in T)$ and for $\tau, t \in T$ denote by

$$
\mathcal{T}_y = \begin{cases} \inf\limits_{t \geq \tau}\big\{t : X(t) > S(t)\big\} & \text{if} \quad X(\tau) = y < S(\tau),\\[2ex] \inf\limits_{t \geq \tau}\big\{t : X(t) < S(t)\big\} & \text{if} \quad X(\tau) = y > S(\tau), \end{cases} \tag{9}
$$

the FPT random variable. By virtue of (4), the FPT pdf is given by

$$
g_X[S(t),t|y,\tau] = \frac{\partial}{\partial t} P\big(\mathcal{T}_y < t\big) = \frac{dr(t)}{dt}\; g_W\big\{S^*[r(t)], r(t)|y^*, r(\tau)\big\}, \tag{10}
$$

where $r(t)$ is defined in (5) and $g_W[S^*(\vartheta),\vartheta|y^*,\vartheta_0]$ is the FPT pdf of $W(\vartheta)$ from y^* at time ϑ_0 to the continuous boundary $S^*(\vartheta)$, with

$$
y^* = \frac{y - m_X[r^{-1}(\vartheta_0)]}{h_2[r^{-1}(\vartheta_0)]}, \qquad S^*(\vartheta) = \frac{S[r^{-1}(\vartheta)] - m_X[r^{-1}(\vartheta)]}{h_2[r^{-1}(\vartheta)]}. \tag{11}
$$

Apart from a few special cases, FPT densities are not known, so that efficient algorithms are developed in the literature to determine $g_X[S(t),t|y,\tau]$ (cf., for instance, [12], [19], [36]). However, in [19] some closed form expressions for FPT pdf are obtained by making use of certain transformations among Gauss-Markov processes. Indeed, by setting

$$
S(t) = m_X(t) + d_1 h_1(t) + d_2 h_2(t) \qquad (d_1, d_2 \in \mathbb{R}) \tag{12}
$$

for all $t \in T$, the FPT pdf through a boundary $S(t)$ is given by

$$
g_X[S(t),t|y,\tau] = \frac{|S(\tau)-y|}{r(t)-r(\tau)}\,\frac{h_2(t)}{h_2(\tau)}\,\frac{dr(t)}{dt}\, f_X[S(t),t|y,\tau], \quad [y < S(\tau)] \text{ or } [y > S(\tau)]. \tag{13}
$$

Furthermore, by choosing $S(t)$ as in (12), if $T = [a,b]$ and $\lim_{t\to b} r(t) = +\infty$, then

$$
\int_\tau^b g_X[S(t),t|y,\tau]\, dt = \begin{cases} 1, & \dfrac{d_1[S(\tau)-y]}{h_2(\tau)} \leq 0\\[3ex] \exp\Big\{-\dfrac{2d_1\,[S(\tau)-y]}{h_2(\tau)}\Big\}, & \dfrac{d_1[S(\tau)-y]}{h_2(\tau)} > 0. \end{cases} \tag{14}
$$

Let us now consider the bridge process $\{Y(t), t \in [t_0, t_1]\}$, obtained by conditioning $\{X(t), t \in T\}$ to start from x_0 at time t_0 and arrive at x_1 at time t_1, where $t_0, t_1 \in T$ and $t_0 < t_1$. The process $\{Y(t), t \in [t_0, t_1]\}$ is a Gauss-Markov process (cf., for instance, [1], [33]) with mean function

$$\begin{aligned} m_Y(t) = E[Y(t)] &= m_X(t) + \big[x_0 - m_X(t_0)\big] \frac{h_1(t_1)\,h_2(t) - h_1(t)\,h_2(t_1)}{h_1(t_1)\,h_2(t_0) - h_1(t_0)\,h_2(t_1)} \\ &+ \big[x_1 - m_X(t_1)\big] \frac{h_1(t)\,h_2(t_0) - h_1(t_0)\,h_2(t)}{h_1(t_1)\,h_2(t_0) - h_1(t_0)\,h_2(t_1)} \quad (t_0 < t < t_1) \end{aligned} \tag{15}$$

and covariance function

$$c_Y(s,t) = \mathrm{E}\{[Y(s) - m_Y(s)]\,[Y(t) - m_Y(t)]\} = H_1(s)\,H_2(t) \quad (t_0 \le s \le t < t_1), \tag{16}$$

where

$$H_1(t) = \frac{h_1(t)\,h_2(t_0) - h_1(t_0)\,h_2(t)}{h_1(t_1)\,h_2(t_0) - h_1(t_0)\,h_2(t_1)}, \qquad H_2(t) = h_1(t_1)\,h_2(t) - h_1(t)\,h_2(t_1). \tag{17}$$

Hence, $\{Y(t), t \in [t_0, t_1]\}$ can be represented in terms of the standard Wiener process $\{W(t), t \ge 0\}$ as

$$Y(t) = m_Y(t) + H_2(t)\,W\big[R(t)\big], \tag{18}$$

where $R(t) = H_1(t)/H_2(t)$ is a monotonically increasing function. The transition pdf $f_Y(x,t|y,\tau)$ of the bridge process is a normal density with conditional mean and variance:

$$\begin{aligned} \mathrm{E}[Y(t)|Y(\tau) = y] &= m_X(t) + [y - m_X(\tau)]\, \frac{h_1(t_1)\,h_2(t) - h_1(t)\,h_2(t_1)}{h_1(t_1)\,h_2(\tau) - h_1(\tau)\,h_2(t_1)} \\ &+ [x_1 - m_X(t_1)]\, \frac{h_1(t)\,h_2(\tau) - h_1(\tau)\,h_2(t)}{h_1(t_1)\,h_2(\tau) - h_1(\tau)\,h_2(t_1)}, \\ & \qquad\qquad\qquad\qquad\qquad\qquad (t_0 \le \tau < t < t_1) \\ \mathrm{Var}[Y(t)|Y(\tau) = y] &= \frac{[h_1(t_1)\,h_2(t) - h_1(t)\,h_2(t_1)]\,[h_1(t)\,h_2(\tau) - h_1(\tau)\,h_2(t)]}{h_1(t_1)\,h_2(\tau) - h_1(\tau)\,h_2(t_1)}. \end{aligned} \tag{19}$$

Furthermore, $f_Y(x,t|y,\tau)$ satisfies the Fokker-Planck equation and the associated delta initial condition, with drift and infinitesimal variance given by:

$$\begin{aligned} B_1(x,t) &= m'_X(t) + [x - m_X(t)]\, \frac{h_1(t_1)\,h'_2(t) - h'_1(t)\,h_2(t_1)}{h_1(t_1)\,h_2(t) - h_1(t)\,h_2(t_1)} \\ &+ [x_1 - m_X(t_1)]\, \frac{h'_1(t)\,h_2(t) - h_1(t)\,h'_2(t)}{h_1(t_1)\,h_2(t) - h_1(t)\,h_2(t_1)}, \\ & \qquad\qquad\qquad\qquad\qquad\qquad (t_0 < t < t_1) \\ B_2(t) &= h_2^2(t)\,r'(t), \end{aligned} \tag{20}$$

showing that the infinitesimal variance of $\{Y(t), t \ge 0\}$ is the same of the process $X(t)$. Moreover, it is easy to prove that $f_Y(x,t|y,\tau)$ is connected to $f_X(x,t|y,\tau)$ by the relation:

$$f_Y(x,t|y,\tau) = f_X(x,t|y,\tau)\, \frac{f_X(x_1,t_1|x,t)}{f_X(x_1,t_1|y,\tau)} \qquad (t_0 \le \tau < t < t_1). \tag{21}$$

We now consider the FPT of the Gauss-Markov bridge $Y(t)$ from $Y(\tau)=y$ to a continuous time-dependent boundary $S(t)$ $(t_0<t<t_1)$. Since $Y(t)$ is a Markov process, the FPT pdf $g_Y[S(t),t|y,\tau]$ is solution of first-kind Volterra integral equation

$$f_Y[x,t|y,\tau]=\int_\tau^t g_Y[S(\vartheta),\vartheta|y,\tau]\,f_Y[x,t|S(\vartheta),\vartheta]\,d\vartheta \tag{22}$$
$$\big([y<S(\tau),x\ge S(t)]\text{ or }[y>S(\tau),x\le S(t)]\big).$$

In order to evaluate $g_Y[S(t),t|y,\tau]$ we make use of the relation (21). Indeed, for $t_0\le\tau<t<t_1$, making use of (21) in (22), one obtains:

$$f_X(x,t|y,\tau)=\int_\tau^t\Big\{g_Y[S(\vartheta),\vartheta|y,\tau]\,\frac{f_X(x_1,t_1|y,\tau)}{f_X[x_1,t_1|S(\vartheta),\vartheta]}\Big\}\,f_X[x,t|S(\vartheta),\vartheta]\,d\vartheta$$
$$\big([y<S(\tau),x\ge S(t)]\text{ or }[y>S(\tau),x\le S(t)]\big),$$

so that, for $t_0\le\tau<t<t_1$, one has:

$$g_Y[S(t),t|y,\tau]=g_X[S(t),t|y,\tau]\,\frac{f_X[x_1,t_1|S(t),t]}{f_X(x_1,t_1|y,\tau)}\qquad\big([y<S(\tau)]\text{ or }[y>S(\tau)]\big) \tag{23}$$

with $g_X[S(t),t|y,\tau]$ given in (10). By virtue of (21), an alternative expression for the FPT pdf of the Gauss-Markov bridge is:

$$g_Y[S(t),t|y,\tau]=\frac{g_X[S(t),t|y,\tau]}{f_X[S(t),t|y,\tau]}\,f_Y[S(t),t|y,\tau]\qquad\big([y<S(\tau)]\text{ or }[y>S(\tau)]\big). \tag{24}$$

In particular, if the threshold of $Y(t)$ is the one given in the right-hand side of (12), with $d_1,d_2\in\mathbb{R}$, by virtue of (13), (21) and (23), for $t_0\le\tau<t<t_1$, one obtains the closed form expression:

$$g_Y[S(t),t|y,\tau]=\frac{|S(\tau)-y|}{r(t)-r(\tau)}\,\frac{h_2(t)}{h_2(\tau)}\,\frac{dr(t)}{dt}\,f_Y[S(t),t|y,\tau]\quad\big([y<S(\tau)]\text{ or }[y>S(\tau)]\big). \tag{25}$$

We note that the threshold $S(t)$ given in (12) can be also rewritten as:

$$S(t)=m_Y(t)+D_1\,H_1(t)+D_2\,H_2(t), \tag{26}$$

with $m_Y(t)$ and $H_i(t)$ $(i=1,2)$ given in (15) and (17), respectively, and where

$$D_1=m_X(t_1)+d_1h_1(t_1)+d_2h_2(t_1)-x_1=S(t_1)-x_1,$$
$$D_2=\frac{\big[m_X(t_0)+d_1h_1(t_0)+d_2h_2(t_0)-x_0\big]}{H_2(t_0)}=\frac{S(t_0)-x_0}{H_2(t_0)}.$$

Hence, an alternative expression for (25) is:

$$g_Y[S(t),t|y,\tau]=\frac{|S(\tau)-y|}{R(t)-R(\tau)}\,\frac{H_2(t)}{H_2(\tau)}\,\frac{dR(t)}{dt}\,f_Y[S(t),t|y,\tau]\quad\big([y<S(\tau)]\text{ or }[y>S(\tau)]\big),$$

that allows to determine the first-passage time probability through (12) for the Gauss-Markov bridge. Indeed, by choosing $S(t)$ as in (12), if $\lim_{t\to t_1}R(t)=+\infty$, then

$$\int_\tau^{t_1}g_Y[S(t),t|y,\tau]\,dt=\begin{cases}1, & \dfrac{[S(t_1)-x_1]\,[S(\tau)-y]}{H_2(\tau)}\le 0\\[2ex] \exp\Big\{-\dfrac{2[S(t_1)-x_1]\,[S(\tau)-y]}{H_2(\tau)}\Big\}, & \dfrac{[S(t_1)-x_1]\,[S(\tau)-y]}{H_2(\tau)}>0.\end{cases} \tag{27}$$

3 Time-non-homogeneous Ornstein-Uhlenbeck bridge Let now consider a Gauss-Markov process $\{X(t), t \geq 0\}$ characterized by mean and covariance functions:

$$
\begin{aligned}
&m_X(t) = \int_0^t \beta(\vartheta) \exp\Big\{ \int_\vartheta^t \alpha(u)\, du \Big\}\, d\vartheta, \\
&c_X(s,t) = \exp\Big\{ \int_s^t \alpha(\vartheta)\, d\vartheta \Big\} \int_0^s \sigma^2(\vartheta) \exp\Big\{ 2 \int_\vartheta^s \alpha(u)\, du \Big\}\, d\vartheta \qquad (0 \leq s \leq t < +\infty),
\end{aligned}
\tag{28}
$$

where $\alpha(t) : \mathbb{R}^+ \to \mathbb{R}$, $\beta(t) : \mathbb{R}^+ \to \mathbb{R}$ and $\sigma(t) : \mathbb{R}^+ \to \mathbb{R}^+$ are continuous functions. Hence, by setting

$$
\varphi(t) = \int_0^t \alpha(\vartheta)\, d\vartheta \quad (t \geq 0), \tag{29}
$$

from (5) and (28), for $t \geq 0$ we can make the following choices:

$$
h_1(t) = e^{-\varphi(t)} \int_0^t \sigma^2(\vartheta)\, e^{2[\varphi(t)-\varphi(\vartheta)]}\, d\vartheta, \;\; h_2(t) = e^{\varphi(t)}, \;\; r(t) = \int_0^t \sigma^2(\vartheta)\, e^{-2\varphi(\vartheta)}\, d\vartheta. \tag{30}
$$

By virtue of (6), we obtain the conditional mean and variance of $X(t)$, having the normal density $f_X(x,t|y,\tau)$:

$$
\begin{aligned}
&\mathrm{E}[X(t)|X(\tau)=y] = y\, e^{\varphi(t)-\varphi(\tau)} + \int_\tau^t \beta(\vartheta)\, e^{\varphi(t)-\varphi(\vartheta)}\, d\vartheta, \\
&\mathrm{Var}[X(t)|X(\tau)=y] = \int_\tau^t \sigma^2(\vartheta)\, e^{2[\varphi(t)-\varphi(\vartheta)]}\, d\vartheta,
\end{aligned}
\qquad (0 \leq \tau < t < +\infty) \tag{31}
$$

and, making use of (8), one derives the infinitesimal moments (2) of the time-non-homogeneous Ornstein-Uhlenbeck diffusion process:

$$
A_1(x,t) = \alpha(t)\, x + \beta(t) \qquad A_2(x,t) = \sigma^2(t), \qquad x \in \mathbb{R},\ t \geq 0.
$$

Furthermore, recalling (25), the FPT pdf through the boundary

$$
S(t) = e^{\varphi(t)} \Big[d_2 + d_1 \int_0^t \sigma^2(\vartheta)\, e^{-2\varphi(\vartheta)}\, d\vartheta + \int_0^t \beta(\vartheta)\, e^{-\varphi(\vartheta)}\, d\vartheta \Big] \qquad (d_1, d_2 \in \mathbb{R}), \tag{32}
$$

for $0 \leq \tau < t < +\infty$ admits a closed form expression:

$$
g_X[S(t),t|y,\tau] = \frac{\sigma^2(t)\, e^{\varphi(t)-\varphi(\tau)}\, |S(\tau)-y|}{\displaystyle\int_\tau^t \sigma^2(\vartheta)\, e^{2[\varphi(t)-\varphi(\vartheta)]}\, d\vartheta}\, f_X[S(t),t|y,\tau] \quad \big(y < S(\tau) \text{ or } y > S(\tau)\big). \tag{33}
$$

When $\lim_{t\to+\infty} r(t) = +\infty$, the FPT probability through (32) follows from (14):

$$
\int_\tau^{+\infty} g_X[S(t),t|y,\tau]\, dt = \begin{cases} 1, & d_1[S(\tau)-y] \leq 0 \\ \exp\Big\{ -2 d_1\, e^{-\varphi(\tau)} [S(\tau)-y] \Big\}, & d_1[S(\tau)-y] > 0. \end{cases} \tag{34}
$$

We now can construct the corresponding Gauss-Markov bridge $\{Y(t), t \in [t_0, t_1]\}$, obtained by conditioning $X(t)$ to start from x_0 at time t_0 and arrive at x_1 at time t_1, where $0 \le t_0 < t_1 < +\infty$. Recalling (15) and (16), for $t_0 \le s \le t < t_1$ mean and covariance functions of $Y(t)$ are:

$$\begin{aligned}
m_Y(t) &= \left[x_0\, e^{\varphi(t)-\varphi(t_0)} + \int_{t_0}^{t} \beta(\vartheta)\, e^{\varphi(t)-\varphi(\vartheta)}\, d\vartheta\right] \frac{\displaystyle\int_t^{t_1} \sigma^2(\vartheta)\, e^{2[\varphi(t_1)-\varphi(\vartheta)]}\, d\vartheta}{\displaystyle\int_{t_0}^{t_1} \sigma^2(\vartheta)\, e^{2[\varphi(t_1)-\varphi(\vartheta)]}\, d\vartheta} \\
&\quad + e^{\varphi(t_1)-\varphi(t)} \left[x_1 - \int_t^{t_1} \beta(\vartheta)\, e^{\varphi(t_1)-\varphi(\vartheta)}\, d\vartheta\right] \frac{\displaystyle\int_{t_0}^{t} \sigma^2(\vartheta)\, e^{2[\varphi(t)-\varphi(\vartheta)]}\, d\vartheta}{\displaystyle\int_{t_0}^{t_1} \sigma^2(\vartheta)\, e^{2[\varphi(t_1)-\varphi(\vartheta)]}\, d\vartheta},
\end{aligned} \tag{35}$$

$$c_Y(s,t) = \frac{e^{\varphi(t)-\varphi(s)} \left[\displaystyle\int_{t_0}^{s} \sigma^2(\vartheta)\, e^{2[\varphi(s)-\varphi(\vartheta)]}\, d\vartheta\right] \left[\displaystyle\int_t^{t_1} \sigma^2(\vartheta)\, e^{2[\varphi(t_1)-\varphi(\vartheta)]}\, d\vartheta\right]}{\displaystyle\int_{t_0}^{t_1} \sigma^2(\vartheta)\, e^{2[\varphi(t_1)-\varphi(\vartheta)]}\, d\vartheta}.$$

For $t_0 < t < t_1$, by setting

$$H_1(t) = \frac{e^{\varphi(t_1)-\varphi(t)} \displaystyle\int_{t_0}^{t} \sigma^2(\vartheta)\, e^{2[\varphi(t)-\varphi(\vartheta)]}\, d\vartheta}{\displaystyle\int_{t_0}^{t_1} \sigma^2(\vartheta)\, e^{2[\varphi(t_1)-\varphi(\vartheta)]}\, d\vartheta},$$

$$H_2(t) = e^{-[\varphi(t_1)-\varphi(t)]} \int_t^{t_1} \sigma^2(\vartheta)\, e^{2[\varphi(t_1)-\varphi(\vartheta)]}\, d\vartheta,$$

we note that the covariance function of $Y(t)$ is such that $c_Y(s,t) = H_1(s)\, H_2(t)$ and $R(t) = H_1(t)/H_2(t)$ is a monotonically increasing function such that $\lim_{t\to t_1} R(t) = +\infty$. From (20) we obtain the drift and the infinitesimal variance of the time-non-homogeneous Ornstein-Uhlenbeck bridge:

$$\begin{aligned}
B_1(x,t) &= \alpha(t)x + \beta(t) + \frac{\sigma^2(t)\, e^{\varphi(t_1)-\varphi(t)} \left[x_1 - x\, e^{\varphi(t_1)-\varphi(t)} - \displaystyle\int_t^{t_1} \beta(\vartheta)\, e^{\varphi(t_1)-\varphi(\vartheta)}\, d\vartheta\right]}{\displaystyle\int_t^{t_1} \sigma^2(\vartheta)\, e^{2[\varphi(t_1)-\varphi(\vartheta)]}\, d\vartheta}, \\
& \hspace{20em} (t < t_1) \\
B_2(t) &= \sigma^2(t).
\end{aligned} \tag{36}$$

Hence, by virtue of (19), for $t_0 \le \tau < t < t_1$ we obtain the mean and the variance of the normal density $f_Y(x,t|y,\tau)$ of the diffusion process (36):

$$\mathrm{E}[Y(t)|Y(\tau)=y] = \Big[y\, e^{\varphi(t)-\varphi(\tau)} + \int_\tau^t \beta(\vartheta)\, e^{\varphi(t)-\varphi(\vartheta)}\, d\vartheta\Big] \frac{\displaystyle\int_t^{t_1} \sigma^2(\vartheta)\, e^{2[\varphi(t_1)-\varphi(\vartheta)]}\, d\vartheta}{\displaystyle\int_\tau^{t_1} \sigma^2(\vartheta)\, e^{2[\varphi(t_1)-\varphi(\vartheta)]}\, d\vartheta}$$

$$+ e^{\varphi(t_1)-\varphi(t)} \Big[x_1 - \int_t^{t_1} \beta(\vartheta)\, e^{\varphi(t_1)-\varphi(\vartheta)}\, d\vartheta\Big] \frac{\displaystyle\int_\tau^{t} \sigma^2(\vartheta)\, e^{2[\varphi(t)-\varphi(\vartheta)]}\, d\vartheta}{\displaystyle\int_\tau^{t_1} \sigma^2(\vartheta)\, e^{2[\varphi(t_1)-\varphi(\vartheta)]}\, d\vartheta},$$

$$(37) \qquad\qquad (t_0 \le \tau < t < t_1)$$

$$\mathrm{Var}[Y(t)|Y(\tau)=y] = \frac{\Big[\displaystyle\int_\tau^{t} \sigma^2(\vartheta)\, e^{2[\varphi(t)-\varphi(\vartheta)]}\, d\vartheta\Big]\Big[\displaystyle\int_t^{t_1} \sigma^2(\vartheta)\, e^{2[\varphi(t_1)-\varphi(\vartheta)]}\, d\vartheta\Big]}{\displaystyle\int_\tau^{t_1} \sigma^2(\vartheta)\, e^{2[\varphi(t_1)-\varphi(\vartheta)]}\, d\vartheta}.$$

Expressions (36) and (37) are in agreement with those obtained by Barczy and Kern in [4]. Furthermore, for the Ornstein-Uhlenbeck bridge, when $\tau < t < t_1$ the FPT pdf through the boundary (32) follows from (25):

$$(38)\quad g_Y[S(t),t|y,\tau] = \frac{\sigma^2(t)\, e^{\varphi(t)-\varphi(\tau)}\, |S(\tau)-y|}{\displaystyle\int_\tau^t \sigma^2(\vartheta)\, e^{2[\varphi(t)-\varphi(\vartheta)]}\, d\vartheta}\, f_Y[S(t),t|y,\tau] \quad \big(y < S(\tau) \text{ or } y > S(\tau)\big).$$

Hence, the FPT probability for the Ornstein-Uhlenbeck bridge through (32) can be obtained from (27):

$$\int_\tau^{t_1} g_Y[S(t),t|y,\tau]\, dt$$

$$(39)\qquad = \begin{cases} 1, & [S(t_1)-x_1]\,[S(\tau)-y] \le 0 \\[2ex] \exp\Big\{-\dfrac{2[S(t_1)-x_1]\,[S(\tau)-y]e^{\varphi(t_1)-\varphi(\tau)}}{\int_\tau^{t_1} \sigma^2(\vartheta)\, e^{2[\varphi(t_1)-\varphi(\vartheta)]}\, d\vartheta}\Big\}, & [S(t_1)-x_1]\,[S(\tau)-y] > 0. \end{cases}$$

In the sequel we specialize the above results to the time-non-homogeneous Wiener bridge and to the time-homogeneous Ornstein-Uhlenbeck bridge.

3.1 Time-non-homogeneous Wiener bridge Let $\{X(t), t \ge 0\}$ a Gauss-Markov process characterized by mean and covariance functions:

$$(40)\qquad m_X(t) = \int_0^t \beta(\vartheta)\, d\vartheta, \qquad c_X(s,t) = \int_0^s \sigma^2(\vartheta)\, d\vartheta \qquad (0 < s \le t < +\infty),$$

with $\beta(t): \mathbb{R}^+ \to \mathbb{R}$ and $\sigma(t): \mathbb{R}^+ \to \mathbb{R}^+$ continuous functions. Relations (40) follow from (28) by setting $\alpha(t) = 0$. By virtue of (8), one obtains the drift $A_1(t) = \beta(t)$ and the infinitesimal variance $A_2(t) = \sigma^2(t)$ of a time-non-homogeneous Wiener diffusion process, whose transition pdf $f_X(x,t|y,\tau)$ is normal with mean and variance:

$$(41)\quad \mathrm{E}[X(t)|X(\tau)=y] = y + \int_\tau^t \beta(\vartheta)\, d\vartheta, \quad \mathrm{Var}[X(t)|X(\tau)=y] = \int_\tau^t \sigma^2(\vartheta)\, d\vartheta \quad (\tau < t).$$

The corresponding Gauss-Markov bridge $\{Y(t), t \in [t_0, t_1]\}$, obtained by conditioning the Wiener process to start from x_0 at time t_0 and arrive at x_1 at time t_1, where $0 \le t_0 < t_1 < +\infty$, is characterized by mean and covariance functions:

$$\begin{aligned} m_Y(t) &= \left[x_0 + \int_{t_0}^{t} \beta(\vartheta)\, d\vartheta\right] \frac{\displaystyle\int_t^{t_1} \sigma^2(\vartheta)\, d\vartheta}{\displaystyle\int_{t_0}^{t_1} \sigma^2(\vartheta)\, d\vartheta} + \left[x_1 - \int_t^{t_1} \beta(\vartheta)\, d\vartheta\right] \frac{\displaystyle\int_{t_0}^{t} \sigma^2(\vartheta)\, d\vartheta}{\displaystyle\int_{t_0}^{t_1} \sigma^2(\vartheta)\, d\vartheta}, \\ c_Y(s,t) &= \frac{\left[\displaystyle\int_{t_0}^{s} \sigma^2(\vartheta)\, d\vartheta\right]\left[\displaystyle\int_t^{t_1} \sigma^2(\vartheta)\, d\vartheta\right]}{\displaystyle\int_{t_0}^{t_1} \sigma^2(\vartheta)\, d\vartheta} \qquad (t_0 \le s \le t < t_1). \end{aligned} \tag{42}$$

The drift and the infinitesimal variance of the time-non-homogeneous Wiener bridge are:

$$B_1(x,t) = \beta(t) + \left[x_1 - x - \int_t^{t_1} \beta(\vartheta)\, d\vartheta\right] \frac{\sigma^2(t)}{\displaystyle\int_t^{t_1} \sigma^2(\vartheta)\, d\vartheta}, \qquad B_2(t) = \sigma^2(t) \quad (t < t_1), \tag{43}$$

and, for $t_0 \le \tau < t < t_1$, we obtain the mean and the variance of the diffusion process (43) having normal density $f_Y(x,t|y,\tau)$:

$$\begin{aligned} \mathrm{E}[Y(t)|Y(\tau)=y] &= \left[y + \int_\tau^{t} \beta(\vartheta)\, d\vartheta\right] \frac{\displaystyle\int_t^{t_1} \sigma^2(\vartheta)\, d\vartheta}{\displaystyle\int_\tau^{t_1} \sigma^2(\vartheta)\, d\vartheta} + \left[x_1 - \int_t^{t_1} \beta(\vartheta)\, d\vartheta\right] \frac{\displaystyle\int_\tau^{t} \sigma^2(\vartheta)\, d\vartheta}{\displaystyle\int_\tau^{t_1} \sigma^2(\vartheta)\, d\vartheta}, \\ \mathrm{Var}[Y(t)|Y(\tau)=y] &= \frac{\left[\displaystyle\int_\tau^{t} \sigma^2(\vartheta)\, d\vartheta\right]\left[\displaystyle\int_t^{t_1} \sigma^2(\vartheta)\, d\vartheta\right]}{\displaystyle\int_\tau^{t_1} \sigma^2(\vartheta)\, d\vartheta}. \end{aligned} \tag{44}$$

Furthermore, for the boundary

$$S(t) = d_2 + d_1 \int_0^t \sigma^2(\vartheta)\, d\vartheta + \int_0^t \beta(\vartheta)\, d\vartheta, \tag{45}$$

with $d_1, d_2 \in \mathbb{R}$, from (38) one obtains the FPT pdf:

$$g_Y[S(t),t|y,\tau] = \frac{\sigma^2(t)\,|S(\tau)-y|}{\displaystyle\int_\tau^t \sigma^2(\vartheta)\, d\vartheta}\, f_Y[S(t),t|y,\tau] \quad ([y < S(\tau)] \text{ or } [y > S(\tau)]). \tag{46}$$

Hence, by virtue of (27), the FPT probability for the Wiener bridge through (45) follows:

$$\int_\tau^{t_1} g_Y[S(t),t|y,\tau]\, dt = \begin{cases} 1, & [S(t_1)-x_1]\,[S(\tau)-y] \le 0 \\[2ex] \exp\left\{\dfrac{-2[S(t_1)-x_1]\,[S(\tau)-y]}{\int_\tau^t \sigma^2(\vartheta)\, d\vartheta}\right\}, & [S(t_1)-x_1]\,[S(\tau)-y] > 0. \end{cases}$$

3.2 Time-homogeneous Ornstein-Uhlenbeck bridge Let now be $\{X(t), t \geq 0\}$ a Gauss-Markov process characterized by mean and covariance functions:

$$m_X(t) = \beta\,\frac{e^{\alpha\,t}-1}{\alpha}, \qquad c_X(s,t) = \sigma^2 e^{\alpha\,t}\,\frac{e^{\alpha\,s}-e^{-\alpha\,s}}{2\alpha}, \qquad 0 < s \leq t < +\infty, \tag{47}$$

with $\alpha, \beta \in \mathbb{R}$ and $\sigma > 0$. Relations (47) are obtained by restricting our attention to the time-homogeneous case and by setting in (28) $\alpha(t) = \alpha, \beta(t) = \beta$ and $\sigma(t) = \sigma$. Making use of (8), one obtains the drift $A_1(x) = \alpha\,x + \beta$ and the infinitesimal variance $A_2(t) = \sigma^2$ of the well-known time-homogeneous Ornstein-Uhlenbeck diffusion process, whose transition pdf $f_X(x, t|y, \tau)$ is normal with mean and variance:

$$\mathrm{E}[X(t)|X(\tau){=}y]{=}ye^{\alpha(t-\tau)} - \frac{\beta}{\alpha}\big(1-e^{\alpha(t-\tau)}\big),\ \mathrm{Var}[X(t)|X(\tau){=}y]{=}\sigma^2\,\frac{e^{2\alpha(t-\tau)}-1}{2\alpha}, \tag{48}$$

with $0 \leq \tau < t < +\infty$

We now consider the Gauss-Markov bridge $\{Y(t), t \in [t_0, t_1]\}$, obtained by conditioning the time-homogeneous Ornstein-Uhlenbeck process to start from x_0 at time t_0 and arrive at x_1 at time t_1, where $0 \leq t_0 < t_1 < +\infty$. For $t_0 \leq s \leq t < t_1$, the process $Y(t)$ has mean and covariance functions:

$$\begin{aligned} m_Y(t) &= \Big[x_0 + \frac{\beta}{\alpha}\big(1-e^{-\alpha(t-t_0)}\big)\Big]\,\frac{\sinh[\alpha(t_1-t)]}{\sinh[\alpha(t_1-t_0)]} \\ &\quad + \Big[x_1 - \frac{\beta}{\alpha}\big(e^{\alpha(t_1-t)}-1\big)\Big]\,\frac{\sinh[\alpha(t-t_0)]}{\sinh[\alpha(t_1-t_0)]}, \\ c_Y(s,t) &= \frac{\sigma^2}{\alpha}\,\frac{\sinh[\alpha(s-t_0)]\,\sinh[\alpha(t_1-t)]}{\sinh[\alpha(t_1-t_0)]}. \end{aligned} \tag{49}$$

The drift and the infinitesimal variance of the time-homogeneous Ornstein-Uhlenbeck bridge are then:

$$B_1(x,t) = -\alpha\,x\,\coth[\alpha(t_1-t)] + \frac{\alpha\,x_1}{\sinh[\alpha\,(t_1-t)]} - \beta\tanh\Big[\frac{\alpha(t_1-t)}{2}\Big], \qquad B_2(t) = \sigma^2, \tag{50}$$

and for $t_0 \leq \tau < t < t_1$ we obtain the mean and the variance of the diffusion process (50) having normal density $f_Y(x, t|y, \tau)$:

$$\begin{aligned} \mathrm{E}[Y(t)|Y(\tau) = y] &= \Big[y + \frac{\beta}{\alpha}\big(1-e^{-\alpha(t-\tau)}\big)\Big]\,\frac{\sinh[\alpha(t_1-t)]}{\sinh[\alpha(t_1-\tau)]} \\ &\quad + \Big[x_1 - \frac{\beta}{\alpha}\big(e^{\alpha(t_1-t)}-1\big)\Big]\,\frac{\sinh[\alpha(t-\tau)]}{\sinh[\alpha(t_1-\tau)]}, \\ \mathrm{Var}[Y(t)|Y(\tau) = y] &= \frac{\sigma^2}{\alpha}\,\frac{\sinh[\alpha(t_1-t)]\,\sinh[\alpha(t-\tau)]}{\sinh[\alpha(t_1-\tau)]}. \end{aligned} \tag{51}$$

Expressions (50) and (51) are in agreement with those obtained in [4] and [26]. Furthermore, for the boundary

$$S(t) = -\frac{\beta}{\alpha} + A\,e^{\alpha\,t} + B\,e^{-\alpha\,t}, \tag{52}$$

with $A, B \in \mathbb{R}$, from (38) one obtains the FPT pdf:

$$g_Y[S(t),t|y,\tau] = \frac{\alpha\,|S(\tau)-y|}{\sinh[\alpha(t-\tau)]}\,f_Y[S(t),t|y,\tau] \quad ([y<S(\tau)] \text{ or } [y>S(\tau)]). \tag{53}$$

Hence, from (27) the FPT probability for the time-homogeneous Ornstein-Uhlenbeck process through (52) is

$$\int_\tau^{t_1} g_Y[S(t),t|y,\tau]\,dt = \begin{cases} 1, & [S(t_1)-x_1]\,[S(\tau)-y]\le 0 \\ \exp\Big\{-\dfrac{2\alpha[S(t_1)-x_1]\,[S(\tau)-y]}{\sigma^2\sinh[\alpha(t_1-\tau)]}\Big\}, & [S(t_1)-x_1]\,[S(\tau)-y]>0. \end{cases}$$

4 FPT for Diffusion Bridge Let $\{X(t), t\ge 0\}$ be a time-non-homogeneous diffusion process, defined in the interval $I=(r_1,r_2)$, with drift $A_1(x,t)$ and infinitesimal variance $A_2(x,t)$. They are assumed to satisfy regularity conditions that guarantee a weakly unique, global solution of the stochastic equation (1), and we denote by $f_X(x,t|y,\tau)$ the transition density of $X(t)$. In the sequel, we derive stochastic bridges by conditioning $X(t)$ to start and finish at specific values at two consecutive times t_0 and t_1 ($t_0<t_1$), and for them we analyze the FPT problem. The procedure is inspired to the section of Karlin and Taylor book, related to the Brownian motion conditioned on its state at time 1 (cf. [30], pp. 267–271).

Let α and β be fixed real numbers, such that $r_1<\alpha<\beta<r_2$. We denote by $\{Z(t), t\in[t_0,t_1]\}$ the process obtained by conditioning $X(t)$ to start from x_0 at time t_0 and to finish in the interval (α,β) at time t_1, i.e. $\alpha<X(t_1)<\beta$. In order to calculate the infinitesimal moments of the conditioned process $Z(t)$, let indicate with $\pi(y,\tau)$ the probability that starting from y at time τ one has $\alpha<X(t_1)<\beta$, so that

$$\pi(y,\tau) = \int_\alpha^\beta f_X(z,t_1|y,\tau)\,dz \qquad (t_0\le\tau<t_1). \tag{54}$$

Furthermore, we indicate with $f_Z(x,t|y,\tau)$ the pdf of the conditioned process $Z(t)$. Due to the Markov property, one obtains:

$$f_Z(x,t|y,\tau) = \frac{\pi(x,t)}{\pi(y,\tau)}\,f_X(x,t|y,\tau) \qquad (t_0\le\tau<t<t_1). \tag{55}$$

With respect to the infinitesimal moments of $Z(t)$, we have:

$$\begin{aligned} B_n(y,t) &= \lim_{\Delta t\downarrow 0}\frac{1}{\Delta t}\,\mathrm{E}\Big\{\big[Z(t+\Delta t)-Z(t)\big]^n|Z(t)=y\Big\} \\ &= A_n(y,t)+\frac{1}{\pi(y,t)}\,\frac{\partial\pi(y,t)}{\partial y}\,A_{n+1}(y,t) \qquad (n=1,2,\ldots). \end{aligned} \tag{56}$$

Hence, $Z(t)$ is still a time-non-homogeneous diffusion process, characterized by drift and infinitesimal variance:

$$\begin{aligned} B_1(x,t) &= A_1(x,t)+\frac{1}{\pi(x,t)}\,\frac{\partial\pi(x,t)}{\partial x}\,A_2(x,t), \\ B_2(x,t) &= A_2(x,t), \end{aligned} \qquad (t_0<t<t_1) \tag{57}$$

respectively. We note that the drift of the conditioned process $\{Z(t), t\in[t_0,t_1]\}$ includes an extra term which forces the diffusion bridge to be in the interval (α,β) at time t_1, whereas the infinitesimal variance is the same of the process $X(t)$.

We now denote with $\{Y(t), t \in [t_0, t_1]\}$ a diffusion bridge process obtained by conditioning $\{X(t), t \geq t_0\}$ to start from x_0 at time t_0 and to arrive at x_1 at time t_1 $(t_1 > t_0)$. To obtain information on $Y(t)$, we distinguish the following cases: *(i)* the state x_1 is an internal point of I; *(ii)* $x_1 = r_1$, where r_1 is an accessible boundary; *(iii)* $x_1 = r_2$, where r_2 is an accessible boundary. In all cases, for $\varepsilon > 0$, we identifies the process $Z(t)$ with $Y^{(\varepsilon)}(t)$, obtained by introducing a dependency on ε in the functions $\pi(x,t)$ and $B_i(x,t)$ $(i = 1,2)$ that appear in (54), (55) and (57).

Case (i) For $\varepsilon > 0$, let be $\alpha = x_1 - \varepsilon$ and $\beta = x_1 + \varepsilon$, with $x_1 \in (r_1, r_2)$. We can achieve the desired bridge as the limit as $\varepsilon \downarrow 0$ of $Y^{(\varepsilon)}(t)$. Since

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\pi_\varepsilon(x,t)} \frac{\partial \pi_\varepsilon(x,t)}{\partial x} = \frac{1}{f_X(x_1, t_1|x,t)} \frac{\partial f_X(x_1, t_1|x,t)}{\partial x},$$

the process $\{Y(t), t \in [t_0, t_1]\}$ is characterized by drift and infinitesimal variance

$$\begin{aligned} B_1(x,t) &= \lim_{\varepsilon \downarrow 0} B_1^{(\varepsilon)}(x,t) = A_1(x,t) + \frac{1}{f_X(x_1, t_1|x,t)} \frac{\partial f_X(x_1, t_1|x,t)}{\partial x} A_2(x,t), \\ B_2(x,t) &= \lim_{\varepsilon \downarrow 0} B_2^{(\varepsilon)}(x,t) = A_2(x,t). \end{aligned} \qquad (t < t_1) \tag{58}$$

Furthermore, since

$$\lim_{\varepsilon \downarrow 0} \frac{\pi_\varepsilon(x,t)}{\pi_\varepsilon(y,\tau)} = \frac{f_X(x_1, t_1|x,t)}{f_X(x_1, t_1|y,\tau)},$$

from (55) one has:

$$f_Y(x,t|y,\tau) = f_X(x,t|y,\tau) \frac{f_X(x_1, t_1|x,t)}{f_X(x_1, t_1|y,\tau)} \qquad (t_0 \leq \tau < t < t_1). \tag{59}$$

Note that (59) is analogue to relation (21) which exists for the Gauss-Markov bridges. Furthermore, if we consider the FPT of the diffusion bridge $Y(t)$ from $Y(\tau) = y$ to a continuous time-dependent boundary $S(t)$, as in the case of Gauss-Markov processes, for $\tau < t < t_1$ one can show that still holds the relation (23), i.e.

$$g_Y[S(t), t|y,\tau] = g_X[S(t), t|y,\tau] \frac{f_X[x_1, t_1|S(t), t]}{f_X(x_1, t_1|y,\tau)} \qquad \big([y < S(\tau)] \text{ or } [y > S(\tau)]\big), \tag{60}$$

where $g_X[S(t), t|y,\tau]$ is the FPT pdf for the diffusion process $X(t)$.

Case (ii) For $\varepsilon > 0$, let be $\alpha = r_1$ and $\beta = r_1 + \varepsilon$. Being

$$\lim_{\varepsilon \downarrow 0} \frac{1}{\pi_\varepsilon(x,t)} \frac{\partial \pi_\varepsilon(x,t)}{\partial x} = \lim_{x_1 \downarrow r_1} \left\{ \frac{1}{f_X(x_1, t_1|x,t)} \frac{\partial f_X(x_1, t_1|x,t)}{\partial x} \right\},$$

the diffusion bridge process $\{Y(t), t \in [t_0, t_1]\}$ is characterized by drift and infinitesimal variance

$$\begin{aligned} B_1(x,t) &= A_1(x,t) + \lim_{x_1 \downarrow r_1} \left\{ \frac{1}{f_X(x_1, t_1|x,t)} \frac{\partial f_X(x_1, t_1|x,t)}{\partial x} \right\} A_2(x,t), \\ B_2(x,t) &= A_2(x,t), \end{aligned} \qquad (t < t_1) \tag{61}$$

provided that such a process exists. Furthermore, since

$$\lim_{\varepsilon\downarrow 0}\frac{\pi_\varepsilon(x,t)}{\pi_\varepsilon(y,\tau)}=\lim_{x_1\downarrow r_1}\frac{f_X(x_1,t_1|x,t)}{f_X(x_1,t_1|y,\tau)},$$

from (55) one obtains:

$$f_Y(x,t|y,\tau)=f_X(x,t|y,\tau)\lim_{x_1\downarrow r_1}\frac{f_X(x_1,t_1|x,t)}{f_X(x_1,t_1|y,\tau)}\qquad(t_0\le\tau<t<t_1). \tag{62}$$

We now consider the FPT of the diffusion bridge $Y(t)$ from $Y(\tau)=y$ to a continuous time-dependent boundary $S(t)$; for $\tau<t<t_1$, one can easily show that

$$g_Y[S(t),t|y,\tau]=g_X[S(t),t|y,\tau]\lim_{x_1\downarrow r_1}\frac{f_X[x_1,t_1|S(t),t]}{f_X(x_1,t_1|y,\tau)}\qquad\big([y<S(\tau)]\text{ or }[y>S(\tau)]\big). \tag{63}$$

The case *(iii)* can be analyzed in a similar way, by setting $\alpha=r_2-\varepsilon$ and $\beta=r_2$.

5 Time-non-homogeneous Feller bridge Let now be $\{X(t),t\ge 0\}$ a diffusion process characterized by drift and infinitesimal variance

$$A_1(x,t)=\alpha(t)\,x+\beta(t),\qquad A_2(x,t)=2\,\xi\,[\beta(t)+\nu\,\alpha(t)]\,(x-\nu),$$

defined in the interval $(\nu,+\infty)$, where $\alpha(t):\mathbb{R}^+\to\mathbb{R}$ and $\beta(t):\mathbb{R}^+\to\mathbb{R}$ are continuous functions, with $\xi>0$, $\nu\in\mathbb{R}$ and $\beta(t)+\nu\,\alpha(t)>0$. Such a process can be reviewed as a time-non-homogeneous Feller-type process. In order to obtain the transition pdf $f_X(x,t|y,\tau)$ of $X(t)$, we recall that it is solution of Fokker–Planck equation

$$\frac{\partial f_X}{\partial t}=-\frac{\partial}{\partial x}\Big\{[\alpha(t)x+\beta(t)]\,f_X\Big\}+\xi\big[\beta(t)+\nu\alpha(t)\big]\frac{\partial^2}{\partial x^2}\Big\{(x-\nu)f_X\Big\}, \tag{64}$$

with the zero–flux condition at $x=\nu$ and the initial delta condition:

$$\begin{aligned}&\lim_{x\downarrow\nu}\Big\{-[\alpha(t)\,x+\beta(t)]\,f_X+\xi\,[\beta(t)+\nu\,\alpha(t)]\frac{\partial}{\partial x}\,\big[(x-\nu)\,f_X\big]\Big\}=0,\\&\lim_{t\downarrow\tau}f(x,t|y,\tau)=\delta(x-y),\end{aligned} \tag{65}$$

respectively. In the sequel for $t\ge 0$ we set:

$$\begin{aligned}\Phi(t)&=\exp\Big\{-\int_0^t\alpha(\vartheta)\,d\vartheta\Big\},\\\Psi(t)&=\int_0^t[\beta(\vartheta)+\nu\alpha(\vartheta)]\,\Phi(\vartheta)\,d\vartheta=\nu[1-\Phi(t)]+\int_0^t\beta(\vartheta)\,\Phi(\vartheta)\,d\vartheta.\end{aligned}$$

The transformation (cf. [14] and [17])

$$\begin{aligned}&\tilde{x}=2(x-\nu)\,\Phi(t),\qquad\tilde{y}=2(y-\nu)\Phi(\tau),\qquad\tilde{t}=2\xi\,\Psi(t),\qquad\tilde{\tau}=2\xi\,\Psi(\tau),\\&f_X(x,t|y,\tau)=2\,\Phi(t)\,f_Z(\tilde{x},\tilde{t}|\tilde{y},\tilde{\tau}),\end{aligned} \tag{66}$$

changes equation (64) and conditions (65) into

$$\frac{\partial f_Z(\tilde{x},\tilde{t}|\tilde{y},\tilde{\tau})}{\partial \tilde{t}} = -\frac{1}{\xi}\frac{\partial f_Z(\tilde{x},\tilde{t}|\tilde{y},\tilde{\tau})}{\partial \tilde{x}} + \frac{\partial^2}{\partial \tilde{x}^2}\Big[\tilde{x}\, f_Z(\tilde{x},\tilde{t}|\tilde{y},\tilde{\tau})\Big], \tag{67}$$

$$\lim_{\tilde{x}\downarrow 0}\Big\{-\frac{1}{\xi}\, f_Z(\tilde{x},\tilde{t}|\tilde{y},\tilde{\tau}) + \frac{\partial}{\partial \tilde{x}}\big[\tilde{x} f_Z(\tilde{x},\tilde{t}|\tilde{y},\tilde{\tau})\big]\Big\} = 0, \tag{68}$$

$$\lim_{\tilde{t}\downarrow\tilde{\tau}} f_Z(\tilde{x},\tilde{t}|\tilde{y},\tilde{\tau}) = \delta(\tilde{x}-\tilde{y}).$$

Equation (67) is the Fokker–Planck equation for the time-homogeneous Feller diffusion process $Z(t)$, having drift $C_1 = 1/\xi$ and infinitesimal variance $C_2(\tilde{x}) = 2\,\tilde{x}$. Equation (68) expresses a zero–flux condition on the boundary $\tilde{x} = 0$. Hence, the diffusion interval of the process $Z(t)$ is $[0,\infty)$. If $0 < \xi \le 1$ then $\tilde{x} = 0$ is an entrance boundary. Further, if $\xi > 1$ then $\tilde{x} = 0$ is a regular boundary and (68) implies that $\tilde{x} = 0$ is a reflecting state. In both cases the transition pdf for $\tilde{x},\tilde{y} > 0$ is given by (cf. [23]):

$$f_Z(\tilde{x},\tilde{t}|\tilde{y},\tilde{\tau}) = \frac{1}{\tilde{t}-\tilde{\tau}}\,\exp\Big\{-\frac{\tilde{x}+\tilde{y}}{\tilde{t}-\tilde{\tau}}\Big\}\Big(\frac{\tilde{x}}{\tilde{y}}\Big)^{\frac{1-\xi}{2\xi}} I_{\frac{1-\xi}{\xi}}\Big(\frac{2\sqrt{\tilde{x}\tilde{y}}}{\tilde{t}-\tilde{\tau}}\Big), \tag{69}$$

where

$$I_\nu(z) = \sum_{k=0}^{\infty}\frac{1}{k!\,\Gamma(\nu+k+1)}\Big(\frac{z}{2}\Big)^{\nu+2k} \tag{70}$$

denotes the modified Bessel function of first kind. Furthermore, when $\xi > 1$ the FPT pdf of $Z(t)$ through zero state is:

$$g_Z(0,\tilde{t}|\tilde{y},\tilde{\tau}) = \frac{1}{(\tilde{t}-\tilde{\tau})\,\Gamma\Big(\frac{\xi-1}{\xi}\Big)}\Big(\frac{\tilde{y}}{\tilde{t}-\tilde{\tau}}\Big)^{1-1/\xi}\exp\Big\{-\frac{\tilde{y}}{\tilde{t}-\tilde{\tau}}\Big\} \qquad (\tilde{y} > 0). \tag{71}$$

Hence, by virtue of (66) and (69) we finally obtain the transition pdf of $X(t)$:

$$\begin{aligned} f_X(x,t|y,\tau) = {} & \frac{\Phi(t)}{\xi[\Psi(t)-\Psi(\tau)]}\,\exp\left\{-\frac{(y-\nu)\Phi(\tau)+(x-\nu)\Phi(t)}{\xi[\Psi(t)-\Psi(\tau)]}\right\}\left[\frac{(x-\nu)\Phi(t)}{(y-\nu)\Phi(\tau)}\right]^{\frac{1-\xi}{2\xi}} \\ & \times I_{\frac{1-\xi}{\xi}}\left(\frac{2\sqrt{(y-\nu)\Phi(\tau)(x-\nu)\Phi(t)}}{\xi[\Psi(t)-\Psi(\tau)]}\right) \qquad (x > \nu, y > \nu). \end{aligned} \tag{72}$$

The classification on the nature of boundary $\tilde{x} = 0$ of the process $Z(t)$ determines the nature of boundary $x = \nu$ of the process $X(t)$. Indeed, for the process $X(t)$ the boundary $x = \nu$ is a reflecting state if $\xi > 1$, whereas is an entrance boundary if $0 < \xi \le 1$. Such a nature also emerges by analyzing the transition pdf $f_X(x,t|y,\tau)$ when x approaches ν:

$$\lim_{x\downarrow\nu} f_X(x,t|y,\tau) = \begin{cases} 0, & 0 < \xi < 1 \\ \dfrac{\Phi(t)}{\Psi(t)-\Psi(\tau)}\exp\Big\{-\dfrac{(y-\nu)\Phi(\tau)}{\Psi(t)-\Psi(\tau)}\Big\}, & \xi = 1 \\ +\infty, & \xi > 1. \end{cases}$$

Making use of (72), we obtain the conditional mean of Feller-type process $X(t)$:

$$E[X(t)|X(\tau) = y] = \nu + (y-\nu)\,\frac{\Phi(\tau)}{\Phi(t)} + \frac{\Psi(t)-\Psi(\tau)}{\Phi(t)} \qquad (y > \nu)\cdot \tag{73}$$

Furthermore, recalling (66) and (71), when $\xi > 1$ the FPT pdf of $X(t)$ through the boundary ν is:

$$g_X(\nu,t|y,\tau) = \frac{\beta(t)+\nu\alpha(t)}{\Gamma\Big(\frac{\xi-1}{\xi}\Big)}\,\frac{\Phi(t)}{\Psi(t)-\Psi(\tau)}\left[\frac{(y-\nu)\Phi(\tau)}{\xi[\Psi(t)-\Psi(\tau)]}\right]^{\frac{\xi-1}{\xi}}$$

$$\times\exp\left\{-\frac{(y-\nu)\Phi(\tau)}{\xi[\Psi(t)-\Psi(\tau)]}\right\}\qquad (y>\nu), \tag{74}$$

so that, for $\xi > 1$, the FPT probability is then:

$$\int_\tau^{+\infty} g_X(\nu,t|y,\tau)\,dt = \begin{cases} 1, & \lim\limits_{t\to+\infty}\Psi(t) = +\infty \\ \dfrac{\Gamma\Big(\dfrac{\xi-1}{\xi},\dfrac{(y-\nu)\Phi(\tau)}{\xi[\Psi_0-\Psi(\tau)]}\Big)}{\Gamma\Big(\dfrac{\xi-1}{\xi}\Big)}, & \Psi_0 = \lim\limits_{t\to+\infty}\Psi(t) < +\infty, \end{cases} \tag{75}$$

where

$$\Gamma(a,x) = \int_x^{+\infty} t^{a-1}\mathrm{e}^{-t}\,dt,\qquad \Gamma(a) = \int_0^{+\infty} t^{a-1}\mathrm{e}^{-t}\,dt\quad (\mathrm{Re}\,a>0).$$

Expressions (72)÷(75) are in agreement with those obtained in [17], with $\nu = 0$, for the continuous approximation of a queueing system.

Starting from the Feller-type process $X(t)$, we construct a diffusion bridge such that $X(t_0) = x_0$ and $X(t_1) = x_1$, with $x_0 \ge \nu$ and $x_1 \ge \nu$. We consider two cases: *(i)* $\xi > 0$ and $x_1 > \nu$ and *(ii)* $\xi > 1$ and $x_1 = \nu$.

Case (i) For $\xi > 0$, let $\{Y(t), t \in [t_0,t_1]\}$ indicate the Feller-type bridge with $X(t_0) = x_0 \ge \nu$ and $X(t_1) = x_1 > \nu$. Making use of (72) in (58), and recalling that (cf. [28], p. 928, n. 8.486.3)

$$\frac{d}{dx}I_\nu(x) = I_{\nu-1}(x) - \frac{\nu}{x}I_\nu(x),$$

one obtains the drift and the infinitesimal variance of Feller-type bridge:

$$B_1(x,t) = \alpha(t)x+\beta(t)+2\,[\beta(t)+\nu\alpha(t)]\left[\xi-1-\frac{(x-\nu)\Phi(t)}{\Psi(t_1)-\Psi(t)}\right]$$

$$+2\,[\beta(t)+\nu\alpha(t)]\,\frac{\sqrt{\Phi(t)(x-\nu)\Phi(t_1)(x_1-\nu)}}{\Psi(t_1)-\Psi(t)}\,\frac{I_{\frac{1-2\xi}{\xi}}\Big(\frac{2\sqrt{(x-\nu)\Phi(t)(x_1-\nu)\Phi(t_1)}}{\xi[\Psi(t_1)-\Psi(t)]}\Big)}{I_{\frac{1-\xi}{\xi}}\Big(\frac{2\sqrt{(x-\nu)\Phi(t)(x_1-\nu)\Phi(t_1)}}{\xi[\Psi(t_1)-\Psi(t)]}\Big)},$$

$$(x>\nu, x_1>\nu, t<t_1) \tag{76}$$

$$B_2(x,t) = 2\xi[\beta(t)+\nu\alpha(t)](x-\nu).$$

Hence, when $x_1 > \nu$, from (59) the transition pdf of $Y(t)$ can be derived:

$$f_Y(x,t|y,\tau) = \frac{\Phi(t)\,[\Psi(t_1)-\Psi(\tau)]}{\xi[\Psi(t)-\Psi(\tau)]\,[\Psi(t_1)-\Psi(t)]}\exp\left\{-\frac{(y-\nu)\Phi(\tau)\,[\Psi(t_1)-\Psi(t)]}{\xi[\Psi(t)-\Psi(\tau)]\,[\Psi(t_1)-\Psi(\tau)]}\right\}$$

$$\times\exp\left\{-\frac{(x-\nu)\Phi(t)\,[\Psi(t_1)-\Psi(\tau)]}{\xi[\Psi(t)-\Psi(\tau)]\,[\Psi(t_1)-\Psi(t)]}\right\}\exp\left\{-\frac{(x_1-\nu)\Phi(t_1)\,[\Psi(t)-\Psi(\tau)]}{\xi[\Psi(t_1)-\Psi(t)]\,[\Psi(t_1)-\Psi(\tau)]}\right\}$$

$$\times\frac{I_{\frac{1-\xi}{\xi}}\Big(\frac{2\sqrt{(y-\nu)\Phi(\tau)(x-\nu)\Phi(t)}}{\xi[\Psi(t)-\Psi(\tau)]}\Big)\,I_{\frac{1-\xi}{\xi}}\Big(\frac{2\sqrt{(x-\nu)\Phi(t)(x_1-\nu)\Phi(t_1)}}{\xi[\Psi(t_1)-\Psi(t)]}\Big)}{I_{\frac{1-\xi}{\xi}}\Big(\frac{2\sqrt{(y-\nu)\Phi(\tau)(x_1-\nu)\Phi(t_1)}}{\xi[\Psi(t_1)-\Psi(\tau)]}\Big)}\qquad (y>\nu, x>\nu). \tag{77}$$

Recalling that (cf. [20], p. 197, no. 22):

$$\int_0^{+\infty} e^{-\lambda t} I_q\big(\sqrt{2\alpha t}\big)\, I_q\big(\sqrt{2\beta t}\big)\, dt = \frac{1}{\lambda}\,\exp\Big\{\frac{\alpha+\beta}{2\lambda}\Big\}\, I_q\Big(\frac{\sqrt{\alpha\beta}}{\lambda}\Big) \qquad (\mathrm{Re}\, q > -1, \mathrm{Re}\,\lambda > 0),$$

it is easy to prove that (77) is integrated to unit in the interval $(\nu,+\infty)$. Furthermore, the conditional mean of the Feller-type bridge $Y(t)$, with $\xi > 0$ and $x_1 > \nu$, can be obtained by means of (77):

$$\begin{aligned} E[Y(t)|Y(\tau)=y] &= \int_\nu^{+\infty} x\, f_Y(x,t|y,\tau)\, dx = \nu + \frac{[\Psi(t)-\Psi(\tau)]\,[\Psi(t_1)-\Psi(t)]}{\Phi(t)\,[\Psi(t_1)-\Psi(\tau)]} \\ &\times\Big\{2\xi - 1 + \frac{(y-\nu)\,\Phi(\tau)\,[\Psi(t_1)-\Psi(t)]}{[\Psi(t_1)-\Psi(\tau)]\,[\Psi(t)-\Psi(\tau)]} + \frac{(x_1-\nu)\,\Phi(t_1)\,[\Psi(t)-\Psi(\tau)]}{[\Psi(t_1)-\Psi(\tau)]\,[\Psi(t_1)-\Psi(t)]} \\ &+ \frac{2\sqrt{(y-\nu)\Phi(\tau)(x_1-\nu)\Phi(t_1)}}{\Psi(t_1)-\Psi(\tau)}\; \frac{I_{\frac{1-2\xi}{\xi}}\Big(\frac{2\sqrt{(y-\nu)\Phi(\tau)(x_1-\nu)\Phi(t_1)}}{\xi[\Psi(t_1)-\Psi(\tau)]}\Big)}{I_{\frac{1-\xi}{\xi}}\Big(\frac{2\sqrt{(y-\nu)\Phi(\tau)(x_1-\nu)\Phi(t_1)}}{\xi[\Psi(t_1)-\Psi(\tau)]}\Big)}\Big\} \qquad (y>\nu). \end{aligned} \tag{78}$$

For the diffusion bridge $Y(t)$, with $x_1 > \nu$ and $\xi > 1$, we now calculate the FPT pdf through the state ν. Making use of (72) and (74) in (60), for $\tau < t < t_1$ one has:

$$\begin{aligned} g_Y(\nu,t|y,\tau) &= \frac{\beta(t)+\nu\alpha(t)}{\Gamma\big(\frac{1}{\xi}\big)\,\Gamma\big(\frac{\xi-1}{\xi}\big)}\; \frac{\Phi(t)\,[\Psi(t_1)-\Psi(\tau)]}{[\Psi(t)-\Psi(\tau)]\,[\Psi(t_1)-\Psi(t)]}\, \Big[\frac{(y-\nu)\Phi(\tau)}{(x_1-\nu)\Phi(t_1)}\Big]^{\frac{\xi-1}{2\xi}} \\ &\times \exp\Big\{-\frac{(y-\nu)\Phi(\tau)\,[\Psi(t_1)-\Psi(t)]}{\xi[\Psi(t)-\Psi(\tau)]\,[\Psi(t_1)-\Psi(\tau)]}\Big\}\, \exp\Big\{-\frac{(x_1-\nu)\Phi(t_1)\,[\Psi(t)-\Psi(\tau)]}{\xi[\Psi(t_1)-\Psi(t)]\,[\Psi(t_1)-\Psi(\tau)]}\Big\} \\ &\times \Big[\frac{\Psi(t_1)-\Psi(t)}{\Psi(t)-\Psi(\tau)}\Big]^{\frac{\xi-1}{\xi}}\, \Big[I_{\frac{1-\xi}{\xi}}\Big(\frac{2\sqrt{(y-\nu)\Phi(\tau)(x_1-\nu)\Phi(t_1)}}{\xi[\Psi(t_1)-\Psi(\tau)]}\Big)\Big]^{-1} \qquad (y>\nu, x_1>\nu). \end{aligned} \tag{79}$$

For $\xi > 1$, the FPT probability for the Feller-type bridge through the state ν can be derived. Indeed, by making use of (79), one has:

$$\begin{aligned} \int_\tau^{t_1} g_Y(\nu,t|y,\tau)\, dt &= \frac{1}{\Gamma\big(\frac{1}{\xi}\big)\,\Gamma\big(\frac{\xi-1}{\xi}\big)}\, \Big[\frac{(y-\nu)\Phi(\tau)}{(x_1-\nu)\Phi(t_1)}\Big]^{\frac{\xi-1}{2\xi}} \\ &\times \Big[I_{\frac{1-\xi}{\xi}}\Big(\frac{2\sqrt{(y-\nu)\Phi(\tau)(x_1-\nu)\Phi(t_1)}}{\xi[\Psi(t_1)-\Psi(\tau)]}\Big)\Big]^{-1} V(t_1|\tau) \qquad (y>\nu, x_1>\nu), \end{aligned}$$

where

$$\begin{aligned} V(t_1|\tau) &= [\Psi(t_1)-\Psi(\tau)] \int_\tau^{t_1} \frac{[\beta(t)+\nu\alpha(t)]\,\Phi(t)}{[\Psi(t)-\Psi(\tau)]\,[\Psi(t_1)-\Psi(t)]}\, \Big[\frac{\Psi(t_1)-\Psi(t)}{\Psi(t)-\Psi(\tau)}\Big]^{\frac{\xi-1}{\xi}} \\ &\times \exp\Big\{-\frac{(y-\nu)\Phi(\tau)}{\xi[\Psi(t_1)-\Psi(\tau)]}\,\frac{\Psi(t_1)-\Psi(t)}{\Psi(t)-\Psi(\tau)} - \frac{(x_1-\nu)\Phi(t_1)}{\xi[\Psi(t_1)-\Psi(\tau)]}\,\frac{\Psi(t)-\Psi(\tau)}{\Psi(t_1)-\Psi(t)}\Big\}\, dt \\ &= \int_0^{+\infty} u^{-1/\xi}\, \exp\Big\{-\frac{(y-\nu)\Phi(\tau)}{\xi[\Psi(t_1)-\Psi(\tau)]}\, u - \frac{(x_1-\nu)\Phi(t_1)}{\xi[\Psi(t_1)-\Psi(\tau)]}\,\frac{1}{u}\Big\}\, du. \end{aligned} \tag{80}$$

The last identity follows by making use of the change of variable $u = [\Psi(t_1)-\Psi(t)]/[\Psi(t)-\Psi(\tau)]$. Hence, recalling that (cf. [28], p. 368, n. 9)

$$\int_0^{+\infty} x^{\nu-1} \exp\Big\{-\frac{\beta}{x} - \gamma\, x\Big\}\, dx = 2\Big(\frac{\beta}{\gamma}\Big)^{\nu/2} K_\nu(2\sqrt{\beta\gamma}) \qquad (\mathrm{Re}\,\beta > 0, \mathrm{Re}\,\gamma > 0), \tag{81}$$

where $K_\nu(z) = K_{-\nu}(z) = (\pi/2)[I_{-\nu}(z) - I_\nu(z)]/\sin(\nu z)$ denotes the modified Bessel function of the third kind, for $\xi > 1$ and $x_1 > \nu$ one finally is lead to:

$$\int_\tau^{t_1} g_Y(\nu, t|y,\tau)\, dt = \frac{2}{\Gamma\left(\frac{1}{\xi}\right)\Gamma\left(\frac{\xi-1}{\xi}\right)} \frac{K_{\frac{1-\xi}{\xi}}\left(\dfrac{2\sqrt{(x_1-\nu)\Phi(t_1)(y-\nu)\Phi(\tau)}}{\xi[\Psi(t_1)-\Psi(\tau)]}\right)}{I_{\frac{1-\xi}{\xi}}\left(\dfrac{2\sqrt{(y-\nu)\Phi(\tau)(x_1-\nu)\Phi(t_1)}}{\xi[\Psi(t_1)-\Psi(\tau)]}\right)}, \tag{82}$$

with $y > \nu$. This relation shows that, when $\xi > 1$ and $x_1 > \nu$, the FPT of the Feller-type bridge through the state ν is not a sure event.

Case (ii) For $\xi > 1$, let $\{Y(t), t \in [t_0, t_1]\}$ indicate the Feller-type bridge with $X(t_0) = x_0 \geq \nu$ and $X(t_1) = x_1 = \nu$. By virtue of (70), we note that

$$\lim_{x_1 \downarrow \nu} \sqrt{x_1 - \nu}\, \frac{I_{\frac{1-2\xi}{\xi}}\left(\frac{2\sqrt{(x-\nu)\Phi(t)(x_1-\nu)\Phi(t_1)}}{\xi[\Psi(t_1)-\Psi(t)]}\right)}{I_{\frac{1-\xi}{\xi}}\left(\frac{2\sqrt{(x-\nu)\Phi(t)(x_1-\nu)\Phi(t_1)}}{\xi[\Psi(t_1)-\Psi(t)]}\right)} = \frac{(1-\xi)\,[\Psi(t_1)-\Psi(t)]}{\sqrt{\Phi(t)\,\Phi(t_1)\,(x-\nu)}},$$

so that making use of (72) in (61), one obtains the drift and the infinitesimal variance of Feller-type bridge:

$$\begin{aligned} &B_1(x,t) = \alpha(t)\, x + \beta(t) - \frac{2\,[\beta(t) + \nu\,\alpha(t)]\,\Phi(t)\,(x-\nu)}{\Psi(t_1) - \Psi(t)}, \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (x > \nu, x_1 = \nu, t < t_1) \\ &B_2(x,t) = 2\xi\,[\beta(t) + \nu\alpha(t)]\,(x-\nu). \end{aligned} \tag{83}$$

We note that the infinitesimal moments (83) have the same functional form of (3), i.e. the drift and the infinitesimal variance are linear with respect to x. Hence, when $x_1 = \nu$, from (62) we derive the transition pdf of $Y(t)$:

$$\begin{aligned} f_Y(x,t|y,\tau) = {} & \frac{\Phi(t)\,[\Psi(t_1)-\Psi(\tau)]}{\xi[\Psi(t)-\Psi(\tau)]\,[\Psi(t_1)-\Psi(t)]} \exp\left\{-\frac{(y-\nu)\Phi(\tau)\,[\Psi(t_1)-\Psi(t)]}{\xi\,[\Psi(t)-\Psi(\tau)]\,[\Psi(t_1)-\Psi(\tau)]}\right\} \\ & \times \exp\left\{-\frac{(x-\nu)\Phi(t)\,[\Psi(t_1)-\Psi(\tau)]}{\xi\,[\Psi(t)-\Psi(\tau)]\,[\Psi(t_1)-\Psi(t)]}\right\} \left[\frac{(x-\nu)\,\Phi(t)}{(y-\nu)\,\Phi(\tau)}\right]^{\frac{1-\xi}{2\xi}} \left[\frac{\Psi(t_1)-\Psi(\tau)}{\Psi(t_1)-\Psi(t)}\right]^{\frac{1-\xi}{\xi}} \\ & \times I_{\frac{1-\xi}{\xi}}\left(\frac{2\sqrt{(y-\nu)\,\Phi(\tau)\,(x-\nu)\,\Phi(t)}}{\xi\,[\Psi(t)-\Psi(\tau)]}\right) \qquad (x > \nu, y > \nu, x_1 = \nu). \end{aligned} \tag{84}$$

Note that relation (84) can be derived also in a different way. Indeed, by setting

$$Q_1(t) = \alpha(t) - \frac{2[\beta(t) + \nu\alpha(t)]\,\Phi(t)}{\Psi(t_1) - \Psi(t)}, \qquad Q_2(t) = 2\xi\,[\beta(t) + \nu\alpha(t)],$$

the transformation

$$\begin{aligned} &\tilde{x} = 2(x-\nu)\,\exp\left\{-2\int_0^t Q_1(\vartheta)\, d\vartheta\right\}, \qquad \tilde{y} = 2(y-\nu)\,\exp\left\{-2\int_0^\tau Q_1(\vartheta)\, d\vartheta\right\}, \\ &\tilde{t} = \int_0^t Q_2(\vartheta)\,\exp\left\{-\int_0^\vartheta Q_1(u)\, du\right\} d\vartheta, \qquad \tilde{\tau} = \int_0^\tau Q_2(\vartheta)\,\exp\left\{-\int_0^\vartheta Q_1(u)\, du\right\} d\vartheta, \\ &f_Y(x,t|y,\tau) = 2\,\exp\left\{-2\int_0^t Q_1(\vartheta)\, d\vartheta\right\} f_Z(\tilde{x}, \tilde{t}|\tilde{y}, \tilde{\tau}), \end{aligned}$$

changes the Fokker-Planck equation, with zero–flux condition in ν, for the transition pdf $f_Y(x,t|y,\tau)$ of the diffusion process (83) into the Fokker–Planck equation, with zero-flux condition in state zero, for the transition pdf $f_Z(\tilde{x},\tilde{t}|\tilde{y},\tilde{\tau})$ of Feller diffusion process $Z(t)$, having drift $C_1 = 1/\xi$ and infinitesimal variance $C_2(\tilde{x}) = 2\,\tilde{x}$. Hence, from (69) relation (84) follows. Recalling that (cf. [20], p. 245, no. 35):

$$\int_0^{+\infty} e^{-\lambda\, t}\, t^{\nu/2}\, I_\nu(2\sqrt{\beta\, t})\, dt = \frac{\beta^{\nu/2}}{\lambda^{1+\nu}}\, e^{\beta/\lambda} \quad (\nu > -1, \lambda > 0),$$

it is easy to prove that (84) is integrated to unit in the interval $(\nu, +\infty)$. Expression (77) and (84) are in agreement with those obtained by Makarov in [32] for the time-homogeneous Bessel process. Furthermore, the conditional mean of the Feller-type bridge $Y(t)$, with $\xi > 1$ and $x_1 = \nu$, can be obtained by means of (84):

$$E[Y(t)|Y(\tau)=y] = \nu + \frac{[\Psi(t)-\Psi(\tau)]\,[\Psi(t_1)-\Psi(t)]}{\Phi(t)\,[\Psi(t_1)-\Psi(\tau)]} + (y-\nu)\,\frac{\Phi(\tau)}{\Phi(t)}\left[\frac{\Psi(t_1)-\Psi(t)}{\Psi(t_1)-\Psi(\tau)}\right]^2, \tag{85}$$

with $\tau < t < t_1$. For the diffusion bridge $Y(t)$, with $\xi > 1$ and $x_1 = \nu$, we now calculate the FPT pdf through the state ν. Making use of (63), for $y > \nu$ and $\tau < t < t_1$, one has:

$$\begin{aligned} g_Y(\nu,t|y,\tau) = {} & \frac{\beta(t)+\nu\,\alpha(t)}{\Gamma\left(\frac{\xi-1}{\xi}\right)}\,\frac{\Phi(t)\,[\Psi(t_1)-\Psi(\tau)]}{[\Psi(t_1)-\Psi(t)]\,[\Psi(t)-\Psi(\tau)]} \\ & \times \left[\frac{(y-\nu)\,\Phi(\tau)\,[\Psi(t_1)-\Psi(t)]}{\xi\,[\Psi(t_1)-\Psi(\tau)]\,[\Psi(t)-\Psi(\tau)]}\right]^{\frac{\xi-1}{\xi}} \exp\left\{-\frac{(y-\nu)\Phi(\tau)\,[\Psi(t_1)-\Psi(t)]}{\xi\,[\Psi(t_1)-\Psi(\tau)]\,[\Psi(t)-\Psi(\tau)]}\right\}, \end{aligned} \tag{86}$$

so that, when $\xi > 1$ and $x_1 = \nu$, the FPT of the Feller-type bridge through the state ν is a sure event, i.e.

$$\int_\tau^{t_1} g_Y(\nu,t|y,\tau)\, dt = 1.$$

We now specialize the above results to the Feller-type bridge with $\xi = 2$.

5.1 A particular Feller bridge Let now consider a diffusion process $\{X(t), t \geq 0\}$ characterized by drift and infinitesimal variance (3), with $\xi = 2$:

$$A_1(x,t) = \alpha(t)\,x + \beta(t), \qquad A_2(x,t) = 4\,[\beta(t)+\nu\,\alpha(t)]\,(x-\nu), \tag{87}$$

defined in the interval $(\nu, +\infty)$, where $\alpha(t) : \mathbb{R}^+ \to \mathbb{R}$ and $\beta(t) : \mathbb{R}^+ \to \mathbb{R}$ are continuous functions, with $\nu \in \mathbb{R}$ and $\beta(t) + \nu\,\alpha(t) > 0$. We assume that the boundary $x = \nu$ is a reflecting state. The choice $\xi = 2$ will allow us to make some simplifications in the formulas of the Feller process and of its bridge, by using some properties of Bessel functions:

$$I_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\,\cosh(x), \quad I_{-3/2}(x) = \sqrt{\frac{2}{\pi x}}\left[\sinh(x) - \frac{\cosh(x)}{x}\right], \quad K_{-1/2}(x) = \sqrt{\frac{\pi}{2\,x}}\,e^{-x}.$$

Indeed, from (72) for $x > \nu$ and $y > \nu$ one has:

$$\begin{aligned} f_X(x,t|y,\tau) = {} & \frac{\Phi(t)}{2[\Psi(t)-\Psi(\tau)]}\sqrt{\frac{\Psi(t)-\Psi(\tau)}{2\pi\,\Phi(t)\,(x-\nu)}}\left[\exp\left\{-\frac{\left[\sqrt{(x-\nu)\Phi(t)}-\sqrt{(y-\nu)\Phi(\tau)}\,\right]^2}{2[\Psi(t)-\Psi(\tau)]}\right\}\right. \\ & \left. + \exp\left\{-\frac{\left[\sqrt{(x-\nu)\Phi(t)}+\sqrt{(y-\nu)\Phi(\tau)}\,\right]^2}{2[\Psi(t)-\Psi(\tau)]}\right\}\right]. \end{aligned} \tag{88}$$

Alternatively, we note that the transformation

$$\tilde{x} = \sqrt{(x-\nu)\Phi(t)}, \quad \tilde{y} = \sqrt{(y-\nu)\Phi(\tau)}, \quad \tilde{t} = \Psi(t), \quad \tilde{\tau} = \Psi(\tau),$$
$$f_X(x,t|y,\tau) = \sqrt{\frac{\Phi(t)}{2(x-\nu)}}\, f_Z(\tilde{x},\tilde{t}|\tilde{y},\tilde{\tau})$$

changes the Fokker-Planck equation, with zero–flux condition in ν, for the transition pdf $f_X(x,t|y,\tau)$ of the diffusion process (87) into the Fokker–Planck equation for the transition pdf $f_Z(\tilde{x},\tilde{t}|\tilde{y},\tilde{\tau})$ of the Wiener diffusion process $Z(t)$, having drift $C_1 = 0$ and infinitesimal variance $C_2 = 1$, restricted to $[0,+\infty)$ by a reflecting condition in zero state.

For $\xi = 2$, let $\{Y(t), t \in [t_0, t_1]\}$ be the Feller-type bridge, with $X(t_0) = x_0 \geq \nu$ and $X(t_1) = x_1$. We first consider the case $x_1 > \nu$. From (76), one obtains the drift and the infinitesimal variance of the Feller-type bridge:

$$B_1(x,t) = \alpha(t)x + \beta(t) + 2\left[\beta(t) + \nu\alpha(t)\right]\left\{-\frac{(x-\nu)\Phi(t)}{\Psi(t_1)-\Psi(t)} + \frac{\sqrt{(x-\nu)\,\Phi(t)\,(x_1-\nu)\,\Phi(t_1)}}{\Psi(t_1)-\Psi(t)}\right.$$
$$\left.\times \tanh\left(\frac{\sqrt{(x-\nu)\,\Phi(t)\,(x_1-\nu)\,\Phi(t_1)}}{\Psi(t_1)-\Psi(t)}\right)\right\},$$
$$(x > \nu, x_1 > \nu, t < t_1)$$
$$B_2(x,t) = 4\left[\beta(t) + \nu\,\alpha(t)\right](x-\nu),$$

and from (77) and (78), for $x_1 > \nu$ and $\tau < t < t_1$, we immediately derive the transition pdf and the conditional mean. Furthermore, for $\tau < t < t_1$ the FPT pdf through the state ν follows from (79):

$$g_Y(\nu,t|y,\tau) = \frac{[\beta(t)+\nu\alpha(t)]\Phi(t)}{\sqrt{2\pi}\,[\Psi(t)-\Psi(\tau)]}\sqrt{\frac{(y-\nu)\Phi(\tau)[\Psi(t_1)-\Psi(\tau)]}{[\Psi(t_1)-\Psi(t)]\,[\Psi(t)-\Psi(\tau)]}}$$
$$\times \exp\left\{-\frac{(y-\nu)\Phi(\tau)\,[\Psi(t_1)-\Psi(t)]}{2[\Psi(t)-\Psi(\tau)]\,[\Psi(t_1)-\Psi(\tau)]}\right\} \exp\left\{-\frac{(x_1-\nu)\Phi(t_1)\,[\Psi(t)-\Psi(\tau)]}{2[\Psi(t_1)-\Psi(t)]\,[\Psi(t_1)-\Psi(\tau)]}\right\}$$
$$\times\left[\cosh\left(\frac{\sqrt{(y-\nu)\Phi(\tau)(x_1-\nu)\Phi(t_1)}}{\Psi(t_1)-\Psi(\tau)}\right)\right]^{-1} \quad (y > \nu, x_1 > \nu), \tag{89}$$

and, recalling (82), the FPT probability through ν is:

$$\int_\tau^{t_1} g_Y(\nu,t|y,\tau)\,dt = 2\left[1+\exp\left\{-\frac{\sqrt{(y-\nu)\Phi(\tau)\,(x_1-\nu)\Phi(t_1)}}{\Psi(t_1)-\Psi(\tau)}\right\}\right]^{-1} \quad (y > \nu, x_1 > \nu).$$

We note that for $\xi = 2$, $x_1 > \nu$ and $\tau < t < t_1$ one has

$$\frac{g_Y(\nu,t|y,\tau)}{g_X(\nu,t|y,\tau)} = \sqrt{\frac{\Psi(t_1)-\Psi(\tau)}{\Psi(t_1)-\Psi(t)}}\exp\left\{-\frac{(x_1-\nu)\Phi(t_1)\,[\Psi(t)-\Psi(\tau)]}{2[\Psi(t_1)-\Psi(t)]\,[\Psi(t_1)-\Psi(\tau)]}\right\}$$
$$\times\exp\left\{\frac{(y-\nu)\Phi(\tau)}{2[\Psi(t_1)-\Psi(\tau)]}\right\}\left[\cosh\left(\frac{\sqrt{(y-\nu)\,\Phi(\tau)\,(x_1-\nu)\,\Phi(t_1)}}{\Psi(t_1)-\Psi(\tau)}\right)\right]^{-1} \quad (y > \nu).$$

Next, when $\xi = 2$, let $\{Y(t), t \in [t_0, t_1]\}$ be a Feller-type bridge with $X(t_0) = x_0 \geq \nu$ and $X(t_1) = \nu$. By virtue of (83) the drift and the infinitesimal variance of $Y(t)$ are:

$$B_1(x,t) = \alpha(t)x + \beta(t) - \frac{2\left[\beta(t)+\nu\alpha(t)\right]\Phi(t)\,(x-\nu)}{\Psi(t_1)-\Psi(t)},$$
$$(x > \nu, x_1 = \nu, t < t_1)$$
$$B_2(x,t) = 4\left[\beta(t)+\nu\alpha(t)\right](x-\nu),$$

and from (84) and (85), for $x_1 = \nu$ and $\tau < t < t_1$, we immediately obtain the transition pdf and the conditional mean. Furthermore, by setting $\xi = 2$, when $\tau < t < t_1$ the FPT pdf through the state ν follows from (86):

$$\begin{aligned} g_Y(\nu, t|y,\tau) &= \frac{[\beta(t) + \nu\,\alpha(t)]\,\Phi(t)}{\sqrt{2\pi}\,[\Psi(t) - \Psi(\tau)]} \sqrt{\frac{(y-\nu)\Phi(\tau)\,[\Psi(t_1) - \Psi(\tau)]}{[\Psi(t_1) - \Psi(t)]\,[\Psi(t) - \Psi(\tau)]}} \\ &\times \exp\left\{ -\frac{(y-\nu)\,\Phi(\tau)\,[\Psi(t_1) - \Psi(t)]}{2\,[\Psi(t) - \Psi(\tau)]\,[\Psi(t_1) - \Psi(\tau)]} \right\} \quad (y > \nu, x_1 = \nu), \end{aligned} \tag{90}$$

so that the FPT through the state ν is a sure event and for $\tau < t < t_1$ one has:

$$\frac{g_Y(\nu, t|y,\tau)}{g_X(\nu, t|y,\tau)} = \sqrt{\frac{\Psi(t_1) - \Psi(\tau)}{\Psi(t_1) - \Psi(t)}}\, \exp\left\{ \frac{(y-\nu)\Phi(\tau)}{2[\Psi(t_1) - \Psi(\tau)]} \right\} \quad (y > \nu, x_1 = \nu).$$

Let us finally mention that some applications of Ornstein-Uhlenbeck bridge and Feller-type bridge to single neuron firing, to volatility of financial assets and to queueing models will be the object of future works.

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