ON THE SIZE OF MINIMAL HALES-JEWETT SETS

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Abstract. A *Hales-Jewett set* is a set of words of a given length on a specified alphabet with the property that whenever it is 2-colored, there must be a monochromatic combinatorial line. We show that any Hales-Jewett set consisting of length 4 words on the alphabet $\{1,2,3\}$ must have at least 25 members and produce an example of a minimal Hales-Jewett set with 37 members.

1 Introduction.

The Hales-Jewett Theorem is one of the fundamental results of Ramsey Theory. In order to describe it, we introduce some terminology. An *alphabet* is simply a set, and a *word* over the alphabet A is a finite sequence of members of A. A member of A is said to *occur* in the word w provided it is one of the terms of the sequence. A *variable word* over A is a word over the alphabet $A \cup \{v\}$ in which v occurs, where v is a *variable* which is not an element of A. Given a variable word $w = w(v)$ and $a \in A$, $w(a)$ is the word obtained by replacing each occurrence of v by a. For example, if $A = \{1,2,3,4\}$ and $w(v) = \langle 2,3,v,1,4,v,v,2\rangle$, then $w(2) = \langle 2,3,2,1,4,2,2,2\rangle$. We write $\mathbb{N}$ for the set of positive integers.

Theorem 1.1 (Hales-Jewett)**.** *Let A be a finite alphabet, let W be the set of words over A, let $r \in \mathbb{N}$, and let $\psi : W \to \{0,1,\ldots,r-1\}$. There exist a variable word w over A and $i \in \{0,1,\ldots,r-1\}$ such that for all $a \in A$, $\psi\big(w(a)\big) = i$.*

Proof. Hales and Jewett [4, Theorem 1]. Or see [3, Section 2.2, Theorem 3] or [5, Corollary 14.8]. □

The function ψ is commonly referred to as an *r-coloring* of W and the set $\{w(a) : a \in A\}$ on which ψ is constant is said to be *monochromatic*. Given $k,t \in \mathbb{N}$, we let C_t^k be the set of length k words over the alphabet $\{1,2,\ldots,t\}$. A set $L \subseteq C_t^k$ is a *combinatorial line* if and only if there is some variable word w over $\{1,2,\ldots,t\}$ such that

$$L = \big\{w(a) : a \in \{1,2,\ldots,t\}\big\}\,.$$

Corollary 1.2. *Let $t,r \in \mathbb{N}$. There exists $n \in \mathbb{N}$ such that if C_t^n is r-colored, then there is a monochromatic combinatorial line.*

Proof. If no such n exists, one may choose for each n, $\psi_n : C_t^n \to \{0,1,\ldots,r-1\}$ for which there is no monochromatic combinatorial line. Let $\psi = \bigcup_{n=1}^{\infty} \psi_n$ and pick by Theorem 1.1 a variable word w over $\{1,2,\ldots,t\}$ such that ψ is constant on $L = \big\{w(a) : a \in \{1,2,\ldots,t\}\big\}$. Let n be the length of w. Then L is a monochromatic combinatorial line with respect to the coloring ψ_n. □

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Many other results of Ramsey Theory assert the existence of some $n \in \mathbb{N}$ which guarantees monochromatic structures for a given r-coloring. There has been widespread interest in determining the smallest n which does the job. Several such numbers have been found for results related to van der Waerden's Theorem and Ramsey's Theorem. (See [3].) Further, Shelah's proof [7] that the number n guaranteed by Corollary 1.2 is a primitive recursive function of $|A|$ and r created substantial interest. It is therefore perhaps surprising that it was only recently shown by Hindman and Tressler in [6] that for $|A| = 3$ and $r = 2$, $n = 4$ is as guaranteed by Corollary 1.2.

The main result of [6] established that if the set C_3^4 is 2-colored, there is a monochromatic combinatorial line contained in C_3^4. We investigate in this paper how small a set can be and have this property. While the notion in the following definition can be made more general, we restrict it to subsets of C_3^4 as well as to 2-colorings here because that is what we are mainly concerned with in this paper. (We shall briefly discuss extensions to higher dimensions at the end of the paper.) And, since we are concerned with words over the alphabet $\{1, 2, 3\}$, we write the members of C_3^4 in the form 1323 rather than $\langle 1, 3, 2, 3\rangle$.

Definition 1.3. A subset A of C_3^4 is a *Hales-Jewett set* if and only if whenever A is 2-colored, there must exist a combinatorial line. It is a *minimal Hales-Jewett set* if and only if it does not properly contain another Hales-Jewett set.

An analogous situation holds with respect to van der Waerden's Theorem. Chvátal [1] showed that if the set $\{1, 2, ..., 35\}$ is 2-colored, there must be a monochromatic length four arithmetic progression, and that this is not true for the set $\{1, 2, ..., 34\}$. In [2], Graham showed that when the set

$$\{1, 4, 7, 8, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 30, 31, 34, 37\}$$

(which has 27 members) is 2-colored, it must contain a monochromatic length 4 arithmetic progression. We are grateful to the referee for pointing out that the corresponding situation does not apply to Ramsey's Theorem itself as shown by the following.

Theorem 1.4. *Let $k, r \in \mathbb{N}$ and let n be the least positive integer with the property that whenever K_n, the complete graph on $\{1, 2, \ldots, n\}$, is r-colored, there must be a monochromatic copy of K_k. If one edge is removed from K_n, the resulting graph can be r-colored with no monochromatic copy of K_k.*

Proof. Let G be the graph consisting of K_n with the edge $\{n-1, n\}$ removed. Let $\psi : K_{n-1} \to \{1, 2, \ldots, r\}$ be an r-coloring with no monochromatic copy of K_k. Define $\varphi : G \to \{1, 2, \ldots, r\}$ as follows for the edge $\{a, b\}$ of G, where $a < b$. If $b < n$, $\varphi(\{a, b\}) = \psi(\{a, b\})$. If $b = n$, $\varphi(\{a, b\}) = \psi(\{a, n-1\})$. □

In Section 2 of this paper we produce a minimal Hales-Jewett set with 37 members. In Section 3 we analyze the structure of the set of words which lie on a combinatorial line with one of 1111, 2222, or 3333. In Section 4, we show that any minimal Hales-Jewett set must have at least 25 members.

2 A minimal Hales-Jewett set. To find minimal Hales-Jewett sets, we utilized a computer program, which would take as input a list of length 4 words over the alphabet $\{1, 2, 3\}$ and find a line free 2-coloring, if one exists. It would assign the first word on the list to color 0. When no new assignments were forced (as was always true after the first assignment) it would find the first unassigned word and assign it to color 0. After each new assignment, it would then see first whether any monochromatic combinatorial lines had been formed.

If so, it would find the last free assignment, erase all assignments that resulted from that free assignment, and assign that word to color 1. After checking for monochromatic lines, it then checked whether any two words from a combinatorial line were the same color, and if the third word from the line was in the set, would assign it to the opposite color. If all words were colored with no monochromatic line, the program announced that it had found a coloring and printed it. If a monochromatic line was found where the first word was the last free assignment, it announced that no line free colorings exist.

The cases listed in the proof of the following theorem were produced by the algorithm described above. The reader need not trust our computer, however, as it is routine to verify that one of these cases must hold. And the fact that any given case forces a monochromatic line can be routinely established by hand in about fifteen minutes.

It should be noted that the algorithm is very sensitive to the order of the elements. For example, if the word 3333 is moved to the end of the list, the number of cases needed to establish that there are no line free colorings goes from 58 to 137. After we have proved the following theorem, we will discuss how the particular minimal Hales-Jewett set was found.

Theorem 2.1. *There is a minimal Hales-Jewett set with* 37 *members.*

Proof. Let $A = \{1111, 2222, 3333, 1222, 1121, 1122, 2111, 1333, 1113, 1313, 2122, 1133, 1131, 1223, 1323, 2223, 2212, 2232, 3313, 3323, 3133, 2121, 2323, 2211, 2233, 3131, 3322, 2131, 2133, 3122, 1123, 2213, 2333, 3111, 3222, 2123\ , 2132\}$.

Suppose we have $\varphi : A \to \{0, 1\}$ with respect to which there are no monochromatic combinatorial lines. We may assume without loss of generality that $\varphi(1111) = 0$. Then one of the 58 cases listed in Table 1 must hold, where, for example, case 3 is the event that 1111, 2222, 1222, and 1113 are assigned to color 0, while 1121 and 1122 are assigned to color 1.

As we mentioned before, it is routine to verify that each case results in a monochromatic line. We illustrate the process by verifying that case 49 yields a monochromatic line. We have that $\varphi(1111) = \varphi(1222) = \varphi(1122) = \varphi(1113) = 0$ and $\varphi(2222) = \varphi(3333) = \varphi(1121) = 1$.

$$\begin{array}{ll}
\varphi(1222) = \varphi(1111) = 0 & \text{so } \varphi(1333) = 1; \\
\varphi(3333) = \varphi(1333) = 1 & \text{so } \varphi(2333) = 0; \\
\varphi(1122) = \varphi(1111) = 0 & \text{so } \varphi(1133) = 1; \\
\varphi(1133) = \varphi(3333) = 1 & \text{so } \varphi(2233) = 0; \\
\varphi(2233) = \varphi(2333) = 0 & \text{so } \varphi(2133) = 1; \\
\varphi(2133) = \varphi(1133) = 1 & \text{so } \varphi(3133) = 0; \\
\varphi(3133) = \varphi(1111) = 0 & \text{so } \varphi(2122) = 1; \\
\varphi(2122) = \varphi(2133) = 1 & \text{so } \varphi(2111) = 0; \\
\varphi(1113) = \varphi(3133) = 0 & \text{so } \varphi(2123) = 1; \\
\varphi(2123) = \varphi(2122) = 1 & \text{so } \varphi(2121) = 0; \\
\varphi(2121) = \varphi(1111) = 0 & \text{so } \varphi(3131) = 1; \\
\varphi(2121) = \varphi(2111) = 0 & \text{so } \varphi(2131) = 1; \\
\varphi(2131) = \varphi(2133) = 1 & \text{so } \varphi(2132) = 0; \\
\varphi(2131) = \varphi(3131) = 1 & \text{so } \varphi(1131) = 0.
\end{array}$$

But then $\varphi(3133) = \varphi(2132) = \varphi(1131) = 0$, a contradiction. □

Case No.	1111	2222	3333	1222	1121	1122	2111	1333	1113	1313	2122
1	0	0		0	0						
2	0	0		0	1	0					
3	0	0		0	1	1			0		
4	0	0		0	1	1			1		
5	0	0		1	0	0					
6	0	0		1	0	1	0		0		
7	0	0		1	0	1	0		1	0	
8	0	0		1	0	1	0		1	1	
9	0	0		1	0	1	1	0	0		
10	0	0		1	0	1	1	0	1	0	
11	0	0		1	0	1	1	0	1	1	
12	0	0		1	0	1	1	1			
13	0	0		1	1	0	0				
14	0	0		1	1	0	1	0			
15	0	0		1	1	0	1	1			
16	0	0		1	1	1	0		0		
17	0	0		1	1	1	0		1		
18	0	0		1	1	1	1	0	0		
19	0	0		1	1	1	1	0	1		
20	0	0		1	1	1	1	1			
21	0	1	0	0	0	0					
22	0	1	0	0	0	1			0		
23	0	1	0	0	0	1			1		
24	0	1	0	0	1	0	0		0	0	
25	0	1	0	0	1	0	0		0	1	
26	0	1	0	0	1	0	0		1		
27	0	1	0	0	1	0	1		0	0	
28	0	1	0	0	1	0	1		0	1	
29	0	1	0	0	1	0	1		1		
30	0	1	0	0	1	1	0		0	0	
31	0	1	0	0	1	1	0		0	1	
32	0	1	0	0	1	1	0		1		
33	0	1	0	0	1	1	1		0	0	
34	0	1	0	0	1	1	1		0	1	
35	0	1	0	0	1	1	1		1		
36	0	1	0	1	0	0	0	0			
37	0	1	0	1	0	0	0	1		0	
38	0	1	0	1	0	0	0	1		1	
39	0	1	0	1	0	0	1			0	
40	0	1	0	1	0	0	1			1	

Table 1

Case No.	1111	2222	3333	1222	1121	1122	2111	1333	1113	1313	2122
41	0	1	0	1	0	1					
42	0	1	0	1	1	0	0		0		
43	0	1	0	1	1	0	0		1		
44	0	1	0	1	1	0	1		0		
45	0	1	0	1	1	0	1		1		
46	0	1	0	1	1	1					
47	0	1	1	0	0	0					
48	0	1	1	0	0	1					
49	0	1	1	0	1	0			0		
50	0	1	1	0	1	0			1		
51	0	1	1	0	1	1					
52	0	1	1	1	0	0	0	0			0
53	0	1	1	1	0	0	0	0			1
54	0	1	1	1	0	0	0	1			
55	0	1	1	1	0	0	1				
56	0	1	1	1	0	1					
57	0	1	1	1	1	0					
58	0	1	1	1	1	1					

Table 1 – Continued

Finding a minimal Hales-Jewett set (when armed with the program described at the start of this section) is routine. We knew by [6] that C_3^4 is a Hales-Jewett set. One may delete one element. If the result is still a Hales-Jewett set one may take that new Hales-Jewett set and delete one element. If the result is not a Hales-Jewett set, restore the deleted element and delete another element. Eventually, one arrives at a Hales-Jewett set with the property that when any of its elements is deleted, there is a line free coloring, so the resulting set is a minimal Hales-Jewett set. Using this process we arrived at the following 44 element minimal Hales-Jewett set. $B = \{1111, 2222, 3333, 1222, 1112, 1121, 1122, 2111, 1211, 1333, 1113, 1313, 1133, 1131, 1223, 1233, 1323, 2223, 2212, 2232, 2122, 3332, 3313, 3323, 3133, 3233, 2112, 2121, 2323, 2211, 2233, 3113, 3223, 3131, 3322, 2131, 2133, 3112, 3122, 1123, 2213, 2333, 3111, 3222\}$.

Call two subsets C and D of C_3^4 *neighbors* provided $|C \setminus D| = |D \setminus C| = 1$. We checked all 1,628 neighbors of B. Of these, 7 were also Hales-Jewett sets. One of these neighbors, namely $(B \setminus \{3223\}) \cup \{2123\}$ had itself several neighbors (all of which added 2132) that could then be reduced to the set A used in the proof of Theorem 2.1. None of the neighbors of A is a Hales-Jewett set.

3 The diagonal set. In this section, we analyze the structure of a subset of C_3^4 which will help us quickly determine whether specified 2-colorings have monochromatic combinatorial lines.

Definition 3.1. The *diagonal* of C_3^4 is $\{1111, 2222, 3333\}$. The *diagonal set*, D_3^4, is the set of words in C_3^4 which lie on a combinatorial line with a member of the diagonal.

Notice that a word w is in the diagonal set if and only if at most two letters occur in w. Our analysis of lines involving members of the diagonal set is based on Table 2, in which the members of the diagonal appear in each 3×3 matrix, and each other member of D_3^4 occurs once. Given $\emptyset \neq X \subseteq \{2,3,4\}$, row i of τ_X is the combinatorial line generated by the variable word $w(v) = a_1a_2a_3a_4$ and column i is the combinatorial line generated by the variable word $u(v) = b_1b_2b_3b_4$ where

$$a_j = \begin{cases} v & \text{if } j \in X \\ i & \text{if } j \notin X \end{cases} \quad \text{and } b_j = \begin{cases} i & \text{if } j \in X \\ v & \text{if } j \notin X \,. \end{cases}$$

We denote by $\tau_X(i,j)$ the entry in row i and column j of array τ_X.

$\tau_{\{2\}}$

1111	1211	1311
2122	2222	2322
3133	3233	3333

$\tau_{\{3\}}$

1111	1121	1131
2212	2222	2232
3313	3323	3333

$\tau_{\{4\}}$

1111	1112	1113
2221	2222	2223
3331	3332	3333

$\tau_{\{2,3\}}$

1111	1221	1331
2112	2222	2332
3113	3223	3333

$\tau_{\{2,4\}}$

1111	1212	1313
2121	2222	2323
3131	3232	3333

$\tau_{\{3,4\}}$

1111	1122	1133
2211	2222	2233
3311	3322	3333

$\tau_{\{2,3,4\}}$

1111	1222	1333
2111	2222	2333
3111	3222	3333

Table 2

Lemma 3.2. *If L is a combinatorial line contained in D_3^4, then one of the following statements holds.*

(a) $L = \{1111, 2222, 3333\}$.

(b) *There exists nonempty $X \subseteq \{2,3,4\}$ such that the elements of L form a row of τ_X.*

(c) *There exists nonempty $X \subseteq \{2,3,4\}$ such that the elements of L form a column of τ_X.*

Proof. Pick a variable word $w(v)$ such that $L = \{w(1), w(2), w(3)\}$ and pick $a_1, a_2, a_3, a_4 \in \{1,2,3,v\}$ such that $w(v) = a_1a_2a_3a_4$. Since $L \subseteq D_3^4$, at most one of 1, 2, and 3 occur in $w(v)$. If $w(v) = vvvv$, then conclusion (a) holds, so assume that we have a unique $k \in \{1,2,3\}$ which occurs in $w(v)$.

Assume first that $a_1 = k$ and let $X = \{i \in \{2,3,4\} : a_i = v\}$. Then L is row k of τ_X.

Now assume that $a_1 = v$ and let $X = \{i \in \{2,3,4\} : a_i = k\}$. Then L is column k of τ_X. □

Lemma 3.3. *If L is a combinatorial line in C_3^4, $L \cap D_3^4 \neq \emptyset$, and $L \setminus D_3^4 \neq \emptyset$, then either*

(a) *there exist $\emptyset \neq X \subsetneq Y \subseteq \{2,3,4\}$ and $i \neq j$ in $\{1,2,3\}$ such that $L \cap D_3^4 = \{\tau_X(i,j), \tau_Y(i,j)\}$ or*

(b) *there exist disjoint nonempty subsets X and Y of $\{2,3,4\}$ and $i \neq j$ in $\{1,2,3\}$ such that $L \cap D_3^4 = \{\tau_X(i,j), \tau_Y(j,i)\}$.*

Proof. Pick a variable word $w(v)$ such that $L = \{w(1), w(2), w(3)\}$ and pick $a_1, a_2, a_3, a_4 \in \{1, 2, 3, v\}$ such that $w(v) = a_1a_2a_3a_4$. Since $L \cap D_3^4 \neq \emptyset$, at most two of 1, 2, and 3 occur in $w(v)$. Since $L \setminus D_3^4 \neq \emptyset$, at least two of 1, 2, and 3 occur in $w(v)$. Let $Z = \{t \in \{1, 2, 3, 4\} : a_t = v\}$.

Assume first that $1 \notin Z$. Let $i = a_1$ and let j be the other member of $\{1, 2, 3\}$ occurring in $w(v)$. Let $X = \{t \in \{2, 3, 4\} : a_t = j\}$ and let $Y = X \cup Z$. Then $w(i) = \tau_X(i, j)$ and $w(j) = \tau_Y(i, j)$.

Now assume that $1 \in Z$. Let i and j be the members of $\{1, 2, 3\}$ that occur in $w(v)$. Let $Y = \{t \in \{2, 3, 4\} : a_t = i\}$ and let $X = \{t \in \{2, 3, 4\} : a_t = j\}$. Then $w(i) = \tau_X(i, j)$ and $w(j) = \tau_Y(j, i)$. □

The converses of Lemmas 3.2 and 3.3 hold as well. That is, given any nonempty $X \subseteq \{2, 3, 4\}$ any row or column of τ_X forms a line contained in D_3^4; given $\emptyset \neq X \subsetneq Y \subseteq \{2, 3, 4\}$ and $i \neq j$ in $\{1, 2, 3\}$, $\tau_X(i, j)$ and $\tau_Y(i, j)$ lie on a line L with $L \setminus D_3^4 \neq \emptyset$; and given disjoint nonempty subsets X and Y of $\{2, 3, 4\}$ and $i \neq j$ in $\{1, 2, 3\}$, $\tau_X(i, j)$ and $\tau_Y(j, i)$, lie on a line L with $L \setminus D_3^4 \neq \emptyset$. We shall not need these assertions so we will not prove them.

4 A lower bound.

We show in this section that any Hales-Jewett set (contained in C_3^4) must have at least 25 members. We do this by introducing a partition of C_3^4 with the property that any Hales-Jewett set must contain a specified number from each cell of the partition. The first result in this direction is quite simple. (We did, however, find it quite surprising that one could get a non-Hales-Jewett set by deleting one member from C_3^4.)

Lemma 4.1. *Any Hales-Jewett set must contain* $\{1111, 2222, 3333\}$.

Proof. Let $A = C_3^4 \setminus \{1111\}$, let $B = \{$1112, 1113, 1121, 1123, 1131, 1132, 1211, 1213, 1222, 1231, 1311, 1312, 1321, 1333, 2111, 2113, 2122, 2131, 2212, 2221, 2222, 2233, 2311, 2323, 2332, 3111, 3112, 3121, 3133, 3211, 3223, 3232, 3313, 3322, 3331, 3333$\}$, and let $C = A \setminus B$.

Table 3 shows the members of B underlined. A glance at the table together with Lemma 3.2 establishes that there are no monochromatic lines contained in D_3^4. One can also routinely verify that there are no monochromatic lines meeting D_3^4. Using Lemma 3.3 one quickly sees, for example, that one needs to verify that the line {1211, 2212 , 3213} is not contained in B and one does not need to worry about the line {1211, 1212 , 1213}.

	$\tau_{\{2\}}$	
	$\underline{1211}$	$\underline{1311}$
$\underline{2122}$	$\underline{2222}$	2322
$\underline{3133}$	3233	$\underline{3333}$

	$\tau_{\{3\}}$	
	$\underline{1121}$	$\underline{1131}$
$\underline{2212}$	$\underline{2222}$	2232
$\underline{3313}$	3323	$\underline{3333}$

	$\tau_{\{4\}}$	
	$\underline{1112}$	$\underline{1113}$
$\underline{2221}$	$\underline{2222}$	2223
$\underline{3331}$	3332	$\underline{3333}$

	$\tau_{\{2,3\}}$	
	1221	1331
2112	$\underline{2222}$	$\underline{2332}$
3113	$\underline{3223}$	$\underline{3333}$

	$\tau_{\{2,4\}}$	
	1212	1313
2121	$\underline{2222}$	$\underline{2323}$
3131	$\underline{3232}$	$\underline{3333}$

	$\tau_{\{3,4\}}$	
	1122	1133
2211	$\underline{2222}$	$\underline{2233}$
3311	$\underline{3322}$	$\underline{3333}$

	$\tau_{\{2,3,4\}}$	
	$\underline{1222}$	$\underline{1333}$
$\underline{2111}$	$\underline{2222}$	2333
$\underline{3111}$	3222	$\underline{3333}$

Table 3

That leaves the lines contained in $C_3^4 \setminus D_3^4$. We do not see a particularly quick way to check these. They are generated by the variable words which have exactly one occurrence each of 1, 2, 3, and v. There are 24 of these, and as far as we can see, one simply has to check them all. One then has verified that neither B nor C contains a combinatorial line, so A is not a Hales-Jewett set. This shows that 1111 must be a member of any Hales-Jewett set. By permuting 1, 2, and 3, we also have that 2222 and 3333 must be members of any Hales-Jewett set. □

In the proof above, we have used the following obvious fact.

Remark 4.2. *Let σ be a permutation of $\{1,2,3\}$ and let τ be a permutation of $\{1,2,3,4\}$. Define $\sigma^* : C_3^4 \to C_3^4$ and $\tau^\diamond : C_3^4 \to C_3^4$ as follows. Given $w = \langle a_1, a_2, a_3, a_4\rangle \in C_3^4$, $\sigma^*(w) = \langle \sigma(a_1), \sigma(a_2), \sigma(a_3), \sigma(a_4)\rangle$ and $\tau^\diamond(w) = \langle a_{\tau(1)}, a_{\tau(2)}, a_{\tau(3)}, a_{\tau(4)}\rangle$. Then σ^* and $\tau^\diamond$ are permutations of C_3^4 that take combinatorial lines to combinatorial lines. Thus if $A \subseteq C_3^4$, the following statements are equivalent.*

(1) *A is a Hales-Jewett set.*

(2) *$\sigma^*[A]$ is a Hales-Jewett set.*

(3) *$\tau^\diamond[A]$ is a Hales-Jewett set.*

Lemma 4.3. *Let $A = \{1123, 1132, 1213, 1312, 1231, 1321\}$, let $B = \{2213,\ 2231,\ 2123,\ 2321,\ 2132,\ 2312\}$, and let $C = \{3312,\ 3321,\ 3132,\ 3231,\ 3123,\ 3213\}$. Any Hales-Jewett set must include two members of $A \cup B \cup C$.*

Proof. We show that any Hales-Jewett set must include one member of $A \cup B$. If σ is the permutation of $\{1,2,3\}$ which interchanges 1 and 3, then $\sigma^*[A] = C$ and $\sigma^*[B] = B$ so by Remark 4.2 it will follow that also any Hales-Jewett set must include one member of $B \cup C$. If ν is the permutation of $\{1,2,3\}$ which interchanges 2 and 3, then $\nu^*[A] = A$ and $\nu^*[B] = C$ so by Remark 4.2 it will follow that also any Hales-Jewett set must include one member of $A \cup C$ and consequently, that any Hales-Jewett set must include two members of $A \cup B \cup C$.

Let $D = \{$1111, 1112, 1121, 1122, 1211, 1212, 1221, 1233, 1323, 1332, 1333, 2111, 2113, 2131, 2133, 2223, 2232, 2233, 2311, 2313, 2322, 2323, 2331, 2332, 3112, 3121, 3123, 3132, 3133, 3211, 3213, 3222, 3223, 3231, 3232, 3312, 3313, 3321, 3322, 3331, 3333$\}$ and let $E = C_3^4 \setminus (A \cup B \cup D)$. Then $C_3^4 \setminus (A \cup B) = D \cup E$. We need to show that neither D nor E contains a combinatorial line.

Table 4 shows the members of D underlined. A glance at the table together with Lemma 3.2 establishes that there are no monochromatic lines contained in D_3^4.

Further, this table along with Lemma 3.3 helps one easily establish that there are no monochromatic lines with two members of D_3^4. For example $\tau_{\{2\}}(3,2) = 3233 \in E$ and $\tau_{\{2,3\}}(3,2) = 3223 \in D$, so one need not worry about 3213. And $\tau_{\{2\}}(1,2) = 1211 \in E$ and $\tau_{\{2,3\}}(1,2) = 1221 \in D$ so one checks 1231 and notes that it is a member of A so is not colored at all.

One observes easily that there are no lines contained in $E \setminus D_3^4 = \{$1223, 1232, 1322, 3122, 3212, 3221$\}$, and with somewhat more effort that there are no lines contained in $D \setminus D_3^4 = \{$3312, 3321, 3132, 3231, 3123, 3213, 2113, 2131, 2311, 3112, 3121, 3211, 1233, 1323, 1332, 2133, 2313, 2331$\}$. □

Lemma 4.4. *Let $A = \{2113,\ 2131,\ 2311,\ 3112,\ 3121,\ 3211\}$, let $B = \{1223,\ 1232,\ 1322,\ 3122,\ 3212,\ 3221\}$, and let $C = \{1233,\ 1323,\ 1332,\ 2133,\ 2313,\ 2331\}$. Any Hales-Jewett set must include two members of $A \cup B \cup C$.*

	$\tau_{\{2\}}$	
$\underline{1111}$	$\underline{1211}$	1311
2122	2222	$\underline{2322}$
$\underline{3133}$	3233	$\underline{3333}$

	$\tau_{\{3\}}$	
$\underline{1111}$	$\underline{1121}$	1131
2212	2222	$\underline{2232}$
$\underline{3313}$	3323	$\underline{3333}$

	$\tau_{\{4\}}$	
$\underline{1111}$	$\underline{1112}$	1113
2221	2222	$\underline{2223}$
$\underline{3331}$	3332	$\underline{3333}$

	$\tau_{\{2,3\}}$	
$\underline{1111}$	$\underline{1221}$	1331
2112	2222	$\underline{2332}$
3113	$\underline{3223}$	$\underline{3333}$

	$\tau_{\{2,4\}}$	
$\underline{1111}$	$\underline{1212}$	1313
2121	2222	$\underline{2323}$
3131	$\underline{3232}$	$\underline{3333}$

	$\tau_{\{3,4\}}$	
$\underline{1111}$	$\underline{1122}$	1133
2211	2222	$\underline{2233}$
3311	$\underline{3322}$	$\underline{3333}$

	$\tau_{\{2,3,4\}}$	
$\underline{1111}$	1222	$\underline{1333}$
$\underline{2111}$	2222	2333
3111	$\underline{3222}$	$\underline{3333}$

Table 4

Proof. We show that any Hales-Jewett set must include one member of $B \cup C$. If σ is the permutation of $\{1,2,3\}$ which sends 1 to 3, 3 to 2, and 2 to 1, then $\sigma^*[B] = A$ and $\sigma^*[C] = B$ so by Remark 4.2 it will follow that also any Hales-Jewett set must include one member of $A \cup B$. Applying σ^* one more time it will follow that any Hales-Jewett set must include one member of $A \cup C$, and consequently, that any Hales-Jewett set must include two members of $A \cup B \cup C$.

Let $D = \{$1111, 1112, 1121, 1123, 1132, 1133, 1211, 1213, 1231, 1312, 1313, 1321, 1331, 1333, 2111, 2113, 2123, 2131, 2132, 2213, 2222, 2223, 2231, 2232, 2311, 2312, 2321, 2322, 3112, 3113, 3121, 3131, 3133, 3211, 3222, 3223, 3232, 3311, 3313, 3322, 3331$\}$ and let $E = C_3^4 \setminus (B \cup C \cup D)$. Then $C_3^4 \setminus (B \cup C) = D \cup E$. We need to show that neither D nor E contains a combinatorial line.

Table 5 shows the members of D underlined. A glance at the table together with Lemma 3.2 establishes that there are no monochromatic lines contained in D_3^4.

	$\tau_{\{2\}}$	
$\underline{1111}$	$\underline{1211}$	1311
2122	$\underline{2222}$	$\underline{2322}$
$\underline{3133}$	3233	3333

	$\tau_{\{3\}}$	
$\underline{1111}$	$\underline{1121}$	1131
2212	$\underline{2222}$	$\underline{2232}$
$\underline{3313}$	3323	3333

	$\tau_{\{4\}}$	
$\underline{1111}$	$\underline{1112}$	1113
2221	$\underline{2222}$	$\underline{2223}$
$\underline{3331}$	3332	3333

	$\tau_{\{2,3\}}$	
$\underline{1111}$	1221	$\underline{1331}$
2112	$\underline{2222}$	2332
$\underline{3113}$	$\underline{3223}$	3333

	$\tau_{\{2,4\}}$	
$\underline{1111}$	1212	$\underline{1313}$
2121	$\underline{2222}$	2323
$\underline{3131}$	$\underline{3232}$	3333

	$\tau_{\{3,4\}}$	
$\underline{1111}$	1122	$\underline{1133}$
2211	$\underline{2222}$	2233
$\underline{3311}$	$\underline{3322}$	3333

	$\tau_{\{2,3,4\}}$	
$\underline{1111}$	1222	$\underline{1333}$
$\underline{2111}$	$\underline{2222}$	2333
3111	$\underline{3222}$	3333

Table 5

As in the proof of Lemma 4.3, one can use Lemma 3.3 to show that there are no monochromatic lines intersecting D_3^4, and check individually that there are no monochromatic lines missing D_3^4. $\square$

Lemma 4.5. *Let $A = \{1122, 1212, 1221, 2112, 2121, 2211\}$, let $B = \{1133, 1313, 1331, 3113, 3131, 3311\}$, and let $C = \{2233, 2323, 2332, 3223, 3232, 3322\}$. Any Hales-Jewett set must include two members of A, two members of B, and two members of C.*

Proof. We shall show that for each $w \in A$, there is a 2-coloring of $C_3^4 \setminus (A \setminus \{w\}) = (C_3^4 \setminus A) \cup \{w\}$ with no monochromatic lines. If σ is the permutation of $\{1, 2, 3\}$ which interchanges 2 and 3 and $u \in B$, then $\sigma^*[A \setminus \{\sigma(u)\}] = B \setminus \{u\}$. If ν is the permutation of $\{1, 2, 3\}$ which interchanges 1 and 3 and $u \in C$, then $\sigma^*[A \setminus \{\nu(u)\}] = C \setminus \{u\}$. So the conclusion will follow from Remark 4.2.

We now claim that it suffices to show that there is a 2-coloring of $C_3^4 \setminus \{1212, 1221, 2112, 2121, 2211\}$ with no monochromatic line. (That is, it suffices to establish the claim above with $w = 1122$.) To see this suppose we have done so, and let u be another member of A. Then there is a permutation τ of $\{1, 2, 3, 4\}$ such that $\tau^\diamond[A \setminus \{1122\}] = A \setminus \{u\}$. (For example, if $u = 2112$, let τ be the permutation of $\{1, 2, 3, 4\}$ which interchanges 1 and 3.)

Let $D = \{1111, 1113, 1122, 1131, 1211, 1223, 1232, 1233, 1313, 1322, 1323, 1331, 1332, 2111, 2123, 2132, 2133, 2212, 2213, 2221, 2231, 2312, 2321, 2322, 2333, 3113, 3122, 3123, 3131, 3132, 3212, 3221, 3222, 3233, 3311, 3312, 3321, 3323, 3332, 3333\}$ and let $E = (C_3^4 \setminus \{1212, 1221, 2112, 2121, 2211\}) \cup D$. Then $C_3^4 \setminus \{1212, 1221, 2112, 2121, 2211\} = D \cup E$. Using Table 6 in which the members of D are underlined, one shows in the same fashion as in the previous few lemmas that neither D nor E contains a line. □

	$\tau_{\{2\}}$	
$\underline{1111}$	$\underline{1211}$	1311
2122	2222	$\underline{2322}$
3133	$\underline{3233}$	$\underline{3333}$

	$\tau_{\{3\}}$	
$\underline{1111}$	1121	$\underline{1131}$
$\underline{2212}$	2222	2232
3313	$\underline{3323}$	$\underline{3333}$

	$\tau_{\{4\}}$	
$\underline{1111}$	1112	$\underline{1113}$
$\underline{2221}$	2222	2223
3331	$\underline{3332}$	$\underline{3333}$

	$\tau_{\{2,3\}}$	
$\underline{1111}$		$\underline{1331}$
	2222	2332
$\underline{3113}$	3223	$\underline{3333}$

	$\tau_{\{2,4\}}$	
$\underline{1111}$		$\underline{1313}$
	2222	2323
$\underline{3131}$	3232	$\underline{3333}$

	$\tau_{\{3,4\}}$	
$\underline{1111}$	$\underline{1122}$	1133
	2222	2233
$\underline{3311}$	3322	$\underline{3333}$

	$\tau_{\{2,3,4\}}$	
$\underline{1111}$	1222	1333
$\underline{2111}$	2222	$\underline{2333}$
3111	$\underline{3222}$	$\underline{3333}$

Table 6

We have saved the messiest lemma for last. (It is also the most powerful, providing the largest number of words that must be in any Hales-Jewett set.) In this proof we reduce to finding colorings of three different sets, rather than the one we have been able to get by with up to this point.

Lemma 4.6. *Let $A = \{1112, 1113, 1121, 1131, 1211, 1311, 2111, 3111\}$, let $B = \{2221, 2223, 2212, 2232, 2122, 2322, 1222, 3222\}$, and let $C = \{3331, 3332, 3313, 3323, 3133, 3233, 1333, 2333\}$. Any Hales-Jewett set must include four members of A, four members of B, and four members of C.*

Proof. As before, using permutations of $\{1, 2, 3\}$ that interchange two members, we see easily that it suffices to establish that any Hales-Jewett set must include four members of A. For this it in turn suffices to show that if K is any three element subset of A, then there is a 2-coloring of $(C_3^4 \setminus A) \cup K$ with no monochromatic lines. Unfortunately, there are 56 choices for K. We claim that it suffices to consider three possibilities, namely $K_1 = \{1112,$ $1113,\ 1121\}$, $K_2 = \{1112,\ 1121,\ 1211\}$, and $K_3 = \{1112,\ 1121,\ 1311\}$. To this end, let σ be the permutation of $\{1, 2, 3\}$ which interchanges 2 and 3. We shall show that if K is any three element subset of A, then there exist $i \in \{1, 2, 3\}$ and a permutation τ of $\{1, 2, 3, 4\}$ such that either $K = \tau^\diamond[K_i]$ or $K = (\sigma^* \circ \tau^\diamond)[K_i]$. By Remark 4.2, this will suffice.

Let $K = \{a_1a_2a_3a_4, b_1b_2b_3b_4, c_1c_2c_3c_4\}$ be a three element subset of A. We may presume that the elements of K are listed in lexicographic order, that is, in the same order as they appear in the listing of the elements of A above. There exist $k, l, m \in \{1, 2, 3, 4\}$ such that $k \geq l \geq m$, $a_k \neq 1$, $b_l \neq 1$, and $c_m \neq 1$. Further, either $k > l$ or $l > m$. We consider six cases.

Case 1. $k = l > m$. Then $a_k = 2$ and $b_k = 3$. Let τ be a permutation of $\{1, 2, 3, 4\}$ such that $\tau(k) = 4$ and $\tau(m) = 3$. If $c_m = 2$, then $K = \tau^\diamond[K_1]$. If $c_m = 3$, then $K = (\sigma^* \circ \tau^\diamond)[K_1]$.

Case 2. $k > l = m$. Then $b_m = 2$ and $c_m = 3$. Let τ be a permutation of $\{1, 2, 3, 4\}$ such that $\tau(k) = 3$ and $\tau(m) = 4$. If $a_k = 2$, then $K = \tau^\diamond[K_1]$. If $a_k = 3$, then $K = (\sigma^* \circ \tau^\diamond)[K_1]$.

Case 3. $k > l > m$ and $a_k = b_l = c_m$. Let τ be a permutation of $\{1, 2, 3, 4\}$ such that $\tau(k) = 4$, $\tau(l) = 3$, and $\tau(m) = 2$. If $a_k = 2$, then $K = \tau^\diamond[K_2]$. If $a_k = 3$, then $K = (\sigma^* \circ \tau^\diamond)[K_2]$.

Case 4. $k > l > m$ and $a_k = b_l \neq c_m$. Let τ be a permutation of $\{1, 2, 3, 4\}$ such that $\tau(k) = 4$, $\tau(l) = 3$, and $\tau(m) = 2$. If $a_k = 2$, then $K = \tau^\diamond[K_3]$. If $a_k = 3$, then $K = (\sigma^* \circ \tau^\diamond)[K_3]$.

Case 5. $k > l > m$ and $a_k \neq b_l = c_m$. Let τ be a permutation of $\{1, 2, 3, 4\}$ such that $\tau(k) = 2$, $\tau(l) = 4$, and $\tau(m) = 3$. If $a_k = 3$, then $K = \tau^\diamond[K_3]$. If $a_k = 2$, then $K = (\sigma^* \circ \tau^\diamond)[K_3]$.

Case 6. $k > l > m$ and $a_k = c_m \neq b_l$. Let τ be a permutation of $\{1, 2, 3, 4\}$ such that $\tau(k) = 4$, $\tau(l) = 2$, and $\tau(m) = 3$. If $a_k = 2$, then $K = \tau^\diamond[K_3]$. If $a_k = 3$, then $K = (\sigma^* \circ \tau^\diamond)[K_3]$.

Now we describe a 2-coloring of $(C_3^4 \setminus A) \cup K_1$ with no monochromatic lines. Let $D = \{1111,$ 1113, 1122, 1212, 1221, 1223, 1232, 1233, 1322, 1323, 1332, 2112, 2121, 2123, 2132, 2133, 2211, 2213, 2221, 2232, 2312, 2313, 2322, 2331, 2333, 3122, 3123, 3132, 3212, 3213, 3222, 3231, 3233, 3312, 3321, 3323, 3332, $3333\}$ and let $E = C_3^4 \setminus \big((A \setminus K_1) \cup D\big)$. Using Table 7 which has members of D underlined, one establishes as before that there are no lines contained in D or E.

	$\tau_{\{2\}}$	
$\underline{1111}$		
2122	2222	$\underline{2322}$
3133	$\underline{3233}$	$\underline{3333}$

	$\tau_{\{3\}}$	
$\underline{1111}$	1121	
2212	2222	$\underline{2232}$
3313	$\underline{3323}$	$\underline{3333}$

	$\tau_{\{4\}}$	
$\underline{1111}$	1112	$\underline{1113}$
$\underline{2221}$	2222	2223
3331	$\underline{3332}$	$\underline{3333}$

	$\tau_{\{2,3\}}$	
$\underline{1111}$	$\underline{1221}$	1331
$\underline{2112}$	2222	2332
3113	3223	$\underline{3333}$

	$\tau_{\{2,4\}}$	
$\underline{1111}$	$\underline{1212}$	1313
$\underline{2121}$	2222	2323
3131	3232	$\underline{3333}$

	$\tau_{\{3,4\}}$	
$\underline{1111}$	$\underline{1122}$	1133
$\underline{2211}$	2222	2233
3311	3322	$\underline{3333}$

	$\tau_{\{2,3,4\}}$	
$\underline{1111}$	1222	1333
	2222	$\underline{2333}$
	$\underline{3222}$	$\underline{3333}$

Table 7

Next we describe a 2-coloring of $(C_3^4 \setminus A) \cup K_2$ with no monochromatic lines. Let

$D = \{1111, 1112, 1121, 1123, 1132, 1133, 1211, 1213, 1222, 1223, 1231, 1232, 1312, 1313, 1321, 1322, 1331, 2113, 2123, 2131, 2132, 2213, 2222, 2231, 2233, 2311, 2312, 2321, 2323, 2332, 2333, 3112, 3113, 3121, 3122, 3131, 3133, 3211, 3212, 3221, 3223, 3232, 3311, 3313, 3322, 3331\}$ and let $E = C_3^4 \setminus \big((A \setminus K_2) \cup D\big)$. Using Table 8 which has members of D underlined, one establishes as before that there are no lines contained in D or E.

	$\tau_{\{2\}}$			$\tau_{\{3\}}$			$\tau_{\{4\}}$	
$\underline{1111}$	$\underline{1211}$		$\underline{1111}$	$\underline{1121}$		$\underline{1111}$	$\underline{1112}$	
2122	$\underline{2222}$	2322	2212	$\underline{2222}$	2232	2221	$\underline{2222}$	2223
$\underline{3133}$	3233	3333	$\underline{3313}$	3323	3333	$\underline{3331}$	3332	3333
	$\tau_{\{2,3\}}$			$\tau_{\{2,4\}}$			$\tau_{\{3,4\}}$	
$\underline{1111}$	1221	$\underline{1331}$	$\underline{1111}$	1212	$\underline{1313}$	$\underline{1111}$	1122	$\underline{1133}$
2112	$\underline{2222}$	$\underline{2332}$	2121	$\underline{2222}$	$\underline{2323}$	2211	$\underline{2222}$	$\underline{2233}$
$\underline{3113}$	$\underline{3223}$	3333	$\underline{3131}$	$\underline{3232}$	3333	$\underline{3311}$	$\underline{3322}$	3333
				$\tau_{\{2,3,4\}}$				
			$\underline{1111}$	$\underline{1222}$	1333			
				$\underline{2222}$	$\underline{2333}$			
				3222	3333			

Table 8

Finally we describe a 2-coloring of $(C_3^4 \setminus A) \cup K_3$ with no monochromatic lines. Let $D = \{1111, 1122, 1212, 1221, 1223, 1232, 1233, 1311, 1322, 1323, 1332, 2112, 2121, 2122, 2133, 2211, 2213, 2223, 2231, 2232, 2312, 2313, 2321, 2331, 2333, 3123, 3132, 3212, 3213, 3221, 3222, 3231, 3233, 3312, 3321, 3323, 3332, 3333\}$ and let $E = C_3^4 \setminus \big((A \setminus K_3) \cup D\big)$. Using Table 9 which has members of D underlined, one establishes as before that there are no lines contained in D or E. □

	$\tau_{\{2\}}$			$\tau_{\{3\}}$			$\tau_{\{4\}}$	
$\underline{1111}$		$\underline{1311}$	$\underline{1111}$	1121		$\underline{1111}$	1112	
$\underline{2122}$	2222	2322	2212	2222	$\underline{2232}$	2221	2222	$\underline{2223}$
3133	$\underline{3233}$	$\underline{3333}$	3313	$\underline{3323}$	$\underline{3333}$	3331	$\underline{3332}$	$\underline{3333}$
	$\tau_{\{2,3\}}$			$\tau_{\{2,4\}}$			$\tau_{\{3,4\}}$	
$\underline{1111}$	$\underline{1221}$	1331	$\underline{1111}$	$\underline{1212}$	1313	$\underline{1111}$	$\underline{1122}$	1133
$\underline{2112}$	2222	2332	$\underline{2121}$	2222	2323	$\underline{2211}$	2222	2233
3113	3223	$\underline{3333}$	3131	3232	$\underline{3333}$	3311	3322	$\underline{3333}$
				$\tau_{\{2,3,4\}}$				
			$\underline{1111}$	1222	1333			
				2222	$\underline{2333}$			
				$\underline{3222}$	$\underline{3333}$			

Table 9

Theorem 4.7. *Any Hales-Jewett set in C_3^4 must contain at least 25 members.*

Proof. The sets in the statements of Lemmas 4.1, 4.3, 4.4, 4.5, and 4.6 partition C_3^4 and establish that any Hales-Jewett set must contain at least $3 + 2 + 2 + 6 + 12$ members. □

One can extend in the obvious way the definition of a Hales-Jewett set to higher dimensions. For $k \geq 4$, let $MHJ(k)$ be the smallest size of a Hales-Jewett set in C_3^k. One has trivially that $MHJ(k+1) \leq MHJ(k)$ because, if $A \subseteq C_3^k$ is a Hales-Jewett set, so is $\{a_1 a_2 \ldots a_k 1 : a_1 a_2 \ldots a_k \in A\}$. Unfortunately, our proof of Theorem 4.7 does not extend to higher dimensions, and we have only very trivial lower bounds for $MHJ(k)$ when $k > 4$. For example, the pigeon hole principle says that $MHJ(k) \geq 5$. One can do slightly better when one uses the fact that combinatorial lines are three element sets, any two of which have only one member in common. But that only allows one to raise the minimum to $MHJ(k) \geq 7$ since, as is well known, if $\{1,2,3,4,5,6,7\}$ is two colored one of the lines in the Fano plane $(\{\{1,2,3\},\{1,4,5\},\{1,6,7\},\{2,4,6\},\{2,5,7\},\{3,5,6\},\{3,4,7\}\})$ must be monochromatic and this is not true for any set of fewer than 7 triples.

Problem 4.8. *Find reasonable bounds for $MHJ(k)$ valid for arbitrarily large k.*

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