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THE STRONG ENDOMORPHISM KERNEL PROPERTY IN DISTRIBUTIVE DOUBLE *P*-ALGEBRAS

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Dedicated to the memory of Professor Kiyoshi Iséki

Abstract. An endomorphism of an algebra $\mathcal{A}$ is said to be *strong* if it is compatible with every congruence on $\mathcal{A}$; and $\mathcal{A}$ is said to have the *strong endomorphism kernel property* if every congruence on $\mathcal{A}$, other than the universal congruence, is the kernel of a strong endomorphism on $\mathcal{A}$. Here we describe, by way of Priestley duality, the distributive double p-algebras that have the strong endomorphism kernel property.

1 Introduction

In the volume published in celebration of Professor Iséki's 80th birthday, a particular open question that was posed in [3] is the following: *For an Ockham algebra $\mathcal{L}$ determine the congruences on $\mathcal{L}$ that are kernels of endomorphisms on $\mathcal{L}$.* Whereas a general solution to this is still an open problem, there has been some progress in investigating kernels of endomorphisms in various lattice-ordered algebras. In this connection, an algebra $\mathcal{A}$ is said to have the *endomorphism kernel property* [1] if every congruence on $\mathcal{A}$, other than the universal congruence, is the kernel of an endomorphism on $\mathcal{A}$. Equivalently, as is shown in [1,Theorem 1], an algebra $\mathcal{A}$ has the endomorphism kernel property if and only if every non-trivial epimorphic image of $\mathcal{A}$ is isomorphic to a subalgebra of $\mathcal{A}$. In [2] a strengthening of this notion was introduced. Specifically, if $\mathcal{A}$ is an algebra and ϑ is a congruence on $\mathcal{A}$ then an endomorphism e on $\mathcal{A}$ is said to be *compatible with* ϑ if

$$(\forall x, y \in \mathcal{A}) \qquad (x, y) \in \vartheta \Longrightarrow \big(e(x), e(y)\big) \in \vartheta.$$

If such an endomorphism e is compatible with every congruence on $\mathcal{A}$ then it is said to be *strong.* Then $\mathcal{A}$ is said to have the *strong endomorphism kernel property* if every congruence on $\mathcal{A}$, other than the universal congruence, is the kernel of a strong endomorphism on $\mathcal{A}$. In [2] the strong endomorphism kernel property is investigated in Ockham algebras by way of Priestley duality. A similar approach is adopted in considering this concept in the context of distributive p-algebras [6]. Here our objective is to investigate the strong endomorphism kernel property in distributive double p-algebras, and this we do also by way of Priestley duality.

We recall that a *double p-algebra* is an algebra $(L; \wedge, \vee, ^\star, ^+, 0, 1)$ in which $(L; \wedge, \vee, 0, 1)$ is a bounded lattice L (with bottom element 0 and top element 1) together with a mapping $^\star : L \to L$ such that $x \wedge y = 0$ if and only if $y \leqslant x^\star$ (the pseudocomplement of x), and a mapping $^+ : L \to L$ such that $x \vee y = 1$ if and only if $y \geqslant x^+$ (the dual pseudocomplement of x). A special subclass of the class of distributive double p-algebras that we shall also consider is the class of *double Stone algebras*, in which the unary operations * and $^+$ are

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such that $x^\star \vee x^{\star\star} = 1$ and $x^+ \wedge x^{++} = 0$ for every $x \in L$. A well-known fact for a double Stone algebra is that both * and $^+$ are dual lattice endomorphisms on L.

2. *dp*-spaces

As is established in [2], in an Ockham algebra $(L; f)$ with dual space $(X; g)$ the strong endomorphism kernel property depends on the set $C_g(X)$ of closed g-subsets of X together with those endomorphisms α of X such that $\alpha(Q) \subseteq Q$ for every $Q \in C_g(X)$. For distributive p-algebras considered in [6], the role of $C_g(X)$ is assumed by that of the set $C_p(X)$ of closed p-subsets. Here we adapt these approaches to determine the analogous closed subsets in the dual space of a distributive double p-algebra that are appropriate for the corresponding strong endomorphism kernel property.

We recall that a *Priestley space* is a compact totally order-disconnected topological space. For the basic properties of such spaces we refer the reader to [5].

Definition. By a *double p-space* we shall mean a Priestley space in which, for every clopen down-set U, the up-set $U^\uparrow$ is clopen; and, for every clopen up-set V, the down-set $V^\downarrow$ is clopen.

From the Priestley duality developed in [7] it follows that in a double p-space X the set $\mathcal{O}(X)$ of clopen down-sets is a distributive double p-algebra, the unary operations being given by $U^\star = X \setminus U^\uparrow$ and $V^+ = (X \setminus V)^\downarrow$; and conversely, if $(L; {}^\star, {}^+)$ is a distributive double p-algebra then the lattice $I_p(L)$ of prime ideals of L forms the double p-space $(I_p(L); \tau, \subseteq)$ where the topology τ has as a base the sets $\{x \in I_p(L) \mid x \ni a\}$ and $\{x \in I_p(L) \mid x \not\ni a\}$ for every $a \in L$. These constructions give $L \simeq \mathcal{O}\big(I_p(L)\big)$ and $X \simeq \big(I_p(\mathcal{O}(X)); \tau, \subseteq\big)$.

Let L be a distributive double p-algebra with dual space X. Throughout what follows we shall denote by $\operatorname{Min} X$ and $\operatorname{Max} X$ the sets of minimal and maximal elements of X. The subsets of *extremal* and *middle* elements are then defined respectively by

$$\operatorname{Ext} X = \operatorname{Min} X \cup \operatorname{Max} X, \quad \operatorname{Mid} X = X \setminus \operatorname{Ext} X.$$

Other subsets of particular importance that we shall consider are

$$(\forall x \in X) \qquad \mathrm{m}(x) = x^\downarrow \cap \operatorname{Min} X, \quad \mathrm{M}(x) = x^\uparrow \cap \operatorname{Max} X.$$

As in [7, Lemma 1], the subsets $\operatorname{Min} X$ and $\operatorname{Max} X$ are closed. Likewise, so are $\mathrm{m}(x)$ and $\mathrm{M}(x)$ for every $x \in X$.

Definition. By a *dp-subset* of X we shall mean a subset Q such that $Q^\downarrow \cap \operatorname{Min} X \subseteq Q$ and $Q^\uparrow \cap \operatorname{Max} X \subseteq Q$.

Clearly, $\operatorname{Ext} X$ is a closed dp-subset, as is every subset that contains $\operatorname{Ext} X$.

We denote by $C_{dp}(X)$ the set of closed dp-subsets of X. That $C_{dp}(X)$ is appropriate to the situation in hand is a consequence of the following observations, the proofs of which are analogous to those in [2].

(1) For every $Q \in C_{dp}(X)$ the relation ϑ_Q defined on $\mathcal{O}(X) \simeq L$ by

$$(A, B) \in \vartheta_Q \iff A \cap Q = B \cap Q$$

is a congruence, and the mapping $\vartheta : C_{dp}(X) \to \operatorname{Con} \mathcal{O}(X)$ given by $\vartheta(Q) = \vartheta_Q$ is a dual lattice isomorphism.

(2) By an endomorphism on X we shall mean a mapping $\alpha : X \to X$ that is isotone, continuous, and such that $\alpha(\mathrm{m}(x)) = \mathrm{m}(\alpha(x))$ and $\alpha(\mathrm{M}(x)) = \mathrm{M}(\alpha(x))$ for every $x \in X$. The monoid of such mappings will be denoted by $\mathrm{End}\, X$.

(3) By an endomorphism on L we shall mean a lattice morphism $\vartheta : L \to L$ that preserves 0 and 1, and commutes with both * and $^+$. The monoid of such mappings will be denoted by $\mathrm{End}\, L$.

(4) The monoids $\mathrm{End}\, X$ and $\mathrm{End}\, L \simeq \mathrm{End}\, \mathcal{O}(X)$ are anti-isomorphic under the assignment $\alpha \mapsto \overline{\alpha}$ where $\overline{\alpha} : \mathcal{O}(X) \to \mathcal{O}(X)$ is given by $\overline{\alpha}(A) = \alpha^{-1}(A)$.

(5) If $\vartheta \in \mathrm{Con}\, L$ and $h \in \mathrm{End}\, L$ then the relation $h \star \vartheta$ defined on L by

$$(x, y) \in h \star \vartheta \iff \big(h(x), h(y)\big) \in \vartheta$$

is a congruence on L, and h is compatible with ϑ if and only if $\vartheta \subseteq h \star \vartheta$.

(6) With the above terminology and notation, the following is a direct translation to distributive double p-algebras of [2,Theorem 5].

Theorem 1. *Let X be a double p-space. If $Q \in C_{dp}(X)$ and $\alpha \in \mathrm{End}\, X$ then*

(1) $\alpha(Q) \in C_{dp}(X)$ *with* $\vartheta_{\alpha(Q)} = \overline{\alpha} \star \vartheta_Q$;

(2) $\vartheta_{\alpha(X)} = \ker \overline{\alpha}$;

(3) $\overline{\alpha}$ *is compatible with* ϑ_Q *if and only if* $\alpha(Q) \subseteq Q$. □

(7) An important subset of $\mathrm{End}\, X$ is the set $\Gamma(X)$ of the endomorphisms α of X such that $\alpha(Q) \subseteq Q$ for all $Q \in C_{dp}(X)$. Relative to this, and corresponding to [2,Theorem 7], we have the following fundamental characterization.

Theorem 2. *Let L be a distributive double p-algebra with dual space X. Then L has the strong endomorphism kernel property if and only if, for every non-empty closed dp-subset Q of X, there exists $\alpha \in \Gamma(X)$ such that* $\mathrm{Im}\, \alpha = Q$. □

3. Structure of the dual space

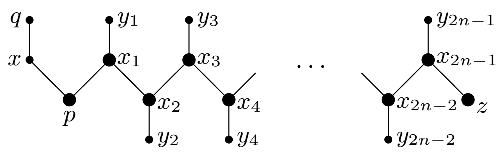
In what follows we suppose that L is a distributive double p-algebra with dual space X. We begin with two important properties of $\mathrm{Ext}\, X$ and $\alpha \in \Gamma(X)$.

Theorem 3. *If the dual space X of L is order connected then a non-empty subset Q of X is a dp-subset if and only if* $\mathrm{Ext}\, X \subseteq Q$.

Proof. $\Rightarrow$: Let Q be a dp-subset and let $x \in Q$. Then there exist $p \in \mathrm{Min}\, X$ and $q \in \mathrm{Max}\, X$ such that $p \leqslant x \leqslant q$. Then $p \in Q^{\downarrow} \cap \mathrm{Min}\, X \subseteq Q$ and $q \in Q^{\uparrow} \cap \mathrm{Max}\, X \subseteq Q$. To see that $\mathrm{Min}\, X \subseteq Q$, let $z \in \mathrm{Min}\, X$. Since X is connected, there exists a finite zig-zag chain from p to z, say

$$p = x_0 < x_1 > x_2 < x_3 > \cdots > x_{2n-2} < x_{2n-1} > x_{2n} = z,$$

and then maximal and minimal elements $y_i \nparallel x_i$ producing the connected subset



Since $p \in Q$ we have $y_1 \in Q^{\uparrow} \cap \mathrm{Max}\, X \subseteq Q$ and then $y_2 \in Q^{\downarrow} \cap \mathrm{Min}\, X \subseteq Q$, and so on. Continuing in this way, we obtain $y_{2n-1} \in Q$ whence $z \in Q^{\downarrow} \cap \mathrm{Min}\, X \subseteq Q$. Consequently, $\mathrm{Min}\, X \subseteq Q$. Similarly, $\mathrm{Max}\, X \subseteq Q$ and therefore $\mathrm{Ext}\, X \subseteq Q$.

$\Leftarrow$: If $\mathrm{Ext}\, X \subseteq Q$ then, as observed above, Q is a dp-subset. ⊡

Theorem 4. *If* $\alpha \in \Gamma(X)$ *then*

(1) $\alpha(\operatorname{Ext} X) \subseteq \operatorname{Ext} X$;

(2) $\alpha(x) = x$ *for every* $x \in \operatorname{Im}\alpha \setminus \operatorname{Ext} X$ *and* $\alpha(x) \in \operatorname{Ext} X$ *otherwise.*

Proof. (1) This is clear from the definition of $\Gamma(X)$.

(2) If $x \in \operatorname{Im}\alpha \setminus \operatorname{Ext} X$ then there exists $y \in X$ such that $x = \alpha(y)$, so that $\alpha^{-1}\{x\} \neq \emptyset$. If now $Q = \alpha^{-1}\{x\} \cup \operatorname{Ext} X$ then we have $Q \in C_{dp}(X)$ and therefore $\alpha(Q) \subseteq Q$. It follows that $x \in \alpha^{-1}\{x\}$ and so $\alpha(x) = x$. On the other hand, if $x \notin \operatorname{Im}\alpha \setminus \operatorname{Ext} X$ then either $x \in \operatorname{Ext} X$ or $x \in X \setminus \operatorname{Im}\alpha$. In the former case, $\alpha(x) \in \operatorname{Ext} X$ by (1). In the latter, $\alpha(x) \neq x$ and $S = \{x\} \cup \operatorname{Ext} X$ is a closed *dp*-subset. Then $\alpha(S) \subseteq S$ gives $\alpha(x) \in \operatorname{Ext} X$. ⊡

When L has the strong endomorphism kernel property, fundamental properties of X are the following.

Theorem 5. *If* L *has the strong endomorphism kernel property then*

(1) X *is connected*;

(2) $\ell(X) \leqslant 3$*, and* $\{x\}$ *is clopen for every* $x \in \operatorname{Mid} X$;

(3) *if* $Q \in C_{dp}(X)$ *and* $\alpha \in \Gamma(X)$ *is such that* $\operatorname{Im}\alpha = Q$ *then* $\alpha(\operatorname{Ext} X) = \operatorname{Ext} X$.

Proof. (1) Let $X = \bigcup_{i \in I} X_i$ in which the X_i are the mutually disjoint order components of X. For each $i \in I$ let $G_i = \operatorname{Ext} X_i$. Then by [7,Lemma 1] each G_i is closed. Since $G_i^{\downarrow} \cap \operatorname{Min} X = G_i^{\downarrow} \cap \operatorname{Min} X_i \subseteq G_i$ and $G_i^{\uparrow} \cap \operatorname{Max} X = G_i^{\uparrow} \cap \operatorname{Max} X_i \subseteq G_i$ we see that each $G_i \in C_{dp}(X)$. Clearly, each G_i is then a minimal element of $C_{dp}(X)$. Consequently, to each G_i there corresponds a unique coatom of $\operatorname{Con} L$. But, as established in [2,Theorem 1], if an algebra $\mathcal{A}$ has the strong endomorphism kernel property then $\mathcal{A}$ has at most one maximal congruence. It follows therefore that we must have $|I| = 1$ whence X is connected.

(2) Suppose, by way of obtaining a contradiction, that $\ell(X) > 3$. Then there exist $p, q, x, y, z \in X$ such that $p < x < y < z < q$ where $p \in \operatorname{Min} X$ and $q \in \operatorname{Max} X$. Consider the closed *dp*-subset $Q = \{x, z\} \cup \operatorname{Ext} X$. By Theorem 2 there exists $\alpha \in \Gamma(X)$ such that $\operatorname{Im}\alpha = Q$. Also, by Theorem 4, $\alpha(x) = x$, $\alpha(z) = z$ and $\alpha(y) \in \operatorname{Ext} X$. Then since $\alpha(\operatorname{Ext} X) \subseteq \operatorname{Ext} X$ we have $\alpha(p) < x \leqslant \alpha(y) \leqslant z < \alpha(q)$ where $\alpha(p), \alpha(q) \in \operatorname{Ext} X$ and $x, z \notin \operatorname{Ext} X$. There follows the contradiction that $\alpha(y) \notin \operatorname{Ext} X$. Consequently, we must have $\ell(X) \leqslant 3$.

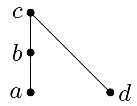
As for the second statement, let $x \in \operatorname{Mid} X$. It is enough to show that $\{x\}$ is open. For this purpose, consider the closed *dp*-subset $Q = \{x\} \cup \operatorname{Ext} X$. By Theorem 2 there exists $\alpha \in \Gamma(X)$ such that $\operatorname{Im}\alpha = Q$. Then $x \in \operatorname{Im}\alpha \setminus \operatorname{Ext} X$ and so, by Theorem 4, $\alpha(x) = x$. To see that $\{x\}$ is open it suffices to show that $X \setminus \{x\}$ coincides with the closed set $\alpha^{-1}(\operatorname{Ext} X)$. For $y \in X \setminus \{x\}$ consider the closed *dp*-subset $S = \{y\} \cup \operatorname{Ext} X$. Since $\alpha(S) \subseteq S$ we have

$$\alpha(y) \in Q \cap S = \big(\{x\} \cup \operatorname{Ext} X\big) \cap \big(\{y\} \cup \operatorname{Ext} X\big) = \operatorname{Ext} X$$

whence $y \in \alpha^{-1}(\operatorname{Ext} X)$ and so $X \setminus \{x\} \subseteq \alpha^{-1}(\operatorname{Ext} X)$. To obtain the reverse inclusion, let $y \in \alpha^{-1}(\operatorname{Ext} X)$. Then $\alpha(y) \in \operatorname{Ext} X$, whence $y \neq x$ since otherwise we would have the contradiction that $x = \alpha(x) = \alpha(y) \in \operatorname{Ext} X$. Hence $\alpha^{-1}(\operatorname{Ext} X) \subseteq X \setminus \{x\}$ and the required equality follows.

(3) If $x \in \operatorname{Min} X$ then, by Theorem 3, $x \in Q = \operatorname{Im}\alpha$. Then there exists $y \in X$ such that $x = \alpha(y)$ whence $x = \alpha(z)$ for all $z \leqslant y$, so that $x \in \alpha(\operatorname{Min} X) \subseteq \alpha(\operatorname{Ext} X)$. Thus $\operatorname{Min} X \subseteq \alpha(\operatorname{Ext} X)$. Similarly, $\operatorname{Max} X \subseteq \alpha(\operatorname{Ext} X)$ and so $\operatorname{Ext} X \subseteq \alpha(\operatorname{Ext} X)$. Equality now follows by Theorem 4(1). ⊡

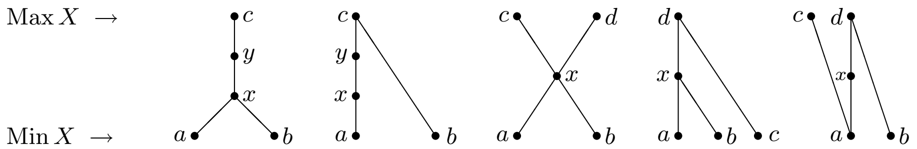
Example 1. Let X be the double p-space with Hasse diagram



Here $\ell(X) = 2$ with $\mathrm{Mid}\, X = \{b\}$. The closed dp-subsets are X itself and $\mathrm{Ext}\, X = X \setminus \{b\}$. If $Q = X$ then $\mathrm{id}_X \in \Gamma(X)$ with $\mathrm{Im}\, \mathrm{id}_X = Q$. If $Q = \mathrm{Ext}\, X$, define $\alpha : X \to X$ by $\alpha(b) = a$ and $\alpha(x) = x$ for $x \neq b$. Then α is clearly continuous and isotone. Observe that $\mathrm{M}(b) = \mathrm{M}(a) = \{c\}$ and $\mathrm{m}(b) = \mathrm{m}(a) = \{a\}$. Then $\alpha(\mathrm{M}(b)) = \{c\} = \mathrm{M}(\alpha(b))$ and $\alpha(\mathrm{m}(b)) = \{a\} = \mathrm{m}(\alpha(b))$. Thus $\alpha \in \Gamma(X)$ with $\mathrm{Im}\, \alpha = \mathrm{Ext}\, X$. It follows by Theorem 2 that the corresponding distributive double p-algebra has the strong endomorphism kernel property.

We now highlight certain subsets of X that are forbidden by the strong endomorphism kernel property.

Theorem 6. *If L has the strong endomorphism kernel property and if its dual space X is such that* $\mathrm{Ext}\, X$ *is finite then X has no subsets of the forms*



(S_1) (S_2) (S_3) (S_4) (S_5)

or their duals.

Proof. (1) Suppose that X has a subset of the form (S_1). Then, since $\ell(X) \leqslant 3$ by Theorem 5, necessarily $a, b \in \mathrm{Min}\, X$, $x, y \in \mathrm{Mid}\, X$ and $c \in \mathrm{Max}\, X$. Let $Q = X \setminus \{x\}$. Then $Q \in C_{dp}(X)$ and by Theorem 2 there exists $\alpha \in \Gamma(X)$ such that $\mathrm{Im}\, \alpha = Q$. By Theorem 4, $\alpha(y) = y$ and $\alpha(x) \in \mathrm{Ext}\, X$. Since α is isotone it follows that $\alpha(x) = z \in \mathrm{m}(y)$. Then $\alpha(\mathrm{m}(x)) = \mathrm{m}(\alpha(x)) = \mathrm{m}(z) = \{z\}$ and therefore $\alpha(b) = \alpha(a) = z$. Since $\alpha(\mathrm{Ext}\, X) = \mathrm{Ext}\, X$ by Theorem 5, and $\mathrm{Ext}\, X$ is finite, there follows the contradiction that $b = a$. Hence no such subset exists, and likewise for its dual.

(2) Suppose that X has a subset of the form (S_2). Let $Q = X \setminus \{y\} \in C_{dp}(X)$ and let $\alpha \in \Gamma(X)$ be such that $\mathrm{Im}\, \alpha = Q$. Then, by Theorem 4, $x = \alpha(x) < \alpha(y) \in \mathrm{Ext}\, X$. Since, by the dual of (1), $|y^{\uparrow}| = 2$ it follows that $\alpha(y) = c = \alpha(c)$. Then $\alpha(b) \in \alpha(\mathrm{m}(c)) = \mathrm{m}(\alpha(c)) = \mathrm{m}(\alpha(y)) = \alpha(\mathrm{m}(y))$ and so $\alpha(b) = \alpha(z)$ for some $z \in \mathrm{m}(y)$. Since α is a bijection on $\mathrm{Ext}\, X$, there follows the contradiction $b = z \in \mathrm{m}(y)$. Hence no such subset exists, and likewise for its dual.

(3) Suppose that X has a subset of the form (S_3). Then clearly $x \in \mathrm{Mid}\, X$. If either c or d does not belong to $\mathrm{Max}\, X$ then by (1) we have a contradiction; and dually when either a or b does not belong to $\mathrm{Min}\, X$. We may therefore suppose that $a, b \in \mathrm{Min}\, X$ and $c, d \in \mathrm{Max}\, X$. Let $Q = X \setminus \{x\} \in C_{dp}(X)$ and let $\alpha \in \Gamma(X)$ be such that $\mathrm{Im}\, \alpha = Q$. Then $\alpha(x) \in \mathrm{Ext}\, X$. If $\alpha(x) = z \in \mathrm{Min}\, X$ then $\alpha(a) = \alpha(b) = z$ and we have the contradiction $b = a$; and, dually, the contradiction $c = d$ if $z \in \mathrm{Max}\, X$. Hence no such subset exists.

(4) Suppose that X has a subset of the form (S_4). Let $Q = X \setminus \{x\} \in C_{dp}(X)$ and let $\alpha \in \Gamma(X)$ be such that $\mathrm{Im}\, \alpha = Q$. There are two possibilities to consider, namely $\alpha(x) \in \mathrm{Min}\, X$ and $\alpha(x) \in \mathrm{Max}\, X$. In the former case we have $\alpha(x) = \alpha(a) = \alpha(b)$ whence the contradiction $a = b$. In the latter case we have $\alpha(x) = \alpha(d)$ whence $\alpha(c) \in \alpha(\mathrm{m}(d)) = \mathrm{m}(\alpha(d)) = \mathrm{m}(\alpha(x)) = \alpha(\mathrm{m}(x))$ and so $\alpha(c) = \alpha(y)$ for some $y \in \mathrm{m}(x)$. There follows the contradiction $c = y \in \mathrm{m}(x)$. Hence no such subset exists, and likewise for its dual.

(5) Suppose that X has a subset of the form (S_5). Let $Q = X \setminus \{x\} \in C_{dp}(X)$ and let $\alpha \in \Gamma(X)$ be such that $\operatorname{Im}\alpha = Q$. Then either $\alpha(x) \in \operatorname{Min} X$ or $\alpha(x) \in \operatorname{Max} X$. In the former case, we have $\alpha(x) = \alpha(a)$ whence $\alpha(c) \in \alpha(\mathrm{M}(a)) = \mathrm{M}(\alpha(a)) = \mathrm{M}(\alpha(x)) = \alpha(\mathrm{M}(x))$ and so $\alpha(c) = \alpha(y)$ for some $y \in \mathrm{M}(x)$ whence there follows the contradiction $c = y \in \mathrm{M}(x)$. In the latter case, $\alpha(x) = \alpha(d)$ whence $\alpha(b) \in \alpha(\mathrm{m}(d)) = \mathrm{m}(\alpha(d)) = \mathrm{m}(\alpha(x)) = \alpha(\mathrm{m}(x))$ and so $\alpha(b) = \alpha(z)$ for some $z \in \mathrm{m}(x)$ whence there follows the contradiction $b = z \in \mathrm{m}(x)$. Hence no such subset exists. ⊡

4. Finite distributive double p-algebras

The finite distributive double p-algebras that have the strong endomorphism kernel property may be characterized by Priestley duality as follows.

Theorem 7. *Let L be a finite distributive double p-algebra with dual space X. Then L has the strong endomorphism kernel property if and only if*

(1) *X is connected with $\ell(X) \leqslant 3$;*

(2) *every $x \in \operatorname{Mid} X$ is such that*

(*a*) $x^{\uparrow} = \{x, q\}$ *with* $\mathrm{m}(x) = \mathrm{m}(q)$,

or (*b*) $x^{\downarrow} = \{x, p\}$ *with* $\mathrm{M}(x) = \mathrm{M}(p)$.

Proof. $\Rightarrow$: Suppose that L has the strong endomorphism kernel property. Then (1) follows from Theorem 5. As for (2), by Theorem 6 subsets of the forms (S_1), (S_1^d), and (S_3) are forbidden. Consequently, for every $x \in \operatorname{Mid} X$, either $|x^{\uparrow}| = 2$ or $|x^{\downarrow}| = 2$. If $|x^{\uparrow}| = 2$ then there exists $q \in \operatorname{Max} X$ such that $x^{\uparrow} = \{x, q\}$. There are two possibilities, namely $\mathrm{m}(x) = \mathrm{m}(q)$ or $\mathrm{m}(x) \neq \mathrm{m}(q)$. If the former holds then so does (2)(a). If the latter holds then since (S_4) is forbidden we must have $x^{\downarrow} = \{x, p\}$, and since (S_2^d) and (S_5) are forbidden it follows that $\mathrm{M}(x) = \mathrm{M}(p)$ which gives (2)(b). The case where $|x^{\downarrow}| = 2$ is dealt with by a similar argument.

$\Leftarrow$: Suppose now that conditions (1) and (2) hold. Let $Q \in C_{dp}(X) \setminus \emptyset$ and define $\alpha : X \to Q$ as follows:

- if $x \in Q$ define $\alpha(x) = x$;
- if $x \notin Q$ then necessarily $x \in \operatorname{Mid} X$ by (1) and Theorem 3 so, in relation to (2)(a)(b), define

$$\alpha(x) = \begin{cases} q & \text{if } x^{\uparrow} = \{x, q\}; \\ p & \text{if } x^{\downarrow} = \{x, p\}. \end{cases}$$

Clearly, α is isotone.

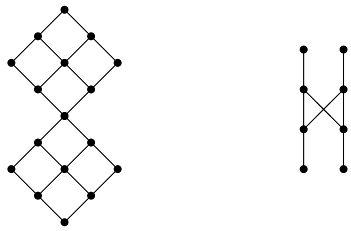
If now $x \in Q$ then $\mathrm{m}(\alpha(x)) = \mathrm{m}(x) = \alpha(\mathrm{m}(x))$ and $\mathrm{M}(\alpha(x)) = \mathrm{M}(x) = \alpha(\mathrm{M}(x))$, whereas if $x \notin Q$ then

$$\mathrm{m}(\alpha(x)) = \begin{cases} \mathrm{m}(q) & \text{if } x^{\uparrow} = \{x, q\}; \\ \mathrm{m}(p) = \{p\} & \text{if } x^{\downarrow} = \{x, p\}, \end{cases}$$

$$\alpha(\mathrm{m}(x)) = \mathrm{m}(x) = \begin{cases} \mathrm{m}(q) & \text{if } x^{\uparrow} = \{x, q\}; \\ \{p\} & \text{if } x^{\downarrow} = \{x, p\}. \end{cases}$$

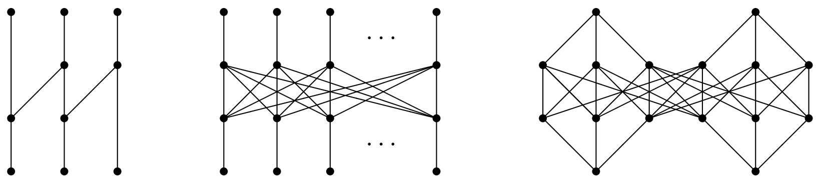
Thus $\mathrm{m}(\alpha(x)) = \alpha(\mathrm{m}(x))$ and similarly $\mathrm{M}(\alpha(x)) = \alpha(\mathrm{M}(x))$. Consequently, $\alpha \in \operatorname{End} X$ with $\operatorname{Im}\alpha = Q$. It follows by Theorem 2 that L has the strong endomorphism kernel property. ⊡

Example 2. The distributive double p-algebra described by the vertical sum $L = \mathbf{3}^2 \,\overline{\oplus}\, \mathbf{3}^2$ and its dual space X have respective Hasse diagrams



Since X satisfies the conditions of Theorem 7, we see that L has the strong endomorphism kernel property.

Example 3. Each of the following Hasse diagrams satisfies the conditions of Theorem 7 and therefore represents the dual space of a finite distributive double p-algebra that has the strong endomorphism kernel property:

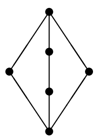


Theorem 7 applies to the special subclass of double Stone algebras. Here there is a considerable simplification. Indeed, as observed in [4], every connected component of the dual space of a Stone algebra has a bottom element. It follows that if L is a double Stone algebra then every order connected component of its dual space X has both a bottom element and a top element. Consequently, X is itself connected if and only if it has both a bottom element and a top element. The finite double Stone algebras that have the strong endomorphism kernel property can therefore be described as follows.

Theorem 8. *Let L be a finite double Stone algebra with dual space X. Then L has the strong endomorphism kernel property if and only if $\ell(X) \leqslant 3$ and X has both a bottom element and a top element.*

Proof. The conditions are necessary by Theorem 7. As for sufficiency, if $\ell(X) \leqslant 3$ and X has both a bottom element 0 and a top element 1 then $\mathrm{m}(x) = 0$ and $\mathrm{M}(x) = 1$ for every $x \in X$, so that condition (2) of Theorem 7 is trivial. ⊡

Example 4. By way of illustration, the finite double Stone algebra $\{0\} \oplus (\mathbf{2}^2 \times \mathbf{3}) \oplus \{1\}$ has dual space the ordered set



and so has the strong endomorphism kernel property.

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